THE BEHAVIOUR OF STATIONARY SOLUTIONS TO SOME NONLINEAR DIFFUSION PROBLEMS

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1.INTRODUCTION. HYPOTHESES

The objective of the present work is the study of the positive solutions to the boundary value problem

$$\begin{cases} -\Delta_p u &= \lambda f(u) \qquad x \in \Omega \\ u &= 0 \qquad\quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where the parameter $\lambda > 0$ becomes large, i.e., when $\lambda \to +\infty$. The operator Δ_p stands for the p-Laplacian, i. e. the operator $\Delta_p := \text{div } (|\nabla u|^{p-2}\nabla u)$, and the number $p > 1$.

It will be assumed that $\Omega \subset \mathbb{R}^N$ is a class $C^{2+\beta}$ bounded domain with $0 < \beta \le 1$. The nonlinearity $f = f(u)$ will be required to be a continuous function $f : \mathbb{R}^+ := [0, +\infty) \to \mathbb{R}$ such that the function $h(u) := \frac{f(u)}{u^{p-1}}$ is C^1 and decreasing in $u > 0$, with $0 < h(0+) \le +\infty$. It will be also supposed that $h = h(u)$ has a positive zero of order k at $u = u_0$, i. e. $h(u) \sim C\ (u_0 - u)^k$ as $u \to u_0$ with $C > 0$ and $k \in \mathbb{N}$ (generically $k = 1$).

For later use, we will set $\sigma = h(0+) = \lim_{u\to 0+} \frac{f(u)}{u^{p-1}}$ (for simplicity we shall only consider the case $\sigma < +\infty$) meanwhile we will term $F = F(u)$ as the zero valued at $u = 0$ primitive of $f(u)$, i. e. $F(u) = \int_0^u f(s)\ ds$.

In the present work the attention will be focused on positive solutions $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to (1.1), where the concept of solution is understood in the weak sense. Namely, $u \in W_0^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi\ dx = \lambda \int_\Omega f(u)\varphi\ dx, \tag{1.2}$$

for each $\varphi \in W_0^{1,p}(\Omega)$. We will be mainly interested in the phenomenom of "homogeneization" of positive solutions $u = u_\lambda(x)$, which in the context of the present work means the study of the limit

$$\lim_{\lambda\to+\infty} u_\lambda(x) = u_0, \tag{1.3}$$

uniformly over compact sets $K \subset \Omega$. Two natural behaviours associated to (1.2) must be analyzed in order to get a complete understanding of the phenomenom. The first one is the arising of boundary layers of $u_\lambda(x)$ near $\partial\Omega$. In fact, due to the strong maximum principle (cf. [13]) we have that positive solutions satisfy $\frac{\partial u_\lambda}{\partial n}(x) < 0$ for each $x \in \partial\Omega$ (n being the outer unit normal field on $\partial\Omega$). (1.2) implies that $\frac{\partial u_\lambda}{\partial n}$ goes to $-\infty$ as $\lambda \to +\infty$ and we want "precisely" estimate the width of the boundary layer. That is, to find out the exact values for the positive constants C_1 and γ which fit in the expression,

$$\frac{\partial u_\lambda}{\partial n} \sim -C_1\ \lambda^\gamma \qquad \text{as } \lambda \to +\infty. \tag{1.4}$$

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The study of homogeneization of stationary solutions to (1.2) has been considered for instance in [2], [4], [5] for the case of (1.1) with $p = 2$, in [12] for $p = 2$ and homogeneous Neumann conditions, in [9], [11] for cooperative systems involving the Laplace operator. However, the very origin of the present work is [8]. There, the special case of (1.1) where $p = 2$ (i. e. Δ_p the Laplace operator) and $f(u) = (m - u^q)u$ with $m > 0$ and $q > 1$ was considered. It is shown that the unique positive solution $u = u_\lambda(x)$ ($\lambda > \lambda_1/m$, λ_1 the first eigenvalue of the Dirichlet problem) satisfies $\lim_{\lambda\to+\infty} u_\lambda(x) = m^{1/q}$ uniformly over compact sets $K \subset \Omega$. In addition, the following exact estimate of the boundary layer near $\partial\Omega$ is given,

$$\frac{\partial u_\lambda}{\partial n} \sim -\left(\frac{q}{q+2} m^{(q+2)/q}\right)\sqrt{\lambda} \qquad \text{as} \quad \lambda \to +\infty, \tag{1.5}$$

uniformly on $\partial\Omega$. In the present work we are providing an extension of (1.5) to Dirichlet problems involving the p-Laplacian operator and a wider class of nonlinearities $f = f(u)$. Our main result is contained in Theorem 2 of §3.

