LARGE SOLUTIONS FOR EQUATIONS INVOLVING THE p-LAPLACIAN AND SINGULAR WEIGHTS

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Abstract. In this paper we consider the boundary blow-up problem $\Delta_p u = a(x)u^q$ in a smooth bounded domain Ω of $\mathbb{R}^N$, with $u = +\infty$ on $\partial\Omega$. Here $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the well-known p-Laplacian operator with $p > 1$, $q > p - 1$, and $a(x)$ is a nonnegative weight function which can be singular on $\partial\Omega$. Our results include existence, uniqueness and exact boundary behavior of positive solutions.

1. Introduction

The objective of the present paper is to describe the set of solutions to the quasilinear boundary blow-up elliptic problem

$$\begin{cases} \Delta_p u = a(x)u^q & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth (say C^2) bounded domain of $\mathbb{R}^N$, Δ_p stands for the p-Laplacian operator defined by $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$ and $q > p - 1$. The weight function $a(x)$ is assumed to be continuous in Ω and nonnegative, but it may be singular on $\partial\Omega$. The boundary condition is to be understood as usual as $u(x) \to +\infty$ when $d(x) \to 0+$, where $d(x)$ stands for the distance function $\mathrm{dist}(x, \partial\Omega)$.

There is a huge amount of works dealing with boundary blow-up problems related with (1.1). They generally deal with the semilinear case $p = 2$ (we quote for instance the pioneering works [4], [23] and [36]). The basic questions which have been considered are existence, uniqueness and boundary behavior of solutions: see [2], [3], [28], [24], [40] or [31] for semilinear problems with general nonlinearities.

The quasilinear elliptic problem

$$\begin{cases} \Delta_p u = f(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

for general nonlinearities $f(u)$ seems to have been first considered in [11]. The questions of existence, uniqueness and boundary behavior of solutions were dealt with there. There have been since then some other papers which included similar results for different types of nonlinearities: we mention for instance [21], [22], [32], [33].

When the right-hand side of (1.2) depends also on x by means of a nonlinearity of the form $a(x)f(u)$, and the weight $a(x)$ is bounded, the problem has been considered by several authors: [34], [13], [14]. But when the weight $a(x)$ is not necessarily bounded, very little is known except for the case $p = 2$, mainly dealing with (1.1): if the growth of a near $\partial\Omega$ is not too strong then solutions to (1.1) exist for $q > 1$ (see [41], [5], [6]). However, solutions may exist with $q < 1$, provided a is singular enough on $\partial\Omega$. We refer the reader to [35] and [5]. We also quote the works [28], [24], [2], [3], [40], [11], [10], [18], [15], [16] and [17] where several interesting features related to (1.1) with $p = 2$ are analyzed (see also [19], where $p = 2$ and q is allowed to be a variable positive function).

We finally mention that some more work has recently been done with regard to boundary estimates and uniqueness of positive solutions to (1.1) when $p = 2$. The basic question is to find sufficient conditions on the weight a to guarantee uniqueness. Some conditions of this type have been found in [7], [8], [9], [29], [30], [17].

In the present work, we extend some of the previous results to the context of the p-Laplacian. Although some of the techniques are an adaptation of the corresponding ones for $p = 2$, the generalization is not straightforward. One of the delicate points is to obtain existence of positive solutions to the equation in (1.1) with finite datum on $\partial\Omega$ when $a(x)$ is singular on $\partial\Omega$. The idea is to truncate the weight, but then some control on the solutions to the truncated problems is needed. In the semilinear case this control is achieved thanks to an existence theorem in [20], which, at the best of our knowledge, is not available for the p-Laplacian.

The second important point is a uniqueness result for (1.1) in a half-space. When $p = 2$, it was obtained in [6], by means of a technique which is not applicable in our case, for linearity was important there. We overcome this difficulty by analyzing first the one-dimensional version of (1.1) in the half-line, but even in this case the proof is not immediate.

We also point out that our hypotheses below on $a(x)$ allow it to be singular at some points of $\partial\Omega$ or to vanish on some others, with a growth rate that may depend on the point. Thus Theorem 1 is new even when $p = 2$.

We finally come to the statement of our main result. Let us mention that we are assuming for simplicity that the functions $Q(x)$ and $\gamma(x)$ involved in the boundary behavior of $a(x)$ are defined in $\overline{\Omega}$.

