REMARKS ON LARGE SOLUTIONS

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This work discusses some aspects of the existence of solutions to the singular Dirichlet problem $-\Delta u = \lambda(x)u - a(x)u^p$, $x \in \Omega$, $u_{|\partial\Omega} = \infty$ under several assumptions on the null set $\{a(x) = 0\}$ of the nonnegative coefficient a. In addition, a detailed analysis of the asymptotic profile of the solution to the finite Dirichlet problem $-\Delta u = \lambda(x)u - a(x)u^p$, $x \in \Omega$, $u_{|\partial\Omega} = u_0$, as $u_0 \to \infty$ is performed.

1. Introduction

The main purpose of this work is discussing and also reviewing some features concerning the existence, non existence and other few qualitative properties of positive solutions to the following class of singular boundary value problems:

$$\begin{cases} -\Delta u = \lambda(x)u - a(x)u^p & x \in \Omega \\ \quad u = \infty & x \in \Omega. \end{cases} \tag{P}$$

It will be assumed that $\Omega \subset \mathbb{R}^N$ is a bounded domain of class $C^{2,\alpha}$, the coefficients $\lambda(x), a(x)$ lying in $C^\alpha(\overline{\Omega})$ with $a(x) \geq 0$ in Ω and the exponent p fixed in the range $p > 1$. All along the work, solutions u will be understood in the classical sense $u \in C^{2,\alpha}(\Omega)$ the boundary condition meaning that:

$$\lim u(x) = \infty,$$

as dist $(x, \partial\Omega) \to 0+$.

In spite of its possibly artificial look, semilinear elliptic problems subjected as (P) to an “infinite” Dirichlet condition have been studied ever since the beginning of the twentieth century, under the presence of different kinds of nonlinearities instead of the power term in (P). It should be mentioned the pioneering works [2], [30], the former on the theory of automorphic functions. A first more modern and N-dimensional approach to (P) goes back to [21], [22], the inspiring problem being in this occasion a question on electrohydrodynamics, together with [28] where the solution to a problem on the classification of Riemann surfaces is given. The study of the power nonlinaerity comes from [29] and [23], but we prefer refering to [7], [14] and [27] for a detailed story and bibliographic quotations on this specific subject. Of course the appeal of (P) goes on and literature on the matter is continuously appearing.

It is worthy of mention that [13], aside of being the origin of our interest on the problem, furnished an unexpected new example showing how (P) arises in the most natural population dynamics model. Under the inspiration of that research we are dealing here with different aspects of the subject of existence of solutions when the “self competition” coefficient $a(x)$ vanishes in Ω in a nontrivial way.

Regarding the structure of the coefficient $a(x)$ the principal feature is that it can vanish in a whole region $\Omega_0 \subset \Omega$ whose boundary $\partial\Omega_0$ could possibly touch somewhere the boundary $\partial\Omega$ of Ω. More precisely, it will be assumed that the null set for a can be expressed as $\{x \in \Omega : a(x) = 0\} = \overline{\Omega}_0$ where $\Omega_0 \subset \Omega$ is open and locally lies on one side of its boundary $\partial\Omega_0$ which in turn is also locally described –perhaps after a rotation– as the graph in $\mathbb{R}^N$ of a real $C^{2,\alpha}$ function. Equivalently, Ω_0 is a finite union of $C^{2,\alpha}$ bounded domains $\Omega_{0,1}, \ldots, \Omega_{0,m}$, their closures $\overline{\Omega}_{0,i}$ being pair-wise disjoint.

This work is preceded by a Section 2 where a brief account of results concerning (P) is presented. Sections 3 and 4 contains the main existence material. Finally, Section 5 develops an analysis of the asymptotic profile to the solutions of the most natural auxiliary problem associated to (P).

For future reference in the course of the exposition we are introducing some notation. For a bounded domain $Q \subset \mathbb{R}^N$ and a potential $q = q(x)$, say $q \in L^\infty(Q)$, $\lambda_1^Q(-\Delta + q)$ will stands for the first eigenvalue λ to $-\Delta\phi + q\phi = \lambda\phi$, $x \in \Omega$, $\phi_{|\partial\Omega} = 0$. It is well known that it is the unique eigenvalue with a positive associated eigenfunction $\phi_1 \in C^{2,\alpha}(\overline{\Omega})$.

We finally remark that the results in sections 3, 4 and a preliminary version of the ones in section 5 were obtained in the spring of 2000 and later presented in the Spanish Meeting on Differential Equations, celebrated in the University of Salamanca on September 2001 (see [17]). At that time we were still so fortunate as to live and enjoy the warm friendship of our dear colleagues *Jesús Esquinas*, *Chema Fraile* and *René Letelier*. This modest work is dedicated to their memory.

2. Preliminaries

The purpose of this section is to describe some basic features concerning problem (P). For historical reasons we are focusing the attention in the case where $a(x) > 0$ in Ω but $a \equiv 0$ on $\partial\Omega$. In fact, this was the precise critical framework in which (P) appeared to us, as a short of "degenerate" perturbation limit in a classical population dynamics model (see [13], and [11] for a preliminary analysis of the associated Dirichlet problem). The main issues to be here reviewed are those of existence, uniqueness and blow-up rates of the solutions at the boundary. It should be stressed that elucidating as better as possible the asymptotic profile of solutions to (P) near $\partial\Omega$ is a crucial step in order to provide the uniqueness of those solutions (see also Remark 2.1 b) below). At the time where [13] was written the uniqueness of positive solutions to (P) with a positive but vanishing on $\partial\Omega$ was still an open question (see [14] for a detailed historical account of results when $a(x)$ is bounded away from zero in $\overline{\Omega}$).

The next result summarizes the main properties of problem (P). We refer to [14] for detailed proofs and more general problems deduced from it by suitable nonlinear perturbations.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ a bounded and $C^{2,\alpha}$ domain, $\lambda, a \in C^\alpha(\overline{\Omega})$ such that $a(x) > 0$ in Ω, $a(x) = 0$ for all $x \in \partial\Omega$. Then the following properties are satisfied.*

i) [Existence] Problem (P) admits a minimal positive solution $\underline{u}(x) \in C^{2,\alpha}(\Omega)$ and a maximal positive solution $\overline{u}(x) \in C^{2,\alpha}(\Omega)$. If in addition $\lambda(x) > 0$ in $\overline{\Omega}$, then every positive solution u to (P) satisfies the lower estimate,

$$u(x) \geq \inf_{\Omega} \left(\frac{\lambda(x)}{a(x)} \right)^{\frac{1}{p-1}}.$$

ii) [Uniqueness] Suppose that $u_1, u_2 \in C^2(\Omega)$ are classical positive solutions to (P) verifying the asymptotic equivalence near $\partial\Omega$,

$$\lim_{d(x)\to 0} \frac{u_1(x)}{u_2(x)} = 1,$$

with $d(x) = dist(x, \partial\Omega)$. Then u_1 and u_2 coincide in Ω.

iii) *[Asymptotic estimates] If the coefficient $a(x)$ decays to zero according to,*

$$\lim_{d(x)\to 0} \frac{a(x)}{d(x)^\gamma} = C_0, \tag{1}$$

for certain positive constants C_0, γ, then every classical positive solution $u \in C^2(\Omega)$ to (P) satisfies,

$$\lim_{d(x)\to 0} \frac{u(x)}{d(x)^{-\alpha}} = A, \tag{2}$$

with exponent $\alpha = (\gamma + 2)/(p - 1)$ and coefficient $A = (\alpha(\alpha + 1)/C_0)^{1/(p-1)}$. In particular, (P) admits a unique positive classical solution $u \in C^2(\Omega)$.