A second aspect of the homogeneization (1.2) comes from the degeneracy of the Δ_p operator when p and the order k of u_0 keep the relation $p > k + 1$. In this case, a critical value $\lambda = \lambda^*(p, \Omega)$ exists so that the positive solution $u = u_\lambda(x)$ degenerates at its maximum possible value $u = u_0$ on a whole set $\mathcal{D}_\lambda := \{x \in \Omega | u_\lambda(x) = u_0\}$ with non empty interior –termed as a "dead core" (see [6])– provided $\lambda > \lambda^*$. The scenario of dead cores to (1.1) will be studied in a forthcoming paper ([10]). An estimate of $\lambda^*(p, \Omega)$ in terms of $f(u)$ and Ω together with an asymptotic estimate of the distance $d(\lambda) = \text{dist}(\mathcal{D}_\lambda, \partial\Omega)$ as $\lambda \to +\infty$ will be given there.

2.PRELIMINARY RESULTS

Under the assumptions made about $f = f(u)$ there are a number of known facts about the problem (1.1) which are next summarized in the following theorem. It is convenient to recall that the nonlinear eigenvalue problem $-\Delta_p u = \lambda |u|^{p-2}u$ in Ω, $u = 0$ on $\partial\Omega$, admits a positive first eigenvalue $\lambda = \lambda_{1,p}(\Omega)$ characterized as the unique value λ such that the problem exhibits a nontrivial signed solution $u = \phi$. $\lambda_{1,p}$ can be expressed by means of the Rayleigh quotient,

$$\lambda_{1,p}(\Omega) = \inf_{\varphi\neq 0} \frac{\int_\Omega |\nabla\varphi|^p \, dx}{\int_\Omega |\varphi|^p \, dx}, \tag{2.1}$$

the infimum being taken over $\varphi \in W_0^{1,p}(\Omega)$ and only attained in a scalar multiple of a unique normalized positive eigenfunction ϕ_1 (cf. [1]). Let us state now a general overview about the problem (1.1).

THEOREM 1.

Let us assume that $f = f(u)$ is continuous in $\mathbb{R}^+$, $f(u) = u^{p-1}h(u)$, with $h = h(u)$ of class C^1, decreasing, $h(0+) = \sigma > 0$ and having a (unique) zero $u = u_0 > 0$. Then,

a) The problem (1.1) does not admit positive solutions for $\lambda \leq \frac{\lambda_{1,p}}{\sigma}$.

b) For each $\lambda > \frac{\lambda_{1,p}}{\sigma}$ (1.1) exhibits a *unique* positive solution $u = u_\lambda(x) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that

$$0 < u_\lambda(x) \leq u_0, \qquad x \in \Omega.$$

Moreover, for a certain $0 < \alpha < 1$

$$u_\lambda \in C^{1+\alpha}(\overline{\Omega}) \cap C^{2+\beta}(\{x | d(x, \partial\Omega) < \epsilon\})$$

where ϵ is a conveniently small positive number.

REMARKS.

The assertion in a) follows from the variational characterization (2.1) of $\lambda_{1,p}$. The estimate by u_0 of all positive solutions follows, say, by introducing the test function $\varphi(x) = (u_0 - u(x))^+$ in (1.2), where $u(x)$ is any nonnegative solution to (1.1). The existence of positive solutions asserted in b) can be obtained by the method of sub and supersolutions (cf. [6], [10]) by taking $u_-(x) = \delta\phi_1(x)$ – choosing $\delta > 0$ small enough– as a subsolution and $u_+(x) = u_0$ as a supersolution (see also [7] for an alternative variational approach). The uniqueness fact follows directly from the well known uniqueness result contained in [7].

On the other hand, the inner $C^{1+\alpha}$ smoothness of the positive solutions follows from well known local $C^{1+\alpha}$ estimates (see for instance [14]). The $C^{1+\alpha}$ regularity up to the boundary is obtained from an argument, reminiscent of the Schwartz reflection principle and the aid of the inner estimates (see [13], [10]). As a consequence of the strong maximum principle in [13], positive solutions to (1.1) satisfy

$$\frac{\partial u_\lambda}{\partial n}(x) < 0 \qquad x \in \partial\Omega. \tag{2.2}$$

Therefore (1.1) becomes strictly elliptic near the boundary and $u_\lambda(x)$ becomes $C^{2+\beta}$ in a region close enough to $\partial\Omega$ (cf. [13], [3]).

