EXISTENCE AND NON-EXISTENCE OF SOLUTIONS TO ELLIPTIC EQUATIONS WITH A GENERAL CONVECTION TERM

SALOMÓN ALARCÓN, JORGE GARCÍA-MELIÁN AND ALEXANDER QUAAS

Abstract. In this paper we consider the nonlinear elliptic problem

$$\begin{cases} -\Delta u + \alpha u = g(|\nabla u|) + \lambda h(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $\alpha \geq 0$, g is an arbitrary C^1 increasing function and $h \in C^1(\overline{\Omega})$ is nonnegative. We completely analyze the existence and nonexistence of (positive) classical solutions in terms of the parameter λ. We show that there exist solutions for every λ when $\alpha = 0$ and the integral $\int_1^\infty \frac{1}{g(s)} ds = \infty$, or $\alpha > 0$ and the integral $\int_1^\infty \frac{s}{g(s)} ds = \infty$. Conversely, when the respectively integrals converge and h is nontrivial on $\partial\Omega$, existence depends on the size of λ. Moreover, nonexistence holds for large λ. Our proofs mainly rely on comparison arguments, and on the construction of suitable supersolutions in annuli. Our results include some cases where the function g is superquadratic and still existence holds without assuming any smallness condition on λ.

1. Introduction

The concern of the present paper is the existence and non-existence of solutions to the following nonlinear elliptic problem:

$$\begin{cases} -\Delta u + \alpha u = g(|\nabla u|) + \lambda h(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of class $C^{2,\eta}$ of $\mathbb{R}^N$ for some $\eta \in (0,1)$, $\alpha \geq 0$ and $g \in C^1(\mathbb{R})$ is increasing with $g(0) = 0$. The function $h \in C^1(\overline{\Omega})$, will be nonnegative, while λ will be regarded as a positive parameter.

We will focus our attention on general functions g, obtaining sharp conditions which imply: either (a) problem (1.1) has a unique solution for every λ or (b) there exists a critical size of λ that divides existence from non-existence for (1.1) when $h \not\equiv 0$ on $\partial\Omega$.

This type of problem has been extensively studied. Here we give a quick review on the topic, other references can be found in the papers quoted below. The pioneering works on the subject seem to be due to Serrin [29], Amann and Crandall [6] and Lions [24]. The case $\alpha > 0$ is considered in [12] and [13], where existence holds when g has at most quadratic growth, see also [14]. The case $\alpha = 0$ and $g(t) = t^2$ was studied for example in [17] and [18] (see also [2] and [20]). For related results see [1] and [11].

More recently also related fully nonlinear equations are considered in [30] (see also the case $\alpha < 0$ in [22], where multiplicity results are obtained).

We finally mention that a starting point of our work can be found in [3]; actually we solve a problem given in that paper, see Remark on page 29 there. More precise information on our contribution with respect to the known results are given in the remarks after our main theorems.

By a solution to (1.1) we mean a function $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ verifying the equation in the classical sense. Remark that, on one hand, standard bootstrapping gives $u \in C^{2,\eta}(\overline{\Omega})$, while on the other hand solutions are strictly positive in Ω by the maximum principle, since $-\Delta u + \alpha u \geq 0$ in Ω. An important remark with regard to problem (1.1) is that uniqueness of solutions holds by the comparison principle (cf. for instance Theorem 10.1 in [19] or the results in [27] and [16]). Thus we only need to show existence and nonexistence of solutions. For other uniqueness results see for example [8], [9] and [10]. Observe also that non-uniqueness holds with less regularity on the solution, see for instance [2].