Remark 2.1.

a) The uniqueness of positive solutions to (P) was first independently obtained in [10] and [14] by means of achieving the blow-up rate (2) from the decaying rate (1). However, a more accurate asymptotic development of u near the boundary was obtained in [14] on the basis of a decaying profile for a of the form,

$$a(x) = C_0 d^\gamma (1 + C_1 d + o(d)) \qquad \text{as} \quad d \to 0,$$

with C_0, C_1 either constant or class C^2 functions on $\partial\Omega$, C_0 positive on $\partial\Omega$ (see Remark 1-a), p. 3595 in [14]). Under this assumption, every positive solution u to (P) can be expressed near $\partial\Omega$ as,

$$u(x) = Ad^{-\alpha}(1 + Bd + o(d)) \qquad \text{as} \quad d \to 0,$$

with α, A as in iii), $B = ((N-1)H - (\alpha+1)C_1)/(\gamma+p+3))$, H the mean curvature of $\partial\Omega$ (see [7] for earlier previous results in this direction). Finally, it must be mentioned that the blow-up rates subject has been recently refined in [27] where (2) is obtained via (1) by allowing both C_0 and γ to vary continuously on $\partial\Omega$.

b) As pointed out in [12], in order to show the uniqueness of positive solutions to (P) (at least when $\lambda = 0$), what one really needs is an approximate measure of the blow-up rate of the solutions at the boundary, not so precise as (2) in Theorem 2.1. It suffices indeed to know that

$$C_1 d(x)^{-\alpha} \le u(x) \le C_2 d(x)^{-\alpha}$$

in a neighborhood of $\partial\Omega$, for some positive constants C_1, C_2.

c) The existence, uniqueness and blow-up rates of problem (P) with $\lambda = 0$, $a > 0$ in Ω and, more importantly, $a(x) \to \infty$ as $d(x) \to 0$ with an asymptotic rate,

$$a(x) \sim C_0 d^{-\gamma} \qquad \text{as } d \to 0, \tag{3}$$

where $d = d(x)$ and C_0 is a positive function defined on $\partial\Omega$, have been deeply studied in [5] (cf. also [4] for a previous study of the radial case which even covers the complementary range $0 < p \leq 1$ for the exponent). Among other results, it is shown there that $\gamma < 2$ is a necessary and sufficient condition for the existence of a positive solution to (P).

d) The extension of results as those in Therorem 2.1 to the case of elliptic systems is a very hard task. Of course, the lack of comparison or suitable variational approaches arise as the main sources of difficulty. In fact, only few particular cases have been properly studied at the present moment. To quote some of them, results on existence, uniqueness and asymptotic rates for,

$$\begin{cases} -\Delta u = f(u,v) & x \in \Omega \\ -\Delta v = g(u,v) & x \in \Omega \end{cases}$$

with $f = -u^p v^q, g = -u^r v^s$, has been produced in [16] for singular boundary conditions $u_{|\partial\Omega} = \infty$, $v_{|\partial\Omega} = \infty$ and even combinations $u_{|\partial\Omega} = \infty$, $v_{|\partial\Omega} =$ finite. The analysis has been performed in the so called subcritical-critical regimes $(p-1)(s-1) - qr \geq 0$. We refer in addition to [15] for recent results concerning the complementary supercritical case $f = -v^q, g = -u^r$ and to [18] for cooperative systems. See also [8], [9], [6], [26] where further systems exhibiting related phenomenologies are considered.

3. Contact null set-boundary

We are now discussing the existence of a positive solution to problem (P) when the null set $\overline{\Omega}_0$ of the coefficient $a(x)$ meets $\partial\Omega$ in a nontrivial way. More precisely, let $\{\Omega_{0,i}\}_{i=1}^m$ the set of connected pieces of Ω_0. We distinguish between "inner components", $\Omega_{0,k}^I$, defined as those $\Omega_{0,i}$ strongly contained in Ω and "boundary components", $\Omega_{0,l}^B$, which are those $\Omega_{0,i}$ whose boundary has common points with $\partial\Omega$. The main assumption in our next result is the existence of interior points to $\partial\Omega \cap \partial\Omega_0$ relatively to $\partial\Omega$. The conclusion is that under these conditions, problem (P) can not support positive (and therefore, finite) solutions.

Theorem 3.1. *Let $a \in C^{\alpha}(\overline{\Omega})$ be nonnegative and nontrivial so that*

$$\{x \in \Omega : a(x) = 0\} = \overline{\Omega}_0 \cap \Omega$$

where $\Omega_0 \subset \Omega$ is smooth in the sense that,

$$\Omega_0 = \left(\cup_{k=1}^{m_1} \Omega_{0,k}^{I}\right) \cup \left(\cup_{l=1}^{m_2} \Omega_{0,l}^{B}\right),$$

being $\Omega_{0,k}^{I}, \Omega_{0,l}^{B}$, $C^{2,\alpha}$-bounded subdomains of Ω whose closures $\overline{\Omega}_{0,k}^{I}, \overline{\Omega}_{0,l}^{B}$ are pair-wise disjoint and satisfy:

$$\overline{\Omega}_{0,k}^{I} \subset \Omega, \qquad \Gamma_l := \partial\Omega_{0,l}^{B} \cap \partial\Omega \neq \emptyset, \tag{4}$$

for each $k \in \{1, \ldots, m_1\}, l \in \{1, \ldots, m_2\}$. Suppose that some $1 \leq l \leq m_2$ exists such that:

$$\overset{\circ}{\Gamma}_l \neq \emptyset, \tag{5}$$

the interior $\overset{\circ}{\Gamma}_l$ of Γ_l being considered with regard to $\partial\Omega$. Then, the following properties are satisfied,

a) Regardless (5), a necessary condition in order that the problem (P) does admit a positive solution u is:

$$\lambda_1^{\Omega_{0,i}}(-\Delta - \lambda(x)) > 0 \qquad \text{for either } \Omega_{0,i} = \Omega_{0,k}^{I} \ \text{ or } \ \Omega_{0,i} = \Omega_{0,k}^{B}, \tag{6}$$

and every $1 \leq k \leq m_1$, $1 \leq l \leq m_2$.

b) Under the validity of condition (6) the auxiliary problem,

$$\begin{cases} -\Delta u = \lambda(x)u - a(x)u^p & x \in \Omega \\ \quad u = m & x \in \partial\Omega, \end{cases} \tag{7}$$

admits, for each $m \in \mathbb{N}$, a unique positive solution $u_m \in C^{2,\alpha}(\overline{\Omega})$.

c) If (6) is satisfied and $u_m \in C^{2,\alpha}(\overline{\Omega})$ is the solution to (7), the limit

$$\lim u_m(x) = \infty,$$

holds uniformly on compact sets of every component $\Omega_{0,l}^{B}$ of Ω_0 fulfilling (5).

Therefore (5) implies that problem (P) can not admit a positive solution u in Ω.

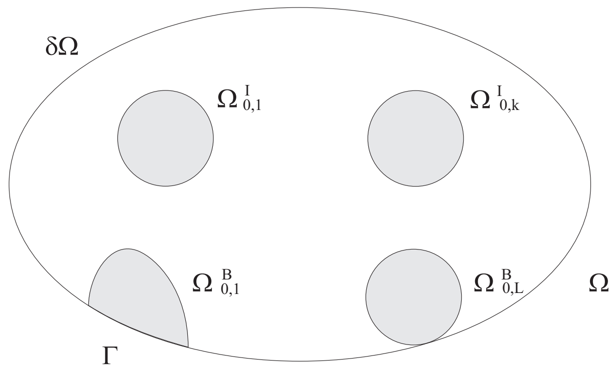

FIGURE 1. A configuration for Ω_0 having inner components $\Omega_{0,k}^I$ and two kind of boundary connected pieces $\Omega_{0,l}^B$. One of them fulfills (5).

Proof. We begin by proving a) and so let $u \in C^2(\Omega)$ be a positive solution to (P). If $\Omega_{0,i}$ is any inner component, $\Omega_{0,i} \subset \Omega$, since u is finite in $\overline{\Omega}_{0,i}$ it defines a strict positive supersolution to the equation

$$-\Delta u - \lambda(x)u = 0, \tag{8}$$

in $\Omega_{0,i}$. It is well-known (see [25]) that this implies,

$$\lambda_1^{\Omega_{0,i}}(-\Delta - \lambda) > 0.$$

Let us suppose now that $\Omega_{0,i}$ is a boundary component and for $\varepsilon > 0$ small enough write $Q = \{x \in \Omega : \mathrm{dist}(x, \Omega_{0,i}) < \varepsilon\}$, $Q \supset \Omega_{0,i}$. Then we have the estimates,

$$\lambda_1^{\Omega_{0,i}}(-\Delta - \lambda) = \lambda_1^{\Omega_{0,i}}(-\Delta - \lambda + q) > \lambda_1^{Q}(-\Delta - \lambda + q), \tag{9}$$

with $q = au^{p-1}$. Take $\delta > 0$ small and define $Q_\delta = \{x \in Q : \mathrm{dist}(x, \partial\Omega) > \delta\}$. Observe that $Q_\delta \subset Q$ together with $\lim_{\delta\to 0} Q_\delta = Q$. Similarly, u defines again a strict positive supersolution to the alternative equation:

$$-\Delta u + (q(x) - \lambda(x))u = 0,$$

in Q_δ for every δ. Thus, $\lambda_1^{Q_\delta}(-\Delta - \lambda + q) > 0$ and

$$0 \le \lim_{\delta\to 0+} \lambda_1^{Q_\delta}(-\Delta - \lambda + q) = \inf_{\delta>0} \lambda_1^{Q_\delta}(-\Delta - \lambda + q) = \lambda_1^{Q}(-\Delta - \lambda + q).$$

This and (9) imply the positivity of $\lambda^{\Omega_{0,i}}(-\Delta - \lambda)$ and the necessity of (6) for the existence of positive solutions to (P) is already shown.