3. THE RESULTS

The main result of the present work is stated now in the following theorem.

THEOREM 2.

Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{2+\beta}$ $(0 < \beta \leq 1)$ domain and let $f = f(u)$ be a continuous function, $f(u) = u^{p-1}h(u)$ with $p > 1$ and $h = h(u)$ being a C^1 decreasing function such that $0 < h(0+) = \sigma < +\infty$ and $h(u)$ having a positive zero $u = u_0$ of order k.

For $\lambda > \frac{\lambda_{1,p}}{\sigma}$ let $u = u_\lambda(x)$ be the unique positive solution to the problem (1.1), which, as stated above, satisfies the estimate $0 < u_\lambda(x) \leq u_0$ for each $x \in \Omega$. Then,

i) The family of solutions u_λ satisfies

$$\lim_{\lambda\to+\infty} u_\lambda(x) = u_0, \tag{3.1}$$

uniformly over compact sets $K \subset \Omega$.

ii) Moreover, the precise estimate of the boundary layer is given by

$$\lim_{\lambda\to+\infty} \lambda^{-\frac{1}{p}} \frac{\partial u_\lambda}{\partial n}(x) = -\left(\frac{p}{p-1}\right)^{\frac{1}{p}} (F(u_0))^{\frac{1}{p}}, \tag{3.2}$$

uniformly in $\partial\Omega$, where $F(u_0) = \int_0^{u_0} f(s)\, ds$.

Sketch of the proof. The proof of theorem 2 proceeds in three steps next outlined in turn. The step I) consists in analysing the one-dimensional version of (1.1). Namely,

$$\left\{ \begin{array}{rcll} -\left(|u'|^{p-2}u'\right)' &=& \lambda f(u) & 0 < x < l \\ u(0) = u(l) &=& 0. & \end{array} \right. \tag{3.3}$$

When observed as a one-dimensional system (cf. [10]), it can be shown that any positive solution $u = u(x)$ to $(3.3)_\lambda$ with $\bar{u} = \max_{0<x<l} u(x)$, $0 < \bar{u} \leq u_0$, must satisfy the identity

$$\lambda = \frac{p-1}{p}\left(\frac{2}{l} I(\bar{u})\right)^p, \tag{3.4}$$

where $I = I(u) = \int_0^u \frac{ds}{(F(u)-F(s))^{1/p}}$. The function $I(u)$ is increasing in $0 < u \le u_0$, $I(0+) = \left(\frac{p}{\sigma}\right)^{1/p} \frac{\pi}{p\sin(\pi/p)}$ and $I(u_0-) = +\infty$ if $p \le k+1$ while $I(u_0-) < +\infty$ provided that $p > k+1$. Thus, (3.3) only exhibits positive solutions when $\lambda > \lambda_{\min} := \frac{p-1}{p}\left(\frac{2}{l}I(0+)\right)^p$. If $p \le k+1$ the solution $u_\lambda(x)$ to (3.3) has a maximum $\bar{u} = \bar{u}_\lambda$, $0 < \bar{u}_\lambda < u_0$, given by the unique solution to (3.4). If $p > k+1$ the solution $u_\lambda(x)$ degenerates and reaches its maximum $\bar{u} = \bar{u}_\lambda = u_0$ in the whole interval $d(\lambda) \le x \le l - d(\lambda)$ provided that $\lambda \ge \lambda^*$, where $d(\lambda) = \lambda^{-1/p}C(f,p)^{1/p}$, $\lambda^* = l^{-p}C(f,p)$ and $C(f,p) = \frac{p-1}{p}(2I(u_0-))^p$. In both cases it can be shown that $\lim_{\lambda\to+\infty}\bar{u}_\lambda = u_0$ and hence (3.1) follows.

With regard to the boundary layer, the relation $u_\lambda'(0) = \left\{\frac{p}{p-1}F(\bar{u}_\lambda)\right\}^{1/p}\lambda^{1/p}$ holds at $x = 0$ (the same holds modulo sign at $x = l$). Therefore,

$$u_\lambda'(0) \sim \left\{\frac{p}{p-1}F(u_0)\right\}^{\frac{1}{p}}\lambda^{\frac{1}{p}} \quad \text{and} \quad u_\lambda'(l) \sim -\left\{\frac{p}{p-1}F(u_0)\right\}^{\frac{1}{p}}\lambda^{\frac{1}{p}}$$

as $\lambda \to +\infty$, what proves (3.2).