Let us state now our main results. We begin with the case $\alpha = 0$. It turns out that the existence of solutions depends on the condition

$$\int_1^\infty \frac{ds}{g(s)} = \infty. \tag{1.2}$$

More precisely:

Theorem 1. *Assume $g \in C^1(\mathbb{R})$ is increasing with $g(0) = 0$, while $h \in C^1(\overline{\Omega})$ is such that $h \geq 0$ in Ω. If $\alpha = 0$ then:*

- (i) *If* (1.2) *holds, there exists a unique solution to* (1.1) *for every $\lambda > 0$.*
- (ii) *If* (1.2) *does not hold and $h \not\equiv 0$ on $\partial\Omega$, then there exists $\Lambda > 0$ such that for $\lambda \in (0, \Lambda)$ problem* (1.1) *has a unique solution, while there are no solutions when $\lambda > \Lambda$.*

Remarks 1. (a) The non-existence part in (ii) is already proved in the case where g is convex (see Theorem 2.1 in [3]). In the particular case $g(t) = t^p$ with $p > 2$, see [21] for existence when h is a measure and λ is small.

(b) Part (ii) of Theorem 1 extends the case $g(t) = t^2$ of the above quoted papers.

We now turn to the case $\alpha > 0$. In this case, the condition for the existence of solutions is

$$\int_1^\infty \frac{s}{g(s)} ds = \infty. \tag{1.3}$$

Observe that condition (1.3) is implied by (1.2). We have:

Theorem 2. *Assume $g \in C^1(\mathbb{R})$ is increasing with $g(0) = 0$, while $h \in C^1(\overline{\Omega})$ is such that $h \geq 0$ in Ω. If $\alpha > 0$, then:*

- (i) *If* (1.3) *holds, there exists a unique solution to* (1.1) *for every $\lambda > 0$.*
- (ii) *If* (1.3) *does not hold and $h \not\equiv 0$ on $\partial\Omega$, then there exists $\Lambda > 0$ such that for $\lambda \in (0, \Lambda)$ problem* (1.1) *admits a unique solution, while no solutions exist when $\lambda > \Lambda$.*

Remarks 2. (a) The non-existence part in (ii) is already proved in the particular case $g(t) = t^p$ with $p > 2$, see Proposition 2.3 in [3].

(b) Part (i) in Theorem 2 applies for instance to $g(t) = t^2 \ln(1 + t)$, which is superquadratic, and so existence holds without the smallness restriction on the right hand side. This means that in the setting of classical solutions, and with smooth data, most of the previous existence theorems are not optimal with respect to the growth condition in g, since at most quadratic behavior is required in the case $\alpha > 0$.

(c) Part (i) in our two theorems answers an open question stated in [3] (see Remark in page 29 of that paper). We are indeed a little bit more precise here, since our optimal conditions are different for the cases $\alpha = 0$ and $\alpha > 0$. When $g(t) = t$ part (i) holds in both theorems, this particular case already being covered in [3] (see Theorem 3.1 there).

(d) The case $g(t) = O(t)$ has been usually the reference case for a general solvability result (Part (i) in Theorem 1 or Theorem 2); see for example [15], and [5] (note that the symmetrization approach reduces the problem to a radial one, which is related to our approach). The superlinear model case $g(t) = |t|^q$, $q > 1$ is deeply studied in [21], in particular as far as necessary conditions for the existence are concerned. In [26], the case $g(t) = t^q$ with the limit $\alpha \to 0$ is described and the maximal constant Λ is characterized in terms of stochastic state constraint ergodic problems. For the superquadratic case, see also [7], where the existence of a generalized viscosity solution is proved when $\alpha > 0$, though this solution is not classical and in particular may not attain the boundary datum.

Let us mention in passing that the positivity condition on h is only imposed in order to simplify the presentation. In particular, it is relevant for the nonexistence results only. For a function h which takes both signs we still may assert the existence of solutions for every $\lambda > 0$ in case (i) in both theorems and for small λ in case (ii), although in this last situation the results are not expected to be optimal. Notice also that when h is negative, the change of u by $-u$ in (1.1) amounts to replacing g by $-g$. In most of the previous works, no distinction is made between the two cases, but the results are far from optimal. Here we have decided to restrict our attention to nonnegative h (hence positive g) for definiteness. Also, the restriction that h is nontrivial on $\partial\Omega$ is probably not necessary for the nonexistence of solutions. The same result is expected to hold for functions h which are nontrivial on $\overline{\Omega}$, although the method of proof has to be modified. We are not pursuing this matter further in the present paper.