Theorem 1. *Let $a \in C(\Omega)$ be a nonnegative function and $q > p - 1$. Assume $a > 0$ in a neighborhood of $\partial\Omega$, and that there exist functions $Q \in C(\overline{\Omega})$, $\gamma \in C^{\mu}(\overline{\Omega})$, $0 < \mu \leq 1$, and $Q(x) > 0$ on $\partial\Omega$ and $\gamma < p$ on $\partial\Omega$ such that*

$$\lim_{x \to x_0} d(x)^{\gamma(x)} a(x) = Q(x_0)$$

for every $x_0 \in \partial\Omega$. Then problem (1.1) admits a unique positive weak solution $u \in W^{1,p}_{\rm loc}(\Omega) \cap C^{1,\eta}_{\rm loc}(\Omega)$ for some $\eta \in (0,1)$, which in addition verifies

$$\lim_{x \to x_0} d(x)^{\alpha(x)} u(x) = \left(\frac{(p-1)\alpha(x_0)^{p-1}(\alpha(x_0)+1)}{Q(x_0)} \right)^{\frac{1}{q-p+1}}, \tag{1.3}$$

for every $x_0 \in \partial\Omega$, where $\alpha(x) = (p-\gamma(x_0))/(q-p+1)$ and $\alpha(x_0)$ is defined analogously.

Remark 1. We remark that condition $\gamma(x) < p$ on $\partial\Omega$ is essential for existence. Indeed, a careful analysis of the proof of Lemma 5 in Section 4 shows that the upper inequality (4.1) remains valid for all positive solutions to the equation $\Delta_p u = a(x) u^q$ even if $\gamma(x_0) \geq p$ at some point $x_0 \in \partial\Omega$. Thus solutions of the equation do not blow-up at x_0.

The paper is organized as follows: in Section 2 we consider two auxiliary important results, one of them dealing with solutions of equation with singular weights and the other one with problem (1.1) in a half-space. Section 3 will be dedicated to prove existence of a positive solution, while the boundary estimates and uniqueness of positive solutions will be the contents of Section 4.

2. Preliminaries

In this section we are proving two basic theorems which will be essential in our proofs in Sections 3 and 4. The first one is a generalization of a result in [20] to the context of the p-Laplacian (see Lemma 4.9 and Exercise 4.6 there).

Theorem 2. *Let $f \in L^\infty_{\rm loc}(\Omega)$ be such that $|f| \leq C_0 d^{-\gamma}$ for some $C_0 > 0$ and $\gamma \in (1,p)$. Then the problem*

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

has a unique weak solution $u \in W^{1,p}_{\rm loc}(\Omega) \cap C^{1,\eta}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$ for some $\eta \in (0,1)$. Moreover, there exists a positive constant C not depending on f such that

$$|u| \leq C C_0^{\frac{1}{p-1}} d^{\frac{p-\gamma}{p-1}} \qquad \text{in } \Omega. \tag{2.2}$$

Proof. We first show that we can construct a supersolution with the appropriate growth near $\partial\Omega$. For this aim, we consider the unique solution ϕ to

$$\begin{cases} -\Delta_p \phi = 1 & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

which is well-known to exist and to verify $\phi \in W_0^{1,p}(\Omega) \cap C^{1,\eta}(\overline{\Omega})$ for some $\eta \in (0,1)$. Moreover, thanks to Hopf's maximum principle (see [39]), we have that

$$C_1 d \leq \phi \leq C_2 d \qquad \text{in } \Omega, \tag{2.3}$$

for some positive constants C_1, C_2. We claim that $\overline{u} = \phi^\alpha$, where $\alpha = \frac{p-\gamma}{p-1}$, is a supersolution to $-\Delta_p u = Cd^{-\gamma}$ for a suitable positive constant C, which vanishes on $\partial\Omega$. Indeed, a calculation shows that

$$-\Delta_p \overline{u} = \alpha^{p-1} \phi^{-\gamma}((\gamma - 1)|\nabla\phi|^p + \phi).$$

We now observe that $|\nabla\phi|^p + \phi > 0$ in $\overline{\Omega}$, since $\phi > 0$ in Ω and $\nabla\phi \neq 0$ on $\partial\Omega$ by Hopf's principle. Since $\gamma > 1$, and taking into account (2.3), we obtain that $-\Delta_p \overline{u} \geq Cd^{-\gamma}$ for some positive constant C.

Next we consider the family of domains $\Omega_k = \{x \in \Omega : \ d(x) > 1/k\}$. Since $f \in L^\infty(\Omega_k)$ for all k, the problem