Next we prove b). As a first remark observe that the uniqueness of positive solutions to (7) follows, for instance, from [3]. To get existence

$\underline{u} = 0$ can always be taken as a subsolution while to get a comparable supersolution $\overline{u}$ one only needs to construct a specific supersolution $u^* \in C^2(\overline{\Omega})$ which is positive in $\overline{\Omega}$. In fact, it can be checked that $\overline{u} = Au^*$ also defines a supersolution for every $A \geq 1$. Hence, supersolutions as large as desired can be produced. Following the approach in [24] we are in fact showing that (5) suffices to construct a positive supersolution u^*. We are assuming for simplicity that $a > 0$ on $\partial\Omega \setminus \overline{\Omega}_0$ (see Remark 3.1).

To construct u^* choose $\delta > 0$ so small as to have

$$\lambda_1^{Q^\delta}(-\Delta - \lambda(x)) > 0,$$

where Q is any of the connected pieces $\Omega_{0,i}$ and $Q^\delta = \{x \in \mathbb{R}^N : \operatorname{dist}(x, Q) < \delta\}$. For every $1 \leq i \leq m$, $Q = \Omega_{0,i}$, let $\phi_i(x)$ be the first positive eigenfunction associated to $\lambda_1^{Q^\delta}(-\Delta - \lambda(x))$ in Q^δ which satisfies $|\phi_i|_\infty = 1$. We define $w \in C^2(\overline{\Omega})$ as,

$$w(x) = \begin{cases} \phi_i(x) & x \in \overline{\Omega}_{0,i}^{\delta/2} \cap \overline{\Omega},\ 1 \leq i \leq m \\ v(x) & x \in \overline{\Omega} \setminus \cup_{i=1}^m \overline{\Omega}_{0,i}, \end{cases}$$

where $v(x)$ is a positive and C^2 extension of $\sum \chi_i \phi_i$ to the whole of $\overline{\Omega}$, and χ_i stands for the characteristic function of $\Omega_{0,i}^{\delta/2}$. Direct computation shows that $u^* = Bw$ defines a positive supersolution to $-\Delta u = \lambda u - au^p$ in Ω provided $B > 0$ is chosen large enough. Thus, the proof of b) is finished.

Before proving *c)* notice that, aside the validity of condition (5), the existence of a positive solution u to (P) easily implies that,

$$u_m(x) < u(x) \qquad x \in \Omega,$$

for every $m \in \mathbb{N}$. Indeed such inequality follows from the fact that $u_m < u$ near the boundary $\partial\Omega$ and that the Dirichlet problem for $-\Delta u = \lambda u - au^p$ is uniquely solvable in the class of positive solutions. In particular,

$$u(x) \geq \lim u_m(x) = \sup u_m(x) \qquad x \in \Omega.$$

Hence, once *c)* is shown the existence of positive solutions to (P) is not possible.

Thus, let us finally proceed to show *c)*. Suppose $\Omega_{0,i} = \Omega_{0,l}^B$ is a connected piece satisfying (5) and choose a connected piece Γ,

$$\Gamma \subset \overset{\circ}{\Gamma}_l\ .$$

Notice that Γ is open in $\partial\Omega$. For $\eta > 0$ small put $\Gamma_\eta = \{x \in \Gamma : \operatorname{dist}(x, \partial\Gamma) > \eta\}$, where $\partial\Gamma$ is the boundary of Γ relative to $\partial\Omega$, and consider a cut off function $\varphi = \varphi(x) \in C_0^{2,\alpha}(\partial\Omega)$, $0 \leq \varphi \leq 1$ such that $\varphi = 1$ in Γ_η, $\varphi = 0$ outside $\Gamma_{\eta/2}$.

In view of Fredholm's alternative (cf. [20]) the problem,

$$\begin{cases} -\Delta u - \lambda(x)u = 0 & x \in \Omega_{0,i} \\ \quad u = \varphi & x \in \partial\Omega_{0,i}, \end{cases}$$

possesses a unique positive solution $u_\Gamma \in C^{2,\alpha}(\overline{\Omega}_{0,i})$. Thanks to (6), the maximum principle holds true in $\Omega_{0,i}$ (see [25]) and weak comparison leads to

$$u_m(x) \geq m\ u_\Gamma(x) \qquad x \in \Omega_{0,i}.$$

This immediately yields

$$u_m(x) \to \infty,$$

uniformly on compacts of $\Omega_{0,i}$. This concludes the proof of Theorem 3.1.□

Remark 3.1.
a) A supersolution u^* to equation $-\Delta u = \lambda(x)u - a(x)u^p$ in Ω, positive in $\overline{\Omega}$, can be still constructed if a vanishes at some points in $\partial\Omega \setminus \partial\Omega_0$. Notice that Ω_0 is defined as the interior of $\{a(x) = 0\}$ in Ω and so the existence of such points is not forbidden. To produce u^* we can assume the more adverse situation where $a \equiv 0$ on $\partial\Omega$. In that case we enlarge Ω to $\Omega^\varepsilon = \{x \in \mathbb{R}^N : \operatorname{dist}(x, \Omega) < \varepsilon\}$, $\varepsilon > 0$ small, and extend $a(x)$, $\lambda(x)$ to $\mathbb{R}^N$ as C^α functions, $a = 0$ outside $\overline{\Omega}$. In this new framework, both the inner components of $\overbrace{\{a(x) = 0\}}^{\circ} \cap\ \Omega$ and the boundary components not satisfying (5) remain unchanged. The remaining boundary components are now part of a larger one $\Omega^B_{0,\varepsilon}$ which also includes the region $\{x \in \mathbb{R}^N : 0 < \operatorname{dist}(x, \partial\Omega) < \varepsilon\}$. Observe that

$$\lambda_1^{\Omega^B_{0,\varepsilon}}(-\Delta - \lambda(x)) \to \min \lambda_1^{\Omega_{0,i}}(-\Delta - \lambda(x)),$$

as $\varepsilon \to 0+$, the minimum only extended to those boundary components $\Omega_{0,i}$ satisfying (5). Thus, $\lambda_1^{\Omega^B_{0,\varepsilon}}(-\Delta - \lambda(x))$ is positive for small ε and the method in the proof of b) can be repeated to get a positive supersolution in Ω^ε whose restriction to Ω is the searched supersolution u^*.

b) It follows from the analysis in section 4 that under condition (6) the sequence u_m of solutions to (7) converges to a function $u \in C^{2,\alpha}(\Omega \setminus \overline{\cup\, \Omega_{0,i}})$, the union being extended to all boundary components. The asymptotic behaviour of u_m on those boundary components $\Omega^B_{0,l}$ not satisfying (5) is still unclear up to complete generality. Further features and explicit examples of this limit behaviour will be included in a work in preparation.

c) An interesting problem that one could pose is whether (P) admits positive solutions if $a \in C^\alpha(\overline{\Omega})$ becomes partially negative in Ω. Suppose, for instance, that $\Omega^\pm = \{x \in \Omega : \pm a(x) > 0\}$, $\Omega_0 = \overbrace{\{a(x) = 0\}}^{\circ}$ are $C^{2,\alpha}$ domains. It is next remarked that the relative position of the components Ω_i^- of Ω^- with respect $\partial\Omega$ is again crucial for this issue. Indeed, if some component Ω_i^- satisfies (5) no positive solutions for (P) are possible. In fact, if such solution u exists we have in that component,

$$\lambda_1^{\Omega_i^-}(-\Delta - \lambda(x)) > \lim_{\delta \to 0+} \lambda_1^{(\Omega_i^-)_\delta}(-\Delta - \lambda(x) + au^{p-1}) \geq 0,$$

$(\Omega_i^-)_\delta = \{x \in \Omega_i^- : \mathrm{dist}(x, \partial\Omega) > \delta\}$, since $\lambda_1^{(\Omega_i^-)_\delta}(-\Delta - \lambda(x) + au^{p-1})$ is positive for δ small. Setting, as in Theorem 3.1, Γ a connected piece of the interior of $\partial\Omega_i^- \cap \partial\Omega$ and choosing φ a $C^{2,\alpha}$ cut-off function as there, then the problem $-\Delta u - \lambda u = 0$ in Ω_i^-, $u = \varphi$ on $\partial\Omega_i^-$ can be solved to obtain a positive solution $u_\Gamma \in C^{2,\alpha}(\overline{\Omega^-})$. Fixed a positive solution u to (P) we get by comparison that for each $m \in \mathbb{N}$, a positive δ_0 exists so that $u > m\ u_\Gamma$ in $(\Omega_i^-)_\delta$ for every $0 < \delta < \delta_0$. Thus $u > m\ u_\Gamma$ in Ω_i^- what is not possible. This proves the remark.