The step II) consists in the analysis of (1.1) in a ball $\Omega = B_R(0) = \{x : |x| < R\}$. By uniqueness, positive solutions to (1.1) must now be radially symmetric, i. e. $u(x) = w(r)$, $r = |x|$, where $w(r)$ is a solution of

$$\begin{cases} -\left\{\left(|w'|^{p-2}w'\right)' + \frac{N-1}{r}\left(|w'|^{p-2}w'\right)\right\} = \lambda f(w) & 0 < r < R \\ w'(0) = w(R) = 0. \end{cases} \tag{3.5}$$

To connect (3.5) with the one-dimensional problem (3.3) it is very convenient to introduce the change of variables $\rho = g(r)$ where $g(r) = \frac{1}{\alpha-1}\left(\frac{1}{r^{\alpha-1}} - \frac{1}{R^{\alpha-1}}\right)$ if $\alpha := \frac{N-1}{p-1} \ne 1$ and $g(r) = \log\left(\frac{R}{r}\right)$ provided that $\alpha = 1$. By setting $\omega(\rho) = w(g^{-1}(\rho))$ (3.5) can be written as the one-dimensional problem,

$$\begin{cases} -\left(|\omega'|^{p-2}\omega'\right)' = \lambda g^{-1}(\rho)^{p\alpha}f(\omega) & 0 < \rho < b \\ \omega(0) = \omega'(b) = 0, \end{cases} \tag{3.6}$$

where $b = \frac{R^{1-\alpha}}{1-\alpha}$ if $\alpha < 1$ and $b = +\infty$ if $\alpha \ge 1$.

Let us fix $a > 0$, $0 < a < b$ and consider the problem,

$$\begin{cases} -\left(|v'|^{p-2}v'\right)' = \lambda g^{-1}(a)^{p\alpha}f(\omega) & 0 < \rho < a \\ v(0) = v(a) = 0. \end{cases} \tag{3.7}$$

For $\lambda > \lambda_{1,p}(0,a)/\left(\sigma g^{-1}(a)^{p\alpha}\right)$ let us designate by $v = v_{\lambda,a}(\rho)$ the positive solution to (3.7). Since any positive solution $\omega_\lambda(\rho)$ to (3.6) is a supersolution to (3.7) we have then $\omega_\lambda(\rho) \ge v_{\lambda,a}(\rho)$ in $0 \le \rho \le a$. In particular, $\frac{d\omega_\lambda}{d\rho}(0) \ge \frac{dv_{\lambda,a}}{d\rho}(0)$. Taking into account the decreasing character of $w_\lambda(r)$ in r, the homogeneization result (3.1) follows then from the corresponding property for $v_{\lambda,a}(\rho)$ as a solution of (3.7).

As for the boundary layer estimate, observe that at any boundary point $x_0 \in \partial\Omega = \partial B_R(0)$ we have,

$$-\frac{\partial u_\lambda}{\partial n}(x_0) = R^{-\alpha}\frac{d\omega_\lambda}{d\rho}(0) \ge g^{-1}(0)^{-\alpha}\frac{dv_{\lambda,a}}{d\rho}(0) = \left(\frac{g^{-1}(a)}{g^{-1}(0)}\right)^\alpha\left(\frac{\lambda p}{p-1}\right)^{\frac{1}{p}}F(\bar{v}_{\lambda,a})^{\frac{1}{p}}, \tag{3.8}$$

where $\bar{v}_{\lambda,a} = \max_{0<x<a} v_{\lambda,a}(x)$. By making simultaneously $a \to 0+$ and $\lambda \to +\infty$ (3.8) gives a first approximation of (3.2). Namely,

$$\liminf_{\lambda\to+\infty} -\lambda^{-\frac{1}{p}}\frac{\partial u_\lambda}{\partial n}(x_0) \ge \left(\frac{p}{p-1}\right)^{\frac{1}{p}}F(u_0)^{\frac{1}{p}}. \tag{3.9}$$

Let us come back to the original problem (1.1) and face the final step III). Choose any open ball $B \subset \Omega$ and define $u = u_{\lambda,B}(x)$ as the positive solution to (1.1) by taking B as the domain Ω for $\lambda > \frac{\lambda_{1,p}(B)}{\sigma}$. Define, $u^-_{\lambda,B}(x) = u_{\lambda,B}(x)$ if $x \in \overline{B}$ and $u^-_{\lambda,B}(x) = 0$ in $\Omega\backslash\overline{B}$. A careful computation (the source of the difficulty coming from the lack of regularity of the weak solutions to (1.1)) shows that $u^-_{\lambda,B}(x)$ is a subsolution to (1.1) (cf. [10]). By the uniqueness of positive solutions to (1.1) we have then,

$$u^-_{\lambda,B}(x) \leq u_\lambda(x) \leq u_0, \qquad x \in \Omega. \tag{3.10}$$