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega_k \\ u = 0 & \text{on } \partial\Omega_k \end{cases} \tag{2.4}$$

has a unique solution $u_k \in W_0^{1,p}(\Omega_k) \cap C^{1,\eta}(\overline{\Omega}_k)$, for some $\eta \in (0,1)$. Moreover, since

$$-\Delta_p u_k = f \leq C_0 d^{-\gamma} \leq -\Delta_p((C_0/C)^{\frac{1}{p-1}} \overline{u}) \qquad \text{in } \Omega_k$$

while $\overline{u} > 0$ on $\partial\Omega_k$, the comparison principle (see [38]) implies $u_k \leq (C_0/C)^{\frac{1}{p-1}} \overline{u}$ in Ω_k. Since a comparison from below can be similarly made, we arrive at

$$|u_k| \leq (C_0/C)^{\frac{1}{p-1}} \phi^{\frac{p-\gamma}{p-1}} \leq C C_0^{\frac{1}{p-1}} d^{\frac{p-\gamma}{p-1}}, \tag{2.5}$$

for a positive constant C that does not depend on k or on f. Thus we obtain uniform local bounds for u_k, and thanks to the interior $C^{1,\eta}$ estimates for the p-Laplacian (see [12], [27] or [37]), we obtain that the sequence $\{u_k\}$ is precompact in C^1, and hence it is standard to obtain that $u_k \to u$ in $C^1_{\text{loc}}(\Omega)$ as $k \to \infty$. Passing to the limit in the weak formulation of (2.4) we obtain that u is a solution to $-\Delta_p u = f$ in Ω, and thanks to (2.5) it also verifies (2.2). Thus u is a weak solution to (2.1), and the comparison principle implies it is the only one. This concludes the proof. □

Remarks 2. (a) We remark that the supersolution $\overline{u}$ belongs to $W^{1,p}_{\text{loc}}(\Omega)$, but $\overline{u} \in W^{1,p}(\Omega)$ only when $\gamma < 2 - 1/p$. A similar conclusion can be made for the solution u to (2.1).

(b) When $0 < \gamma \leq 1$, Theorem 2 can still be applied, since $|f| \leq C_0 d^{-\gamma}$ implies $|f| \leq C_0 d^{-\theta}$ for every $\theta > \gamma$. However, we observe that the estimate (2.2) for the solution is not optimal in this case. It is indeed possible to prove that $|u| \leq Cd$, at least if $0 < \gamma < 1$.

(c) When f verifies $\liminf_{x\to x_0} d(x)^\gamma f(x) > 0$ for some $x_0 \in \partial\Omega$ and $\gamma \geq p$, bounded solutions to $-\Delta_p u = f$ do not exist.

Our next result deals with problem (1.1) in a half-space $D = \{x \in \mathbb{R}^N : \ x_1 > 0\}$, where for $x \in \mathbb{R}^N$ we write $x = (x_1, x')$. This problem will arise in the proof of estimates (1.3). We remark that the semilinear version of Theorem 3 below was proved in [6] by entirely different methods, which do not apply here since linearity was essential. Also, the result there was slightly weaker, because a growth restriction on the solutions was needed.

Theorem 3. *Let $Q_0 > 0$, $\gamma < p$, $q > p-1$ and $u \in W^{1,p}_{\rm loc}(D)$ a positive weak solution to the problem*

$$\left\{ \begin{array}{ll} \Delta_p u = Q_0 x_1^{-\gamma} u^q & \text{in } D, \\ u = +\infty & \text{on } \partial D. \end{array} \right. \tag{2.6}$$

Then

$$u(x) = \left(\frac{(p-1)\alpha^{p-1}(\alpha+1)}{Q_0} \right)^{\frac{1}{q-p+1}} x_1^{-\alpha}, \tag{2.7}$$

where $\alpha = (p-\gamma)/(q-p+1)$.

The proof of Theorem 3 relies on a similar uniqueness result for the one-dimensional version of problem (2.6). We consider this problem first.

Lemma 4. *Let $q > p-1$, $Q_0 > 0$ and $\gamma < p$, and $u \in W^{1,p}_{\rm loc}(0,\infty)$ a positive weak solution to the problem*

$$\left\{ \begin{array}{ll} (|u'|^{p-2}u')' = Q_0 x^{-\gamma} u^q & \text{in } x > 0, \\ u(0) = +\infty. \end{array} \right. \tag{2.8}$$

Then

$$u(x) = \left(\frac{(p-1)\alpha^{p-1}(\alpha+1)}{Q_0} \right)^{\frac{1}{q-p+1}} x^{-\alpha},$$

where $\alpha = (p-\gamma)/(q-p+1)$.

Proof. We first prove that all positive solutions to (2.8) verify

$$C_1 x^{-\alpha} \leq u \leq C_2 x^{-\alpha} \qquad \text{in } x > 0, \tag{2.9}$$

for some positive constants C_1, C_2 (the proof is similar to that of Lemma 5 in Section 4). Fix $x > 0$ and consider $v(y) = x^\alpha u(x + xy)$. Then $(|v'|^{p-2}v')' = Q_0(1+y)^{-\gamma} v^q$ in $|y| < 1$. Thus $v \leq U$, where U is the solution to $(|U'|^{p-2}U')' = Q_0(1+y)^{-\gamma}U^q$ in $|y| < 1$ with $U(\pm 1) = +\infty$. Setting $y = 0$ we obtain $x^\alpha u(x) \leq U(0)$, which proves the upper inequality in (2.9). The lower inequality is obtained similarly, since $v \geq V$, which is the unique solution to $(|V'|^{p-2}V')' = Q_0(1+y)^{-\gamma}V^q$ in $|y| < 1$ with $V(-1) = 1$, $V(1) = 0$. Thus (2.9) is proved.