As a conclusion, positive solutions to (P) are only possible when $\Omega^- \subset \overline{\Omega^-} \subset \Omega$ (of course, together with $\overline{\Omega}_0 \subset \Omega$). On the other hand, the existence of positive solutions can be shown, in symmetric cases, provided that Ω^- and Ω^+ keep a suitable balance. Such results, belonging to a work in progress with Prof. C. Flores, will be diffused in a next future.

4. Inner null set

We are exploring in this section the existence of positive solutions to (P) when the refuge Ω_0, the interior in Ω of the null set $\{a(x) = 0\}$, is strongly contained in Ω. Suppose that Ω_0 is also a $C^{2,\alpha}$ domain and so it only exhibits finitely many connected pieces $\Omega_{0,1}, \dots, \Omega_{0,m}$ all of them of class $C^{2,\alpha}$. Then, if (P) admits a positive solution $u \in C^{2,\alpha}(\Omega)$ we immediately find (see Theorem 3.1),

$$\lambda_1^{\Omega_{0,i}}(-\Delta - \lambda(x)) > 0 \qquad i = 1, \dots, m. \tag{10}$$

Our next result states that (10) is also sufficient to ensure the existence of a positive solution to (P).

Theorem 4.1. *Let $a, \lambda \in C^\alpha(\overline{\Omega})$, $a \geq 0$ with $\overline{\Omega}_0 = \{a(x) = 0\}$, $\Omega_0 \subset \Omega$ a $C^{2,\alpha}$ domain with connected pieces $\Omega_{0,i}, \dots, \Omega_{0,m}$ satisfying:*

$$\overline{\Omega}_{0,i} \subset \Omega \qquad i = 1, \dots, m.$$

Then (10) *is a necessary and sufficient condition for the existence of a positive solution to (P).*

Proof. Only the sufficient character of (10) need to be proved. To construct a positive solution to (P) we first analyze the auxiliary problem (7). In view of (10) and the proof of Theorem 3.1, a positive strict supersolution $u^* \in C^2(\overline{\Omega})$ to $-\Delta u = \lambda(x)u - a(x)u^p$ can be obtained (see section 5, Remarks 5.3 for an alternative construction of u^*). Thus, problem (7) admits a unique positive solution $u_m \in C^{2,\alpha}(\overline{\Omega})$ for every $m \in \mathbb{N}$. Notice that, thanks to Remark 3.1 a), this assertion holds true even in the case where a vanishes partially or totally on $\partial\Omega$.

On the other hand, u_m is increasing and admits the estimate:

$$u_m(x) < u_D(x) \qquad x \in D, \tag{11}$$

with $D = \{x \in \Omega : a(x) > 0\}$, $u_D \in C^{2,\alpha}(D)$ being the minimal solution to the singular problem,

$$\begin{cases} -\Delta u = \lambda(x)u - a(x)u^p & x \in D \\ \quad u = \infty & x \in D. \end{cases}$$

Thus, by using local L^∞ estimates, $W^{2,q}$ estimates and bootstrapping in a standard way we obtain that $u_m \to u$ in $C^{2,\alpha}(D)$. It is also easily checked that $u \to \infty$ as $\mathrm{dist}(x, \partial\Omega) \to 0+$.

To complete the existence proof we only need to extend the convergence of u_m to an open neighborhood of Ω_0. Let us proceed separately on each component and so, choose a single component $\Omega_{0,i}$ and $\delta > 0$ small so that,

$$\lambda_1^{(\Omega_{0,i})^\delta}(-\Delta - \lambda(x)) > 0,$$

with $(\Omega_{0,i})^\delta = \{x \in \mathbb{R}^n : \mathrm{dist}(x, \Omega_{0,i}) < \delta\}$. Let $\phi_i \in C^{2,\alpha}(\overline{(\Omega_{0,i})^\delta})$ the associated normalized positive eigenfunction such that, say $|\phi_i|_\infty = 1$. Since $\Gamma_{\delta/2} = \{\mathrm{dist}(x, \Omega_{0,i}) = \frac{\delta}{2}\} \subset D$, thanks to (11) we know that:

$$\sup_m u_m(x) \leq c < \infty \qquad x \in \Gamma_{\delta/2}.$$

Hence, a positive constant A not depending on m can be found such that,

$$u_m(x) < c < A\phi_i(x) \qquad x \in \Gamma_{\delta/2},$$

while for every $m \in \mathbb{N}$,

$$-\Delta u_m - \lambda u_m \leq -\Delta A\phi_i - \lambda A\phi_i \qquad \text{in } \Omega_{0,i}^{\delta/2}.$$

Finally, the weak comparison principle holds in $\Omega_{0,i}^{\delta/2}$ since

$$\lambda_1^{(\Omega_{0,i})^{\delta/2}}(-\Delta-\lambda(x)) > \lambda_1^{(\Omega_{0,i})^{\delta}}(-\Delta-\lambda(x)) > 0.$$

Therefore,

$$u_m(x) \leq A\phi_i(x) \qquad x \in \overline{\Omega_{0,i}^{\delta/2}}.$$

As seen before, this estimate implies the convergence of u_m to certain u in $C^{2,\alpha}(\Omega_{0,i}^{\delta/2})$. In conclusion, $u_m \to u$ in $C^{2,\alpha}(\Omega)$. Notice in addition that u defines the minimal solution to (P). The proof of Theorem 4.1 is thus concluded. □

Remark 4.1.
a) Under the present assumptions on the weight $a(x)$, a maximal solution can also be constructed.

b) Estimates for positive solutions near the boundary (and therefore uniqueness) can be achieved by assuming that $a > 0$ on $\partial\Omega$ or $a = 0$ on $\partial\Omega$ with a convenient decay (see Theorem 2.1 iii)).

5. A normal derivative estimate

As has been shown, the solution $u(\cdot, u_0) \in C^{2,\alpha}(\overline{\Omega})$ to the problem,

$$\begin{cases} -\Delta u = \lambda(x)u - a(x)u^p & x \in \Omega \\ \quad u = u_0 & x \in \partial\Omega, \end{cases} \tag{12}$$

converges to the minimal solution u to (P) as $u_0 \to \infty$ provided $a > 0$ in an open neighborhood $\{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \varepsilon\}$ of $\partial\Omega$ relatively to Ω.

Solutions to (12) can be used to construct, say, supersolutions in larger domains, when they are matched outside Ω with solutions of a Dirichlet problem with datum u_0 (see Remarks 5.3 below). As observed in [1], to succeed in this construction it is essential to achieve an exact estimate on the growth of the normal derivative $\dfrac{\partial u(\cdot, u_0)}{\partial \nu}$ as $u_0 \to \infty$. In our next results, such a fundamental information is obtained under two different assumptions on $a(x)$. In a first instance it is supposed that a is positive at the point x_0 where the normal derivative will be estimated. In a second improvement, the estimate will be measured if a vanishes at x_0 but according to a suitable –and in some sense rather natural– decaying rate (see (28)). These are the objectives of the following statements, which deserve full interest by their own. We will be working under the hypotheses of section 4. In fact, problem

(12) only leads to a finite solution to (P) under such assumptions. On the other hand, it can be checked that $\dfrac{\partial u(\cdot,u_0)}{\partial\nu}$ grows linearly in u_0 when measured in regions $\Gamma\subset\partial\Omega\cap\partial\Omega_0$ corresponding to boundary components $\Omega_{0,i}$ of Ω_0 satisfying (6).

Theorem 5.1. *Let $a\in C^\alpha(\overline{\Omega})$ be a nonnegative function such that its null set $\{a(x)=0\}=\overline{\Omega}_0$ where $\Omega_0\subset\overline{\Omega}_0\subset\Omega$ is $C^{2,\alpha}$ and exhibits m connected pieces $\Omega_{0,1},\dots,\Omega_{0,m}$. Suppose in addition that condition* (10) *holds, i. e.,*

$$\lambda_1^{\Omega_{0,i}}(-\Delta-\lambda)>0,$$

$1\le i\le m$. Then, for every $x_0\in\partial\Omega$ such that $a(x_0)>0$ we have the following exact estimate,

$$\frac{\partial u(x_0,u_0)}{\partial\nu}\sim\sqrt{\frac{2a(x_0)}{p+1}}u_0^{\frac{p+1}{2}},\tag{13}$$

as $u_0\to\infty$.