On the other hand, we believe that the proofs can be adapted to deal with some more general operators than the Laplacian, for instance the p-Laplacian or even some fully nonlinear operators which depend on the second derivatives of the solution.

The basic idea to prove existence of solutions to (1.1) comes from [24] (see also [25] and [23]). It consists in truncating the term $g(|\nabla u|)$ in order to obtain a problem in a classical setting, i. e. with subquadratic growth in the gradient. Then the standard method of sub and supersolutions can be used to get a solution to the truncated problem, and the final step is to show that the solution to the truncated problem is indeed a solution to the original one. This can be achieved by obtaining appropriate estimates for the gradient ∇u of the solution u in $\overline{\Omega}$. By an adaptation of the classical method of Bernstein (see [29] or [24]), these estimates are a consequence of a kind of maximum principle for $|\nabla u|^2 + u^2$, so that everything is reduced to estimating $|\nabla u|$ on $\partial\Omega$. This in turn can be done by comparing with a suitable supersolution.

It is important to notice that our approach does not rely in obtaining a supersolution $\bar{u}$ to (1.1) which vanishes on the whole $\partial\Omega$, something which is required to apply Theorem III.1 in [24]. Rather, we construct the supersolution by analyzing problem (1.1) in an annulus which after a suitable translation is tangent to $\partial\Omega$ at every fixed $x_0 \in \partial\Omega$. This enables us to deal with a radial problem which is in some sense integrable, so we are able to find conditions which are both necessary and sufficient for existence.

The rest of the paper is organized as follows: in Section 2 we construct supersolutions to (1.1) in the particular case where Ω is an annulus. Section 3 is dedicated to show nonexistence of solutions to (1.1) when Ω is a ball. Finally, in Section 4 we deal with the proofs of Theorems 1 and 2.

2. Supersolutions for problems in annuli

It will be proved in Section 4 that the existence of a radial supersolution to problem (1.1) posed in an annulus when h is constant suffices to ensure the existence of a solution to (1.1). Thus this section will be dedicated to construct a positive radial function u verifying

$$\begin{cases} -(r^{N-1}u')' \geq r^{N-1}(-\alpha u + g(|u'|) + c), & R_1 < r < R_2 \\ u(R_1) = 0, \ u(R_2) \geq 0, \end{cases} \tag{2.1}$$

for suitable values of c, depending on whether $\alpha = 0$ or $\alpha > 0$ and also on the integrability conditions on g at infinity considered in the Introduction. In what follows, $R_2 > R_1 > 0$ will be fixed.

Lemma 3. *Assume $g \in C^1(\mathbb{R})$ is increasing with $g(0) = 0$ and $\alpha = 0$. Then, if*

$$\int_1^\infty \frac{ds}{g(s)} = \infty, \tag{2.2}$$

for every $c > 0$ there exists a positive radial function u verifying (2.1)*. If* (2.2) *does not hold, the existence of such a function also follows provided c is small enough.*

Proof. Introducing the change of variables

$$s = \begin{cases} \log r & N = 2 \\ -\dfrac{1}{N-2}\dfrac{1}{r^{N-2}}, & N \geq 3, \end{cases} \tag{2.3}$$

and denoting $u(r) = v(s)$, (2.1) is transformed into

$$\begin{cases} -v'' \geq r^{2(N-1)} \left(g\left((\frac{1}{r^{N-1}} |v'|) \right) + c \right) \\ v(a) = 0, \ v(b) \geq 0, \end{cases}$$

where $a = \log R_1$, $b = \log R_2$ when $N = 2$, while $a = -\frac{1}{N-2}\frac{1}{R_1^{N-2}}$, $b = -\frac{1}{N-2}\frac{1}{R_2^{N-2}}$ if $N \geq 3$. Since g is increasing and positive, it suffices to have

$$\begin{cases} -v'' \geq R_2^{2(N-1)}\left(g\left(\frac{1}{R_1^{N-1}}|v'|\right) + c\right) \\ v(a) = 0, \ v(b) \geq 0. \end{cases}$$