Let us now show that estimates (2.9) imply similar estimates for the derivative u'. Observe first that $u' < 0$ in $(0,\infty)$, as is standard to show. Also, since $|u'|^{p-2}u'$ is increasing we obtain that u' is increasing, and thus it has a limit at infinity. On the other hand, (2.9) implies $u \to 0$ at infinity, and thus $u' \to 0$ as $x \to +\infty$.

We have

$$(|u'|^{p-2}u')' = Q_0 x^{-\gamma} u^q \le Q_0 C_2^q x^{-\gamma-\alpha q} = Q_0 C_2^q x^{-(\alpha+1)(p-1)-1},$$

and integrating between two arbitrary values x and y, $x < y$, we arrive at

$$|u'(y)|^{p-2}u'(y) - |u'(x)|^{p-2}u'(x) \le \frac{Q_0 C_2^q}{\alpha(p+1)}(x^{-(\alpha+1)(p-1)} - y^{-(\alpha+1)(p-1)}).$$

Letting $y \to \infty$ we have

$$-|u'(x)|^{p-2}u'(x) \le C x^{-(\alpha+1)(p-1)},$$

for every $x > 0$, and some positive constant C (from now on, we are using the letter C to denote different constants, not necessarily the same everywhere). Thus

$$-u'(x) \le C x^{-\alpha-1} \qquad \text{for } x > 0.$$

It is similarly proved that a lower inequality of this type also holds. Hence,

$$C_1 x^{-\alpha-1} \le -u'(x) \le C_2 x^{-\alpha-1} \qquad \text{for } x > 0. \tag{2.10}$$

Now notice that since $u' < 0$ then $u \in C^2(0,\infty)$, so that it verifies $(p-1)|u'|^{p-2}u'' = Q_0 x^{-\gamma} u^q$ in $(0,\infty)$. Letting $u = x^{-\alpha} v$, we find that v solves the equation

$$(p-1)(\alpha v - x v')^{p-2}(x^2 v'' - 2\alpha x v' + \alpha(\alpha+1)v) = Q_0 v^q \tag{2.11}$$

in $(0,\infty)$, and the function v is bounded and bounded away from zero. Performing the change of variables:

$$t = -\alpha \log x,$$

and $w(t) = v(x)$, the equation (2.11) gets transformed into

$$(p-1)\alpha^{p-1}(w' + w)^{p-2}(\alpha w'' + (2\alpha+1)w' + (\alpha+1)w) = Q_0 w^q \tag{2.12}$$

in $\mathbb{R}$, where the prime denotes now differentiation with respect to t. We notice that condition (2.10) is transformed into

$$C_1 \le w' + w \le C_2 \tag{2.13}$$

for some positive constants C_1, C_2. Since w is bounded and bounded away from zero, we obtain from (2.13) that w' is bounded.

Our objective is to show that $w = A$, where $A = ((p-1)\alpha^{p-1}(\alpha + 1)/Q_0)^{\frac{1}{q-p+1}}$. For this aim we are proving first that

$$\lim_{t \to \pm\infty} w(t) = A. \tag{2.14}$$

We now make the important observation that equation (2.12) has no bounded positive periodic solutions in $\mathbb{R}$, except for the constant A. Indeed, if such a solution w exists, then there are points $t_1, t_2 \in \mathbb{R}$ such that w attains its global maximum at t_1 and its global minimum at t_2. Thus $w'(t_1) = w'(t_2) = 0$, $w''(t_1) \le 0$, $w''(t_2) \ge 0$. It follows immediately from (2.12) that $w(t_1) \le A \le w(t_2)$, which is impossible unless w is constant.

We write equation (2.12) as a first order system of autonomous differential equations:

(2.15)

$$\begin{cases} w' = z \\ z' = \dfrac{1}{\alpha}\left(-(2\alpha+1)z + \dfrac{Q_0}{(p-1)\alpha^{p-1}}\dfrac{w^q}{(z+w)^{p-2}} - (\alpha+1)w\right), \end{cases}$$

and notice that every bounded solution w to (2.12) verifying (2.13) gives rise to a bounded solution (w, z) to the system (2.15). Thus thanks to Poincaré-Bendixon theorem, it follows that (w, z) tends to the unique equilibrium point $(A, 0)$ of (2.15) as both $t \to \pm\infty$, since there are no periodic solutions. Thus we have shown that (2.14) holds.