Proof. We are first proving that

$$\sup_\Omega u(\cdot,u_0)=O(u_0),\tag{14}$$

as $u_0\to\infty$ (see also Remark 5.1 below). In fact, observe that condition (10) provides the existence of a positive supersolution $u^*\in C^2(\overline{\Omega})$ to equation $-\Delta u=\lambda u-au^p$ in Ω. Recall that cu^* is also a supersolution if $c\ge 1$. This implies that

$$u(x,u_0)\le u_0c_0u^*(x)\qquad x\in\Omega,$$

with $c_0=\{\inf_{\partial\Omega}u^*\}^{-1}$ provided that $u_0c_0\ge 1$. This shows the assertion.

By means of a rotation followed by a translation if necessary it can be assumed without loss of generality that $x_0=0$ while the outward unit normal to Ω at x_0 is $\nu(x_0)=-e_N$. By hypothesis there exist $\varepsilon_0,\varepsilon>0$, a real function $f\in C^{2,\alpha}(B(0,\varepsilon_0)\cap\mathbb{R}^{N-1})$ ($B(0,r)=$ the open ball in R^N centered at 0 with radius r) and a neighborhood V_0 of $x_0=0$ in $\mathbb{R}^N$ such that:

$$V_0\cap\Omega=\{x=(x',x_N)\in\mathbb{R}^N:0<f(x')<x_N<f(x')+\varepsilon,\quad |x'|<\varepsilon_0\},$$

and,

$$V_0\cap\partial\Omega=\{x=(x',x_N)\in\mathbb{R}^N:x_N=f(x'),\quad |x'|<\varepsilon\},$$

with $f(0)=0$, $\nabla_{x'}f(0)=0$, $x'=(x_1,\dots,x_{N-1})$.

In order to straightening $\partial\Omega$ near 0 consider, as usual, the change of coordinates $y = h(x)$ given by,

$$\begin{cases} y' = x' \\ y_N = x_N - f(x'), \end{cases} \qquad x \in V_0. \tag{15}$$

Then, problem (12), when observed in V_0 is transformed into,

$$\begin{cases} -\Delta_y u + \sum_{i=1}^{N-1} b_i(y') \dfrac{\partial^2 u}{\partial y_i \partial y_N} + c(y') \dfrac{\partial u}{\partial y_N} = \lambda(y) u - a(y) u^p, & y \in V_1 \\ u(y', 0) = u_0 & |y'| < \varepsilon_0, \end{cases} \tag{16}$$

where $V_1 = h(V_0) = \{y \in \mathbb{R}^N : |y'| < \varepsilon_0,\ 0 < y_N < \varepsilon\}$,

$$b_i(y') = 2 \frac{\partial f}{\partial x_i}(x'), \qquad c(y') = \Delta_{x'} f(x'), \qquad x = h^{-1}(y),$$

and where we have retained u, λ, a to designate the corresponding transformed functions of y.

Taking into account the estimate (14) we are next using the blow-up approach in [19] (see also [5] and [16] in this context). Thus, introduce the change,

$$z = \sigma y, \qquad \sigma = u_0^{\frac{p-1}{2}}, \tag{17}$$

together with,

$$u(y) = u_0 v(z). \tag{18}$$

Then, problem (16) can be written as,

$$\begin{cases} -\Delta_z v + \sum_{i=1}^{N-1} \tilde{b}_i(z') \dfrac{\partial^2 v}{\partial z_i \partial z_N} + \dfrac{1}{\sigma} \tilde{c}(z') \dfrac{\partial v}{\partial z_N} = \dfrac{1}{\sigma^2} \tilde{\lambda}(z) v - \tilde{a}(z) v^p, \\ v(z', 0) = 1, \qquad |z'| < \sigma \varepsilon_0, \end{cases} \tag{19}$$

where $z \in \sigma V_1 = \{\sigma y : y \in V_1\}$ and functions with a tilde stand for the former functions evaluated either at $y' = \sigma^{-1} z'$ or $y = \sigma^{-1} z$, for instance $\tilde{b}_i(z') = b_i(\sigma^{-1} z')$ and $\tilde{a}(z) = a(\sigma^{-1} z)$.

As a conclusion, we have the family of functions $v(\cdot, \sigma) \in C^{2,\alpha}(\sigma \overline{V}_1)$, $v(\cdot, \sigma)$ solving (19), being the family $v(\cdot, \sigma)$ uniformly bounded in σV_1,

$$v(z, \sigma) \le C \qquad z \in \sigma V_1, \quad \sigma \ge \sigma_0,$$

with $C = \{\inf_{\partial\Omega} u^*\}^{-1} \sup_\Omega u^*$ and $\sigma_0 = \{\inf_{\partial\Omega} u^*\}^{(p-1)/2}$.

For arbitrary $R > 0$, let $Q \subset R^N_+$ a smooth subdomain such that $\overline{Q} \cap \{x_N = 0\} = B(0,R) \cap \{x_N = 0\}$. Since $\overline{Q} \subset \sigma \overline{V}_1$ for σ large then,

$$\mathcal{L}_\sigma v(\cdot,\sigma) = f_\sigma \qquad z \in Q,$$

where $\mathcal{L}_\sigma = -\Delta_z + \sum_{i=1}^{N-1} \tilde{b}_i \partial^2_{z_i z_N} + \tilde{c}\partial_{z_N} - \tilde{\lambda}$, $f_\sigma = \tilde{a} v(\cdot,\sigma)^p$. The L^q estimates of Agmon-Douglis-Nirenberg state in this case that for all $q > 1$,

$$|v|_{W^{2,q}(Q)} \leq C_q(|v|_{\infty,Q} + |f_\sigma|_q + |\mathcal{B}u|_{W^{2-1/q,q}(T)}),$$

with $v = v(x,\sigma)$, $T = \overline{Q} \cap \{x_N = 0\}$, $\mathcal{B}u = u_{|T}$ in the sense of traces, and where C_q only depends on Q, N, q and the L^∞ norms of the coefficients of $\mathcal{L}_\sigma$. Combining the uniform boundedness of $v(\cdot,s)$ with Sobolev's embeddings we get the estimate,

$$|v(\cdot,\sigma)|_{C^{1,\beta}(\overline{Q})} \leq M_\beta \tag{20}$$

for $0 < \beta < 1$ arbitrary. This can be used together with the up to the boundary partial Schauder's estimate (cf. [20]),

$$|v|_{C^{2,\alpha}(Q\cup T)} \leq C_\alpha(|v|_{\infty,Q} + |f_\sigma|_{C^\alpha(Q\cup T)} + |\mathcal{B}v|_{C^{2,\alpha}(Q\cup T)}),$$

with $v = v(z,\sigma)$, to get a subfamily $v(\cdot,\sigma')$ converging to a certain $v = v(z)$ in $C^{2,\alpha}(Q\cup T)$. In fact, it suffices with using (20) to extract such convergent subfamily in $C^{1,\beta}(\overline{Q})$, $\beta > \alpha$ and then employing the Schauder's estimate to get a Cauchy's condition in $C^{2,\alpha}(Q \cup T)$.

Finally, by making $Q \to R^N_+$ $(R \to \infty)$ and using a standard diagonal convergence device it is possible to obtain a subfamily still denoted $v(\cdot,\sigma')$ of $v(\cdot,\sigma)$ such that,

$$v(z,\sigma') \to v(z), \qquad \sigma' \to \infty,$$

in $C^{2,\alpha}_{\text{loc}}(\overline{\mathbb{R}}^N_+)$. Moreover, $v = v(z)$ is a *bounded* solution of the problem:

$$\begin{cases} -\Delta v = -a_0 v^p & z \in \mathbb{R}^N_+ \\ \quad v = 1 & z_N = 0. \end{cases} \tag{21}$$

Observe in addition that being $v(z)$ bounded and sub harmonic in $\mathbb{R}^N_+$ we additionally have that,

$$0 < v(z) < 1 \qquad z_N > 0. \tag{22}$$

It follows that problem (21) has a unique bounded classical positive solution $v = v(z)$ which in addition depends only on z_N:

Lemma 5.1. *Let $a_0 > 0$. Then problem (21) has a unique bounded positive solution v, which is a function of z_N only, and is explicitly given by:*

$$v(z) = \left(1 + \frac{p-1}{2} A z_N\right)^{\frac{2}{p-1}} ,$$

where A is given by

$$A = \sqrt{\frac{2a_0}{p+1}} \ .$$

For convenience, we postpone the proof of Lemma 5.1 until the end of this one. Observe that Lemma 5.1 implies in particular that the whole family,

$$v(z,\sigma) \to v(z) = \left(1 + \frac{p-1}{2} A z_N\right)^{\frac{2}{p-1}} , \tag{23}$$

in $C^{2,\alpha}_{\text{loc}}(\overline{\mathbb{R}}^N_+)$ as $\sigma \to \infty$, where A is as in the statement of Lemma 5.1. As a main consequence,

$$\frac{\partial u(\cdot, u_0)}{\partial \nu}_{|x=x_0} = -\frac{\partial u(\cdot, u_0)}{\partial x_N}_{|x=0} = -u_0^{\frac{p+1}{2}} \frac{\partial v(\cdot,\sigma)}{\partial z_N}_{|z=0} =$$
$$- u_0^{\frac{p+1}{2}} (v_1'(0) + o(1)) = A u_0^{\frac{p+1}{2}} + o\left(u_0^{\frac{p+1}{2}}\right),$$

as $u_0 \to \infty$. This finishes the proof of Theorem 5.1. □

We proceed now to the Proof of Lemma 5.1.