Setting $w(s) = v(s + a)$, this suggests to consider the one-dimensional autonomous initial value problem

$$\begin{cases} -w'' = R_2^{2(N-1)}\left(g\left(\frac{1}{R_1^{N-1}}|w'|\right) + c\right) \\ w(0) = 0, \ w'(0) = \gamma > 0, \end{cases} \tag{2.4}$$

which has a unique solution for every $\gamma > 0$, and find a positive solution in $(0, b - a)$. Observe that solutions to (2.4) verify on one hand $w'' \leq -cR_2^{2(N-1)}$, so that an integration provides $w(s) \leq s(\gamma - cR_2^{2(N-1)}s/2)$. On the other hand, since w' is decreasing we have $w'(s_0) = 0$ for some $s_0 > 0$, and it follows by the symmetry of the problem that w is symmetric with respect to s_0 and $w(2s_0) = 0$. Letting $2s_0$ be the first zero of w and integrating the equation in $(0, s_0)$ we obtain

$$s_0 = \left(\frac{R_1}{R_2^2}\right)^{N-1} \int_0^{\frac{\gamma}{R_1^{N-1}}} \frac{dt}{g(t) + c}.$$

We conclude that s_0 is an increasing function of γ which verifies

$$s_0 \to \left(\frac{R_1}{R_2^2}\right)^{N-1} \int_0^{\infty} \frac{dt}{g(t) + c} \tag{2.5}$$

as $\gamma \to +\infty$. Therefore in the case where (2.2) holds, since g is increasing, then the integral in (2.5) diverges. So, we can always choose γ large enough so that $s_0 > (b - a)/2$, and this provides with a positive solution of (2.1). When the integral converges, we can also obtain $s_0 > (b - a)/2$ if we select c small enough, since the integral

$$\int_0^{\infty} \frac{dt}{g(t)} \tag{2.6}$$

diverges at 0, due to $g(0) = 0$ and $g \in C^1(\mathbb{R})$. This concludes the proof. □

Lemma 4. *Assume $g \in C^1(\mathbb{R})$ is increasing with $g(0) = 0$ and $\alpha > 0$. Then, if*

$$\int_1^{\infty} \frac{s}{g(s)} ds = \infty, \tag{2.7}$$

for every $c > 0$ there exists a positive radial function u verifying (2.1)*. If* (2.7) *does not hold and c is small enough, such a function also exists.*

Proof. Setting $z = c/\alpha - u$, we look for a function verifying

$$\begin{cases} (r^{N-1}z')' \geq r^{N-1}(\alpha z + g(|z'|)) \\ z(R_1) = \dfrac{c}{\alpha}, \ z(R_2) \leq \dfrac{c}{\alpha}. \end{cases}$$

We will look for a positive solution z to this inequality. With the change of variables (2.3), and letting $v(s) = z(r)$, we find as before that v is a supersolution provided for instance that

$$\begin{cases} v'' \geq R_2^{2(N-1)}(\alpha v + g(\frac{1}{R_1^{N-1}}|v'|)) \\ v(a) = \dfrac{c}{\alpha},\ v(b) = 0. \end{cases}$$

Setting $w(s) = v(b-s)$, it is thus natural to consider the initial value problem

$$\begin{cases} w'' = R_2^{2(N-1)}\left(\alpha w + g\left(\frac{1}{R_1^{N-1}}|w'|\right)\right) \\ w(0) = 0,\ w'(0) = \gamma > 0, \end{cases} \tag{2.8}$$

which has a unique solution for every $\gamma > 0$, and see if we can select γ so that $w(b-a)$ is as large as we please.