Now assume $\sup w > A$. Thanks to (2.14), there must exist a point t_0 such that w attains its global maximum at t_0. We have already observed that this implies $w(t_0) \le A$, which is impossible. Thus $\sup w \le A$, and it is similarly proved that $\inf w \ge A$. Hence $w = A$, and this proves the lemma. □

Proof of Theorem 3. Denote by u_0 the one-dimensional solution to (2.6) given by the right-hand side in (2.7). By using this solution we are going to show that problem (2.6) has a maximal and a minimal solution, both of them depending on the single variable x_1.

We first show that (2.6) admits a maximal solution $u_{\max}$. Let $\{D_k\}$ be a sequence of smooth bounded domains such that $D_k \subset\subset D_{k+1}$ and $D = \cup_{k=1}^{\infty} D_k$. Since $x_1^{-\gamma}$ is bounded and positive in D_k for every k, the problem

$$\begin{cases} \Delta_p u = Q_0 x_1^{-\gamma} u^q & \text{in } D_k, \\ u = +\infty & \text{on } \partial D_k, \end{cases}$$

has a unique positive weak solution u_k. Also, since $u_{k+1} < +\infty$ on ∂D_k, we obtain by comparison that $u_{k+1} \le u_k$ in D_k. It also follows by comparison that $u_0 \le u_k$ in D_k. Thus the sequence u_k converges pointwise to a limit $u_{\max} \ge u_0$. By arguing as in the proof of Theorem 2, we obtain that the convergence is in $C^1_{\text{loc}}(D)$. Hence $u_{\max}$ is a weak solution to (2.6). We claim that it is the maximal solution. Indeed, if v is another solution to (2.6), it follows by comparison that $v \le u_k$ in D_k for every k, and thus $v \le u_{\max}$ in D.

Now we make the crucial observation that $u_{\max}$ must depend only on x_1. Indeed, let $t \in \mathbb{R}^{N-1}$ be arbitrary. It is easily seen that the function $w(x) = u_{\max}(x_1, x' + t)$ is a solution to (2.6) and so $w \le u_{\max}$, that is

$u_{\max}(x_1, x' + t) \le u_{\max}(x_1, x')$ for every $x_1 > 0$, $x' \in \mathbb{R}^{N-1}$. Since t is arbitrary, this implies that $u_{\max}$ must depend only on x_1. Hence $u_{\max}$ is a solution to (2.8), and thanks to Lemma 4 we obtain $u_{\max} = u_0$.

We can construct in a similar way a minimal solution, by choosing a sequence of smooth bounded domains $\{D'_k\}$ with $D'_k \subset D'_{k+1}$ and $D = \cup_{k=1}^{\infty} D'_k$. We take every D'_k with the additional property that $B_{3k}(0) \cap \partial D \subset \partial D'_k \cap \partial D$.

We now fix a positive integer k and choose a smooth function ψ_k with $0 \le \psi_k \le 1$, $\psi_k = 1$ on $\partial D'_k \cap \partial D \cap B_k(0)$ and $\psi_k = 0$ on $\partial D'_k \setminus (B_{2k}(0) \cap \partial D)$. The problem

$$\begin{cases} \Delta_p v = Q_0 x_1^{-\gamma} v^q & \text{in } D'_k, \\ v = n\psi_k & \text{on } \partial D'_k, \end{cases}$$

has a unique positive solution $v_{k,n}$ for every positive integer n. This is not straightforward due to the singularity of the weight on $\partial D'_k$ when $\gamma > 0$, but the proof is essentially the same as that in Theorem 1, so we will skip the details.

We again have by comparison that $v_{k,n}$ is increasing in n, and that $v_{k,n} \le u_0$ in D'_k. If moreover we select ψ_k so that $\psi_{k+1} \ge \psi_k$ on $\partial D'_k \cap \partial D'_{k+1} \cap \partial D$, then we also have that $v_{k,n}$ is increasing in k.

Hence we can let $k \to +\infty$ to obtain that $v_{k,n}$ converges in $C^1_{\rm loc}(D)$ to a function v_n which is a solution to

$$\begin{cases} \Delta_p v = Q_0 x_1^{-\gamma} v^q & \text{in } D, \\ v = n & \text{on } \partial D, \end{cases}$$

and verifies $v_n \le u_0$. Since v_n is increasing in n, we can now let $n \to \infty$ to arrive at v_n converges to a solution $u_{\min}$ to (2.6) in $C^1_{\rm loc}(D)$.

It is easily seen by comparison that $u_{\min}$ is the minimal solution to (2.6), since $v_{k,n} \le u$ in D'_k for every solution u to (2.6). If follows as for $u_{\max}$ that $u_{\min}$ must depend only on the variable x_1, and then $u_{\min} = u_0$ by Lemma 4. Since the maximal and the minimal solution to (2.6) coincide, this problem has a unique positive solution which is u_0. This concludes the proof. □

3. Existence

In this section, we deal with the issue of existence of positive solutions to problem (1.1). We are following the usual strategy of constructing solutions with finite datum n on $\partial\Omega$ and then letting $n \to \infty$. However, this problem with finite datum can only have solutions when $\gamma < p$, and even in this case the existence is not at all straightforward.