Proof of Lemma 5.1. Fix R, h positive and choose $D_{R,h}$ any bounded $C^{2,\alpha}$ sub domain of R^N_+ containing $\{x : |x'| < R,\ 0 < x_N < h\}$, for instance a regularization of the latter domain which contains it. Consider the auxiliary problem,

$$\begin{cases} -\Delta v = -a_0 v^p & z \in D_{R,h} \\ \quad v = 1 & z \in \partial D_{R,h}. \end{cases} \tag{24}$$

By using a standard weak comparison argument it follows that problem (24) admits at most a unique positive solution. Such solution can be obtained by the method of sub and supersolutions by taking $\underline{v} = 0$ as subsolution,

$\overline{v} = 1$ as a supersolution. Thus (24) admits a unique positive solution $v_{R,h} \in C^{2,\alpha}(\overline{D}_{R,h})$ which satisfies,

$$0 < v_{R,h}(z) < 1, \qquad z \in D_{R,h}.$$

Now observe that if $v = v(z) \in C^{2,\alpha}_{\text{loc}}(\overline{\mathbb{R}}^N_+)$ is any nonnegative bounded solution to (21) (and so satisfying (22)), then

$$v_{|D_{R,h}}$$

defines a subsolution to problem (24) which, as a consequence of weak comparison, satisfies $v(z) \le v_{R,h}(z)$ in $D_{R,h}$. Hence,

$$0 < v(z) < v_{R,h}(z) \qquad z \in D_{R,h},$$

where strong comparison has been also employed.

For $n \in \mathbb{N}$ set $D_n := nD_{R,h} = \{nz : z \in D_{R,h}\}$ while $v_n(z)$ designates the solution to problem (24) in D_n. Just for the same argument as the one given above we arrive at:

$$0 < v(z) < v_{n+k}(z) < v_n(z) < 1 \qquad z \in D_n,$$

for every $n, k \in \mathbb{N}$. In fact, the restriction of v_{n+k} to D_n defines a strict subsolution to (24) in D_n that can be strongly compared with v_n.

Using the L^q and Schauder's estimates as has been already done we get the convergence,

$$\overline{v}(z) = \lim v_n(z) = \inf v_n(z) \qquad z \in \overline{\mathbb{R}}^N_+,$$

in $C^{2,\alpha}_{\text{loc}}(\overline{\mathbb{R}}^N_+)$. In addition,

$$0 < v(z) \le \overline{v}(z) \qquad z \in \overline{\mathbb{R}}^N_+,$$

for any arbitrary bounded positive solution v to (21) satisfying (22). Since $\overline{v}(z)$ is also a solution to (21), this means that $\overline{v}(z)$ is in fact the *maximal* positive solution to (21) in such class.

Observe now that for every vector $\tau \in \mathbb{R}^{N-1}$,

$$\overline{v}_\tau(z) = \overline{v}(z + (\tau, 0)),$$

is again as positive solution to (21) bounded by 1. In particular

$$\overline{v}(z + (\tau, 0)) \le \overline{v}(z) \qquad z \in \mathbb{R}^N_+.$$

This immediately implies that

$$\overline{v}(z' + \tau, z_N) = \overline{v}(z', z_N),$$

for all $z \in \mathbb{R}^N_+$ and $\tau \in R^{N-1}$. Hence, $\overline{v}$ depends only on z_N.

It can be shown in the same way that problem (21) has a minimal solution $\underline{v}$ in the class of positive solutions bounded by 1, which only depends on z_N. In fact, consider the problem,

$$\begin{cases} -\Delta v = -a_0 v^p & z \in D_{R,h} \\ \quad v = \zeta & z \in \partial D_{R,h}, \end{cases} \tag{25}$$

where $D_{R,h}$ is as before while $\zeta \in C^{2,\alpha}(\partial D_{R,h})$, $0 \le \zeta \le 1$, satisfying in addition $\zeta(z) = 1$ for $z_N = 0$, $|z'| < R - 2\varepsilon$, $\varepsilon > 0$ small enough, and $\zeta = 0$ outside $\{|z'| < R - \varepsilon,\ z_N = 0\}$ in the whole of $\partial D_{R,h}$. Problem (25) and the corresponding ones in D_n admit a unique positive classical solution $w_n(z)$ such that the relation

$$0 < w_n(z) < w_{n+k}(z) < v(z),$$

holds for any $n, k \in \mathbb{N}$, $z \in D_n$ and an arbitrary nonnegative solution v to (21) satisfying (22). Then, the limit,

$$\underline{v}(z) = \lim w_n(z) = \sup w_n(z)$$

exists in $C^{2,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}}^N_+)$ and defines the minimal solution to (21). Therefore, $\underline{v}$ only depends on z_N.

Finally, observe that the one-dimensional problem,

$$\begin{cases} v'' = a_0 v^p & t > 0 \\ v(0) = 1, \end{cases} \tag{26}$$

has a unique positive solution $v = v_1(t)$ with the property of being defined in the whole interval $[0, \infty)$. Such solution is explicitely given by,

$$v_1(t) = \left(1 + \frac{p-1}{2} A t\right)^{\frac{2}{p-1}}, \qquad A = \sqrt{\frac{2a_0}{p+1}},$$

with

$$v_1'(0) = A.$$

In particular, the maximal and minimal solutions satisfy:

$$\underline{v}(z) = \overline{v}(z) = v_1(z_N) \qquad z \in \mathbb{R}^N_+,$$

and problem (21) has $v_1(z_N)$ as its unique positive bounded solution.

Remark 5.1.
a) In many cases it can be shown that

$$\sup_{\Omega} u(\cdot, u_0) = u_0, \tag{27}$$

for u_0 large. In fact, notice that since $u(x, u_0)$ becomes finite in every $x \in \Omega$ as $u_0 \to \infty$ (Theorem 4.1), then $\mathrm{dist}(x_{u_0}, \Omega) \to 0$ as $u_0 \to \infty$ where x_{u_0} is any point in $\overline{\Omega}$ where $u(\cdot, u_0)$ achieves its maximum. On the other hand $\lambda(x) - a(x)u(x, u_0)^{p-1} \geq 0$ at $x = x_{u_0}$ if $x_{u_0} \in \Omega$.

If, for instance, $a > 0$ on $\partial\Omega$ then it follows that $x_{u_0} \in \partial\Omega$ (and so (27) holds) if u_0 is large. Otherwise,

$$\sup u(\cdot, u_0) \leq \sup_{d(x) \leq \delta} \left(\frac{\lambda(x)}{a(x)} \right)^{1/(p-1)}$$

with $d(x) = \mathrm{dist}(x, \partial\Omega)$ and δ some sufficiently small positive number. That is not possible since $\sup u(\cdot, u_0) \to \infty$ as $u_0 \to \infty$.

b) It should be stressed that regarding the asymptotic rate (13), $a(x)$ is allowed to freely vary on $\partial\Omega$ provided it keeps positive there. On the other hand, the limit (23) not only provides asymptotic information on $u(\cdot, u_0)$ at x_0 but also in nearby points of the form $x = x_0 + \sigma^{-1}y$, y fixed, with $\sigma = u_0^{(p-1)/2}$ and $u_0 \to \infty$.