The homogeneization result (3.1) can now be easily shown by using (3.10) on variable balls $B \subset \Omega$ with arbitrarily small chosen and fixed radius. To show the estimate (3.2) let us assume, for simplicity, the case Ω convex (see [10] for the situation of general smooth domains). If $x_0 \in \Omega$ and $n(x_0)$ is the outward unit normal at x_0, set $\nu = -n(x_0)$. For a certain positive $b_1 > 0$ the domain Ω is contained in the strip $0 < \nu(x - x_0) < b_1$. Thus, consider the one-dimensional auxiliary problem,

$$\begin{cases} -\left(|v_1'|^{p-2}v_1'\right)' &= \lambda f(v_1) \qquad 0 < \xi < b_1 \\ v_1(0) = v_1(b_1) &= 0. \end{cases} \tag{3.11}$$

Designate by $v_1 = v_{\lambda,1}(\xi)$, $\lambda > \frac{\lambda_{1,p}(0,b_1)}{\sigma}$, its positive solution. Then, a direct computation yields that $u^+_\lambda(x) = v_1(\nu(x - x_0))$ is a supersolution to (1.1). Therefore, $u^+_\lambda(x) \geq u_\lambda(x)$ for $x \in \Omega$, and so

$$-\frac{dv_{\lambda,1}}{d\xi}(0) = -\frac{\partial u^+_\lambda}{\partial n}(x_0) \geq -\frac{\partial u_\lambda}{\partial n}(x_0).$$

Hence, by using again the one-dimensional results, we arrive at

$$\limsup_{\lambda\to+\infty} \; -\lambda^{-\frac{1}{p}}\frac{\partial u_\lambda}{\partial n}(x_0) \leq \left(\frac{p}{p-1}\right)^{\frac{1}{p}} F(u_0)^{\frac{1}{p}}. \tag{3.12}$$

Let us choose now $B_0 \subset \Omega$ as some small inner tangent ball to $\partial\Omega$ at x_0. From (3.9) and (3.12) we finally arrive at

$$\begin{aligned} \left(\tfrac{p}{p-1}\right)^{\frac{1}{p}} F(u_0)^{\frac{1}{p}} &\leq \liminf_{\lambda\to+\infty} \; -\lambda^{-\frac{1}{p}}\tfrac{\partial u_{\lambda,B_0}}{\partial n}(x_0) \leq \liminf_{\lambda\to+\infty} \; -\lambda^{-\frac{1}{p}}\tfrac{\partial u_\lambda}{\partial n}(x_0) \\ &\leq \limsup_{\lambda\to+\infty} \; -\lambda^{-\frac{1}{p}}\tfrac{\partial u_\lambda}{\partial n}(x_0) \leq \left(\tfrac{p}{p-1}\right)^{\frac{1}{p}} F(u_0)^{\frac{1}{p}}. \end{aligned}$$

This completes the proof of (3.2) and theorem 2.

REMARKS

As an application of our results, a complete description of the asymptotic behaviour as $\lambda \to +\infty$ of the positive solution $v = v_\lambda(x)$ to the logistic problem $-\Delta_p v = \lambda m|v|^{p-2}v - v^q$ in Ω, $v = 0$ on $\partial\Omega$, $m > 0$, $q > p - 1 > 0$ can be given. In fact, after the scaling $v = \lambda^{\frac{1}{(q-p+1)}}u$ we arrive at $-\Delta_p u = \lambda(m - u^{q-p+1})u^{p-1}$ in Ω, $u = 0$ on $\partial\Omega$, which falls in the scope of theorem 2. In this case $h = m - u^{q-p+1}$, $u_0 = m^{\frac{1}{(q-p+1)}}$. Moreover, a more detailed analysis permits to show that *all* positive solutions to the perturbation of that logistic problem, namely $-\Delta_p v = \lambda m|v|^{p-2}v - v^q + g(v)$ in Ω, $v = 0$ on $\partial\Omega$, behave as $\lambda \to +\infty$ exactly as the solution $v_\lambda(x)$ of the unperturbed problem provided that the perturbation term satisfies, say, $\lim_{u\to 0+} \frac{g(u)}{u^{p-1}} = \lim_{u\to+\infty} \frac{g(u)}{u^q} = 0$ (see [10]).

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