Proof of existence in Theorem 1. Consider the problem

$$\begin{cases} \Delta_p u = a(x) u^q & \text{in } \Omega, \\ u = n & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

for a positive integer n. Since a may be singular on $\partial\Omega$, to obtain a solution we truncate the weight as follows: choose a smooth function $\psi(t)$ such that $0 \le \psi \le 1$, $\psi = 0$ in $[0,1]$ and $\psi = 1$ in $[2,+\infty)$, and for a positive integer k let $a_k(x) = \psi(kd(x))a(x)$. Since a is nonnegative, the sequence $\{a_k\}$ is increasing in k and $a_k(x) \le a(x) \le Cd(x)^{-\gamma(x)}$, for a positive constant C, while in addition $a_k \in L^\infty(\Omega)$.

Hence we consider the truncated problem

$$\begin{cases} \Delta_p u = a_k(x)u^q & \text{in } \Omega,\\ u = n & \text{on } \partial\Omega, \end{cases}$$

which is easily seen to have a solution since $\underline{u} = 0$ is a subsolution while $\overline{u} = n$ is a supersolution. This solution is moreover unique, thanks to the comparison principle. We denote it by $u_{k,n}$.

Our next task is to show that we can pass to the limit as $k \to +\infty$. It is at this point that the condition $\gamma(x) < p$ is essential. If we define $u_{k,n} = v_{k,n} + n$, then $v_{k,n}$ solves

$$\begin{cases} \Delta_p v = a_k(x)(v+n)^q & \text{in } \Omega,\\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Observe that the right hand side of (3.2) is bounded by $Cn^q d(x)^{-\gamma(x)}$, with $\gamma(x) < p$. Set $\gamma_0 = \max_{\partial\Omega}\gamma < p$, and let ϕ be the unique solution to

$$\begin{cases} -\Delta_p \phi = d^{-\gamma_0} & \text{in } \Omega,\\ \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

which exists thanks to Theorem 2 (see also Remarks 2 (b)), and is positive by the maximum principle. By comparison we arrive at $|v_{k,n}| \le Cn^q\phi(x)$. This gives local bounds for the sequence $v_{k,n}$. Arguing as in the proof of Theorem 2, we obtain that the sequence $\{v_{k,n}\}$ is precompact in C^1, and hence it is standard to obtain that $v_{k,n} \to v_n$ in $C^1_{\rm loc}(\Omega)$ as $k \to \infty$. The function v_n verifies $|v_n| \le Cn^q\phi(x)$, and thus $v_n = 0$ on $\partial\Omega$. Passing to the limit in the weak formulation of (3.2), we obtain that v_n is a solution to

$$\begin{cases} \Delta_p v = a(x)(v+n)^q & \text{in } \Omega,\\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

and hence $u_n = v_n + n$ is a solution to (3.1). The uniqueness of u_n is also obtained directly through the comparison principle. We remark that the sequence $\{u_n\}$ is increasing thanks to uniqueness.

Now we are obtaining local uniform bounds for the solutions u_n. Since $a > 0$ in a neighborhood of $\partial\Omega$, we can choose $\delta > 0$ such that $a > 0$ in $\Omega_\delta = \{x \in \Omega : \ d(x) < \delta\}$. Choose $\varepsilon < \delta$ and a point $x_0 \in \Omega_\delta$ with $d(x_0) = \varepsilon/2$. Since $a(x) \ge a_0 > 0$ in $B(x_0, \varepsilon/4)$, we have for u_n:

$$\Delta_p u_n \ge a_0 u_n^q \qquad \text{in } B(x_0,\varepsilon/4).$$

It follows by comparison that $u_n \le U$, where U is the unique solution to $\Delta_p U = a_0 U^q$ in $B(x_0, \varepsilon/4)$, $U_{|\partial B(x_0,\varepsilon/4)} = +\infty$. This shows that u_n is uniformly bounded in $B(x_0, \varepsilon/8)$. A compactness argument then shows that u_n is uniformly bounded on the set $\{x \in \Omega : \ d(x) = \varepsilon/2\}$. Notice that $\Delta_p u_n \ge 0$, and then the comparison principle implies

$$u(x) \le \sup_{d=\varepsilon/2} u, \qquad x \in \Omega \setminus \Omega_{\varepsilon/2}.$$

Since ε is arbitrarily small, we have obtained uniform interior local bounds for the solutions u_n. We reason as before to deduce that, passing to a subsequence, $u_n \to u$ in $C^1_{\rm loc}(\Omega)$ for a certain function u, which will be a weak solution to $\Delta_p u = a(x)u^q$ in Ω. Since u is increasing, it follows that $u = \sup_n u_n$, and then the whole sequence u_n converges to u. Finally, $u = +\infty$ on $\partial\Omega$, thanks to the monotonicity of the sequence, and u is a solution to (1.1). This concludes the proof. □

4. Boundary behavior and uniqueness

In this final section we deal with the boundary estimates (1.3) of all possible positive solutions to (1.1). We follow the approach in [6] (which has also been used in [1]). For this aim some a priori "rough" estimates of solutions are needed. They are provided by the next lemma (see [3], [25], [26], [6], [17] or [19] for related results).