We are now studying the asymptotic behavior of the solution $u(\cdot, u_0)$ to (12) near a point $x_0 \in \partial\Omega$ where the coefficient a vanishes. To this proposal we are allowing $a(x)$ to decay as a power of the distance to the boundary, with a variable rate, possibly depending on the location of the reference point $x_0 \in \partial\Omega$. Since it will be assumed that Ω is $C^{2,\alpha}$ no generality is lost if it is assumed as before that $x_0 = 0$ while the outward unit normal to $\partial\Omega$ at $x = 0$ is $\nu = -e_N$. Let us introduce the decaying restriction on the coefficient a to be used in the forthcoming analysis. The notation of the proof of Theorem 5.1 describing $\partial\Omega$ near $x = 0$ as $x_N = f(x')$ is kept. Supposing $a(0) = 0$ it will be assumed the existence of functions a_0, a_1 and g defined on a neighborhood of $x_0 = 0$ and a positive constant γ such that,

$$a(x) = a_0(x') + a_1(x')y_N^{\gamma} + g(x', y_N)y_N^{\gamma}, \qquad y_N = x_N - f(x'), \tag{28}$$

where a_0, a_1 are nonnegative, $a_0(x') = o(|x'|^{\gamma})$ as $|x'| \to 0$, $a_1(0) \neq 0$ and $g(x', y_N) \to 0$ uniformly in x' ($|x'| \leq \delta$) as $y_N \to 0$. The meaning of the condition is clarified by observing that y_N has the status of the "vertical distance" to the boundary $\partial\Omega$, a_0 gives the restriction $a_{|\partial\Omega}$, i.

e. $a_0(x') = a(x', f(x'))$, while a_1 measures the rate of deviation of $a(x)$ regarding its value in the boundary as the distance y_N tends to 0.

Remark 5.2. Condition (28) is commonly used in the literature under the more restrictive form,

$$a(x) = Ad^\gamma + o(d^\gamma), \qquad d \to 0,$$

$d(x) = \text{dist}(x, \partial\Omega)$ with a coefficient A either constant or variable but positive (see [10], [14], [27], and references quoted there). This corresponds to set $a_0(x') \equiv 0$ in (28). On the other hand it is remarked that $a_1(x')$ is allowed to be variable in (28).

We can already state the following improved version of Theorem 5.1.

Theorem 5.2. *Let $a \in C^\alpha(\overline{\Omega})$ be nonnegative such that its null set $\{a(x) = 0\} = \overline{\Omega}_0$, with $\Omega_0 \subset \overline{\Omega}_0 \subset \Omega$ is a $C^{2,\alpha}$ subdomain of Ω fulfilling the conditions of Theorem 5.1.*

If $a(x_0) = 0$ at $x_0 \in \partial\Omega$ and $a(x)$ satisfies the decaying condition (28) at x_0 then the normal derivative of the solution $u(\cdot, u_0)$ to problem (12) exhibits the exact asymptotic behavior,

$$\frac{\partial u(\cdot, u_0)}{\partial \nu}_{|x=x_0} \sim \kappa a_1(x_0)^{\frac{1}{2+\gamma}} u_0^{\frac{p+1+\gamma}{2+\gamma}}, \qquad u_0 \to \infty.$$

where κ is a universal constant depending only on $p > 1$.

Proof. Recall we are taking $x_0 = 0$ with $\{x_N = 0\}$ the tangent hyperplane to $\partial\Omega$ at $x = 0$. The course of the proof of Theorem 5.1 can be followed, firstly rectifying $\partial\Omega$ near $x = 0$ by means of the change (15) (new coordinates y) and then performing the alternative blow-up scaling,

$$z = \sigma y, \qquad \sigma = u_0^{\frac{p-1}{2+\gamma}}, \qquad u(y) = u_0 v(z).$$

Problem (12) is thus transformed near zero into,

$$\begin{cases} -\Delta_z v + \sum_{i=1}^{N-1} \tilde{b}_i \partial^2_{z_i z_N} v + \sigma^{-1}\tilde{c}\partial_{z_N} v = \sigma^{-2}\tilde{\lambda} v - \tilde{a} v^p, \\ v(z', 0) = 1, \qquad |z'| < \sigma\varepsilon_0, \end{cases} \tag{29}$$

with $|z'| < \sigma\varepsilon_0$, $0 < z_N < \sigma\varepsilon$, where

$$\tilde{a}(z) = \sigma^\gamma a_0(\frac{z'}{\sigma}) + a_1(\frac{z'}{\sigma}) z_N^\gamma + g(\frac{z}{\sigma}) z_N^\gamma.$$

Since the family $v(z,\sigma)$ is uniformly bounded in σV_1 and $\sigma\overline{V}_1 \to \overline{\mathbb{R}}^N_+$ as $\sigma \to \infty$, by arguing as in the proof of Theorem 5.1 a subfamily $v(z,\sigma')$ is found such that

$$v(z,\sigma') \to v(z) \qquad \sigma' \to \infty, \tag{30}$$

in $C^{2,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}}^N_+)$. Moreover, $v(z)$ solves the problem,

$$\begin{cases} -\Delta v = -a_1 z_N^\gamma v^p & z \in \mathbb{R}^N_+ \\ \quad v(z',0) = 1, \end{cases} \tag{31}$$

with $a_1 = a_1(0)$. The same reasoning used in Lemma 5.1 ensures that (31) has a minimal solution $\underline{v}(z)$ and a maximal solution $\overline{v}(z)$, both positive and exclusively depending on z_N. In addition, the following lemma (whose proof is delayed to the end of the current one) holds.

Lemma 5.2. *The initial value problem,*

$$\begin{cases} v'' = t^\gamma |v|^p & t > 0 \\ v(0) = 1, \end{cases} \tag{32}$$

has a unique solution $v(t)$ with the property of being defined in the whole interval $[0,\infty)$. In addition, $v(t)$ is positive, convex, decreasing and,

$$v(t) = At^{-\theta} + O(t^{-(\mu+\theta)}) \qquad as \;\; t \to \infty,$$

with $\theta = \dfrac{2+\gamma}{p-1}$, $A = (\theta(\theta+1))^{\frac{1}{p-1}}$ and every μ satisfying

$$0 < \mu < \frac{1}{2}\{\sqrt{4p\theta(\theta+1)+1} - (2\theta+1)\}.$$

Thus the whole family $v(z,\sigma) \to v(z_N)$ in $C^{2,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}}^N_+)$ with $v(z_N) = v_1(a_1^{1/(2+\gamma)} z_N)$, $v_1(t)$ being the unique solution to (32) whose existence is ensured by Lemma 5.2. Set, in particular,

$$\kappa = -v_1'(0) > 0.$$

Then we finally get,

$$\frac{\partial u(\cdot,u_0)}{\partial \nu}_{|x=x_0} = -u_0^{\frac{p+1+\gamma}{2+\gamma}} \frac{\partial v(\cdot,\sigma)}{\partial z_N}_{|z=0} = \kappa a_1^{\frac{1}{2+\gamma}} u_0^{\frac{p+1+\gamma}{2+\gamma}} + o(u_0^{\frac{p+1+\gamma}{2+\gamma}}),$$

as $u_0 \to \infty$. Thus, the proof of Theorem 5.2 is concluded. □

Proof of Lemma 5.2. Consider the initial value problem,

$$\begin{cases} v'' = t^\gamma |v|^p & t > 0 \\ v(0) = 1,\ v'(0) = \sigma, \end{cases} \tag{33}$$

σ regarded as a parameter and set $v(t,\sigma)$ its unique non continuable solution. We claim that a solution to (33) defined for all $t > 0$ must have $v'(t) < 0$ in $0 \le t < \infty$. As a consequence of this fact and arguing by convexity one finds that $\lim v = \lim v' = 0$ as $t \to \infty$ and so $0 < v(t) < 1$ for $t > 0$.

To show the claim suppose $v'(t_0) \ge 0$ at some t_0. If $v(t_0) \ge 0$ (of course, the case $v = v' = 0$ at $t = t_0$ is discarded by uniqueness) then $v' > 0$ together with $v''v' \ge t_0^\gamma v^p v'$ for $t > t_0$. This implies,

$$v'^2 \ge \frac{2t_0^\gamma}{p+1}(v^{p+1} - v_0^{p+1}),$$

and v must blow-up at a finite $t_1 > t_0$. If $v(t_0) < 0$, v' is positive at the right of t_0 and by convexity v becomes positive with positive derivative at finite time. Thus, blow-up occurs again.

Next, let us show the uniqueness of a solution defined in $[0,\infty)$. It can be checked that $v(t,\sigma_1) < v(t,\sigma_2)$ if $\sigma_1 < \sigma_2$ in any interval $(0,b)$ where both solutions are positive. In particular, this must be true if $b = \infty$ and they are two possible different solutions in the conditions of the statement. By integrating in (33) using their behavior at $t = \infty$ one gets,

$$-\sigma_1 = \int_0^\infty t^\gamma v(t,\sigma_1)^p < \int_0^\infty t^\gamma v(t,\sigma_2)^p = -\sigma_2,$$

which contradicts the assumption on the σ_i's.

To construct the solution define $t_{\min} = t_{\min}(\sigma)$, $\sigma < 0$, as that $t > 0$ where $\inf v(\cdot,\sigma)$ is achieved. It can be proved that $t_{\min}$ increases when σ decreases provided that $v(\cdot,\sigma)$ keeps positive. Set

$$\sigma^* = \inf\{\sigma < 0 : \inf v(\cdot,\sigma) > 0\}.$$

Then $-\infty < \sigma^* < 0$ while the solution $v(t,\sigma^*)$ is defined in all $t \ge 0$ and provides the desired solution. In fact, $\inf v(\cdot,\sigma) > 0$ for $\sigma < 0$ small, by smooth perturbation, and so $\sigma^* < 0$. On the other hand, for $0 < t < t_{\min}(\sigma)$ and $\sigma > \sigma^*$ one gets,

$$0 < v(t,\sigma) < 1 + \sigma t + \frac{t^{\gamma+2}}{(\gamma+1)(\gamma+2)}.$$

Since such inequality can not be true if $\sigma \to -\infty$ with t fixed then it follows that $\sigma^* > -\infty$. Finally we achieve

$$t^* := \lim_{\sigma \to \sigma^*+} t_{\min}(\sigma) = \sup_{\sigma \to \sigma^*+} t_{\min}(\sigma) = \infty.$$

Otherwise, $v(t, \sigma^*) = \inf_{\sigma > \sigma^*} v(t, \sigma)$ would vanish together with its derivative at $t = t^*$ what is impossible.

To finish the proof we study the asymptotic behavior of $v(t, \sigma^*)$ ($v(t)$ for short) as $t \to \infty$. By performing the scaling (see [15], [4]),

$$v(t) = At^{-\theta} z(t),$$

with the exponent and coefficient θ, A introduced in the statement, the normalized solution $z(t)$ observed in $t > 0$ satisfies,

$$t^2 z'' - 2\theta t z' = \theta(\theta + 1)(z^p - z).$$

After Euler's change $\tau = \log t$, $z = z(\tau)$ solves the autonomous equation,

$$z'' - (2\theta + 1)z' + \theta(1 + \theta)(z - z^p) = 0, \qquad ' = \frac{d}{d\tau}, \tag{34}$$

in the whole $-\infty < \tau < \infty$. Let us study the phase space of (34) to show that,

$$\lim_{\tau \to \infty} z(\tau) = 1,$$

which yields the desired asymptotic behavior. To achieve this observe that $(z, z_1) = (0, 0)$ is an unstable node while $(z, z_1) = (1, 0)$ defines a saddle for the associated equation,

$$\begin{cases} z' = z_1 \\ z_1' = (2\theta + 1)z_1 - \theta(1 + \theta)(z - z^p). \end{cases} \tag{35}$$

Also, the function $E(z, z_1) = \dfrac{z_1^2}{2} + \theta(1+\theta)\left(\dfrac{z^2}{2} - \dfrac{z^{p+1}}{p+1}\right)$ strictly increases on solutions.

Now, orbits Γ entering the quadrant $z > 0, z_1 > 0$ from $(0, 0)$ at $\tau = -\infty$ can only exhibit the following behaviors (see Figure 2).

i) The orbit Γ keeps in the region $z_1 > \dfrac{\theta(1+\theta)}{(2\theta+1)}(z - z^p) =: h(z)$ for $0 < z < 1$. Since $E(z, z_1)$ increases on the orbit this implies that the parameterizing solution $z(\tau)$ is increasing and blows-up at finite time.

ii) The orbit Γ reaches $z_1 = h(z)$ at some $0 < z < 1$, enters the region $0 < z < 1$, $0 < z_1 < h(z)$ remaining there for all future times τ.

iii) Orbit Γ behaves as in ii) but enters the region $z_1 < 0$ at some $0 < z < 1$. In this case $z(\tau)$ vanishes at some finite τ.

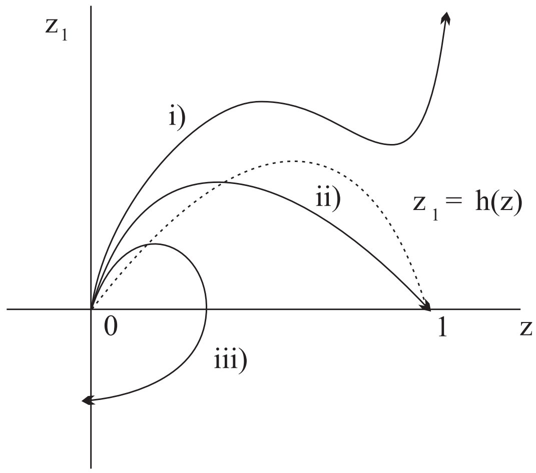

FIGURE 2. Phase space configuration for equation (35).

Observe now that the solution $z(\tau)$ obtained from our global solution $v(t) = v(t, \sigma^*)$ satisfies $z(\tau) = A^{-1}e^{\theta\tau}v(e^\tau) \sim A^{-1}e^{\theta\tau}$ as $\tau \to -\infty$, $z'(\tau) = A^{-1}e^{\theta\tau}(\theta v + e^\tau v'(e^\tau)) \sim \theta A^{-1}e^{\theta\tau}$ as $\tau \to -\infty$. Thus, its orbit enters $z > 0, z_1 > 0$ but does not vanish, nor blows-up at finite time. Therefore such orbit fulfills ii), which means that it necessarily defines a branch of the stable manifold corresponding to $(1, 0)$. This implies that,

$$w(\tau) = 1 + O(e^{-\mu\tau}) \qquad \text{as} \quad \tau \to \infty,$$

for every $\mu \in (0, -\mu_-)$ with μ_- the negative eigenvalue of the linearization of (34) at the saddle $(1, 0)$. This concludes the proof of the lemma. $\square$

Remark 5.3.

a) As an application of the results in this section we are performing an alternative construction of the finite supersolution u^* appearing in sections 3 and 4. Assume that $\Omega_0 \subset \overline{\Omega}_0 \subset \Omega$ consists of m connected pieces $\{\Omega_{0,i}\}_{i=1}^m$ fulfilling condition (10). For $\delta > 0$ small introduce $Q_i = (\Omega_{0,i})^\delta$ and $D =$

$\Omega \setminus \cup_{i=1}^m Q_i$, all of them being $C^{2,\alpha}$ domains. The problem

$$\begin{cases} -\Delta u = \lambda(x)u - a(x)u^p & x \in D \\ \quad u = u_0 & x \in \partial D, \end{cases} \tag{36}$$

has a unique positive solution $u_D(\cdot, u_0)$ while each problem,

$$\begin{cases} -\Delta v - \lambda(x)v = 0 & x \in Q_i \\ \quad v = 1 & x \in \partial Q_i, \end{cases} \tag{37}$$

has also a unique positive solution $v_i(x) \in C^{2,\alpha}(\overline{Q}_i)$, the existence being ensured by the conditions $\lambda_1^{Q_i}(-\Delta - \lambda) > 0$, $1 \leq i \leq m$.

Define,

$$u^*(x) = \begin{cases} u_D(x, u_0) & x \in \overline{D} \\ u_0 v_i(x) & x \in Q_i, \quad 1 \leq i \leq m. \end{cases} \tag{38}$$

Then u^* defines a supersolution to $-\Delta u = \lambda u - au^p$ both in D and in each Q_i. Thanks to Theorem 5.1, the normal derivative $\dfrac{\partial u_D(x, u_0)}{\partial \nu}$ at $x \in \partial Q_i$ grows faster than $-u_0 \dfrac{\partial v_i(x)}{\partial \nu}$ as $u_0 \to \infty$. According to [1] this implies that u^* defines a positive supersolution for u_0 large.

b) A positive supersolution u^* to $-\Delta u = \lambda u - au^p$ in Ω having $u_{|\partial\Omega} = \infty$ can be constructed exactly in the same way by replacing $u_D(x, u_0)$ in (38) by the solution $\tilde{u}_D(x, u_0)$ to,

$$\begin{cases} -\Delta u = \lambda(x)u - a(x)u^p & x \in D \\ \quad u = u_0 & x \in \partial D \setminus \partial\Omega \\ \quad u = \infty & x \in \partial\Omega. \end{cases} \tag{39}$$

With an appropriate handling, Theorems 5.1 and 5.2 can be adapted to this new scenario to show that $\dfrac{\partial \tilde{u}_D}{\partial \nu}$ satisfies the same growth estimates regarding u_0, as $u_0 \to \infty$. The use of this supersolution permits to obtain the existence assertion in Theorem 4.1 under an alternative approach.

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