Lemma 5. *Let u be a positive solution to* (1.1)*. There exists a neighborhood $\mathcal{U}$ of $\partial\Omega$ and positive constants C_1, C_2 such that*

$$C_1 d(x)^{-\alpha(x)} \le u(x) \le C_2 d(x)^{-\alpha(x)} \tag{4.1}$$

in $\mathcal{U}$, where $\alpha(x) = (p - \gamma(x))/(q - p + 1)$.

Proof. Fix $x_0 \in \partial\Omega$, and for $x \in \Omega$ introduce the function $v(y) = d(x)^{\alpha(x)} u(x + d(x)y)$, where $y \in B_1(0)$. Then v verifies

$$\Delta_p v = d(x)^{\gamma(x)} a(x + d(x)y) v^q \qquad \text{in } B_1(0).$$

Since

$$a(x + d(x)y) \ge C d(x + d(x)y)^{-\gamma(x+d(x)y)} \ge C' d(x)^{-\gamma(x+d(x)y)}$$

for x close to x_0, we have $\Delta_p v \ge C' d(x)^{\gamma(x)-\gamma(x+d(x)y)} v^q$ in $B_1(0)$. Thanks to the Hölder condition on γ, we have

$$|\gamma(x) - \gamma(x + d(x)y)| \le C d(x)^{\mu}, \tag{4.2}$$

and thus if $d(x)$ is small enough then $\Delta_p v \ge C v^q$ in $B_1(0)$. It follows that $v \le U$, the unique solution to $\Delta_p U = C U^q$ in $B_1(0)$ with $U = +\infty$ on $\partial B_1(0)$. Setting $y = 0$ we obtain

$$u(x) \le U(0) d(x)^{-\alpha(x)},$$

which is the upper inequality in (4.1).

We next prove the lower inequality. Let $x_0 \in \partial\Omega$ be fixed. For a point $x \in \Omega$ close to x_0, denote by $\bar{x}$ its projection onto $\partial\Omega$. We denote $c_x = \bar{x} + d(x)\nu(\bar{x})$, where $\nu(\bar{x})$ is the outer unit normal to $\partial\Omega$ at $\bar{x}$. Notice that $c_x \notin \Omega$ if x is close enough to x_0. Now consider the annulus with center in c_x, inner radius $d(x)$ and outer radius $3d(x)$, namely $A_x = \{y \in \mathbb{R}^N : \ d(x) < |y - c_x| < 3d(x)\}$. We define

$$w(y) = d(x)^{\alpha(x)} u(c_x + d(x)y)$$

where $y \in Q_x = \{y \in A : \ c_x + d(x)y \in \Omega\}$ and $A = \{y \in R^N : \ 1 < |y| < 3\}$ is the "normalized annulus". Then w verifies

$$\Delta_p w = d(x)^{\gamma(x)} a(c_x + d(x)y) w^q$$

in Q_x, and it is shown as before that this implies the existence of a positive constant C such that $\Delta_p w \le C w^q$ in Q_x.

On the other hand, it can be proved that the problem

$$\begin{cases} \Delta_p z = z^q & \text{in } A \\ z = +\infty & \text{on } |y| = 1 \\ z = 0 & \text{on } |y| = 3, \end{cases}$$

admits a unique positive weak solution z, which is indeed radially symmetric, and by comparison $w \ge z$ in Q_x, since $w > 0$ on $|y| = 3$. This gives $w(-2\nu(\bar{x})) \ge z(-2\nu(\bar{x}))$, and since z is a radial function, we obtain $u(x) \ge z(2) d(x)^{-\alpha(x)}$, which is the lower inequality in (4.1). The inequality is shown to be valid in a whole neighborhood of $\partial\Omega$ by means of a standard compactness argument. This concludes the proof of the lemma. □

We now proceed to the proof of the estimates. The argument is to perform a local analysis of solutions by means of a scaling, taking into account inequalities (4.1).

Proof of estimates (1.3). Let $x_0 \in \partial\Omega$. With no loss of generality we may assume that $x_0 = 0$ and $\nu(x_0) = -e_1$, where e_1 is the first vector in the canonical basis of $\mathbb{R}^N$. Take an arbitrary sequence $\{x_n\} \subset \Omega$, $x_n \to 0$, and denote by ξ_n the projection of x_n onto $\partial\Omega$. If $d_n = d(x_n)$, $\alpha_n = \alpha(x_n)$, we define the function

$$v_n(y) = d_n^{\alpha_n} u(\xi_n + d_n y)$$

for $y \in \Omega_n := \{y \in \mathbb{R}^N : \ \xi_n + d_n y \in \mathcal{U}\}$, where $\mathcal{U}$ is the neighborhood of $\partial\Omega$ given by Lemma 5. Observe that $\Omega_n \to D$ as $n \to \infty$. The function v_n verifies the equation

$$\Delta_p v = d_n^{\gamma(x_n) - \gamma(\xi_n + d_n y)} d_n^{\gamma(\xi_n + d_n y)} a(\xi_n + d_n y) v_n^q \qquad \text{in } \Omega_n. \tag{4.3}$$

Notice that (4.2) implies that $(\gamma(x_n) - \gamma(\xi_n + d_n y))/d_n^{\mu}$ is bounded for y in bounded subsets of D, and hence the first term in the right hand side of (4.3) tends to 1 as $n \to \infty$, uniformly for y in bounded subsets of D. Also, since $d(\xi_n + d_n y)/d_n \to y_1$, we obtain again thanks to the Hölder condition on γ that

$$d_n^{\gamma(\xi_n+d_n y)} a(\xi_n + d_n y) \to Q_0 y_1^{-\gamma_0}$$

as $n \to \infty$, uniformly for y in bounded subsets of D, where $Q_0 = Q(0)$ and $\gamma_0 = \gamma(0)$.

We now use the estimates provided by Lemma 5: there exists a positive constant C_2 such that

$$v_n(y) \le C_2 \left(\frac{d_n}{d(\xi_n + d_n y)} \right)^{\alpha_n} d(\xi_n + d_n y)^{\alpha_n - \alpha(\xi_n+d_n y)}, \tag{4.4}$$

and a similar inequality from below. Since the right-hand side of (4.4) converges to $C_2 y_1^{-\alpha_0}$ as $n \to \infty$, where $\alpha_0 = \alpha(0)$, we obtain that the sequence v_n is uniformly bounded in compact subsets of D. Thus we can pass to the limit to get that v_n converges in $C^1_{\rm loc}(D)$ to a function v which verifies

$$\begin{cases} \Delta_p v = Q_0 y_1^{-\gamma_0} v^q & \text{in } D, \\ v = +\infty & \text{on } \partial D, \end{cases}$$

verifying in addition $C_1 y_1^{-\alpha_0} \le v \le C_2 y_1^{-\alpha_0}$. Thanks to Theorem 3, we have

$$v(x) = \left(\frac{(p-1)\alpha_0^{p-1}(\alpha_0+1)}{Q_0} \right)^{\frac{1}{q-p+1}} y_1^{-\alpha_0},$$

and setting $y = e_1$ we arrive at

$$d_n^{\alpha_n} u(x_n) \to \left(\frac{(p-1)\alpha_0^{p-1}(\alpha_0+1)}{Q_0} \right)^{\frac{1}{q-p+1}},$$

as we wanted to see. This concludes the proof. □

As usual in the literature, when the exact boundary behavior of all possible positive solutions is elucidated, uniqueness is an easy consequence of the monotonicity of the right-hand side in (1.1).

Proof of uniqueness in Theorem 1. Let u, v be positive weak solutions to (1.1). According to estimates (1.3), we have

$$\lim_{x \to x_0} \frac{u(x)}{v(x)} = 1$$

for every $x_0 \in \partial\Omega$. By compactness, this limit holds uniformly, and thus for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(1-\varepsilon)v(x) \le u(x) \le (1+\varepsilon)v(x) \tag{4.5}$$

if $d(x) < \delta$. Denote $\Omega^\delta = \{x \in \Omega : \ d(x) > \delta\}$, and consider the problem

$$\begin{cases} \Delta_p w = a(x) w^q & \text{in } \Omega^\delta, \\ w = u & \text{on } \partial\Omega^\delta, \end{cases} \tag{4.6}$$

which has a unique solution thanks to the comparison principle, since $a(x)$ is bounded and nonnegative. This solution is of course $w = u$. On the other hand, since $q > p - 1$, the functions $(1-\varepsilon)v$ and $(1+\varepsilon)v$ are respectively a sub- and a supersolution to (4.6). Hence (4.5) holds also in Ω^δ, that is

$$(1-\varepsilon)v(x) \le u(x) \le (1+\varepsilon)v(x) \qquad \text{in } \Omega.$$

Letting $\varepsilon \to 0$ we arrive at $u = v$, and this proves uniqueness. □

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