Notice that $w'' \geq 0$ as long as $w \geq 0$, so that it is not hard to see that solutions are positive, increasing and convex for $s > 0$. For every $\gamma > 0$, the solution is defined in an interval $[0, T(\gamma))$, and when $T(\gamma) < \infty$ we have

$$\lim_{s \to T(\gamma)} w(s) = +\infty \qquad \text{or} \quad \lim_{s \to T(\gamma)} w'(s) = +\infty. \tag{2.9}$$

Let us see that when the integral condition (2.7) is satisfied, we always have both conditions in (2.9). Indeed, the first one implies the second, and if we had $w(T(\gamma)) < +\infty$, then

$$w'' \leq R_2^{2(N-1)}\left(\alpha w(T(\gamma)) + g\left(\frac{1}{R_1^{N-1}}w'\right)\right).$$

Multiplying by w' and integrating we arrive at

$$\int_0^T \frac{w'w''}{\alpha w(T(\gamma)) + g(\frac{1}{R_1^{N-1}}w')} \leq R_2^{2(N-1)}w(T(\gamma)),$$

which yields

$$\int_{\frac{\gamma}{R_1^{N-1}}}^{\infty} \frac{s}{\alpha w(T(\gamma)) + g(s)}ds \leq \frac{R_2^{2(N-1)}}{R_1^{N-1}}w(T(\gamma)),$$

contradicting condition (2.7). Thus $w, w' \to \infty$ as $s \to T(\gamma)$.

We have two cases to consider: either $T(\gamma_0)$ is infinite for some $\gamma_0 > 0$ or $T(\gamma)$ is finite for every $\gamma > 0$. In the first case, let us see that this implies $T(\gamma) = \infty$ for every $\gamma > 0$. Observe first that when u, v are two solutions to the equation in (2.8) with $u(0) \geq v(0)$, $u'(0) = \gamma_1 > \gamma_2 = v'(0)$, then $u > v$ in the common interval of definition, hence $T(\gamma_1) \leq T(\gamma_2)$.

In particular, $T(\gamma) = \infty$ for $\gamma < \gamma_0$. If $\gamma > \gamma_0$ and we temporarily denote by w_γ the unique solution to (2.8), there exists $\delta > 0$ such that $w'_{\gamma_0}(\delta) > \gamma$, since w'_{γ_0} is increasing and converges to infinity. Let

$$\bar{w}(x) = w_{\gamma_0}(x + \delta).$$

Then $\bar{w}$ is a solution to the same equation with initial data $\bar{w}(0) = w_{\gamma_0}(\delta) > 0$, $\bar{w}'(0) = w'_{\gamma_0}(\delta) > \gamma$. It follows by the previous observation that $\bar{w} > w_\gamma$, and in particular $T(\gamma) = \infty$. Thus all solutions are global in this case and it is easy to conclude: since $w(x) \geq \gamma x$ by convexity, we can have $w(b-a)$

as large as we please, so that a supersolution can be constructed with large values of c.

The second possibility is that all solutions blow-up in finite time, i. e. $T(\gamma) < \infty$ for every $\gamma > 0$. Let us see that in such case $T(\gamma)$ is a continuous function of γ. Take $\gamma_n \downarrow \gamma$. By comparison we have $T(\gamma_n) < T(\gamma)$. Moreover, we can choose $\delta_n \downarrow 0$ such that $w'_\gamma(\delta_n) > \gamma_n$. Arguing as before, $w_\gamma(x+\delta_n) > w_{\gamma_n}(x)$, so that $T(\gamma)-\delta_n < T(\gamma_n)$ and we obtain $T(\gamma_n) \to T(\gamma)$. When $\gamma_n \uparrow \gamma$ the proof is similar.

Next, we claim that $T(\gamma) \to \infty$ as $\gamma \to 0$. Indeed, assume $T(\gamma) \le T_0$ when $\gamma \to 0$. Since $w'' \ge 0$, we obtain $w \le T_0 w'$, so that

$$w'' \le R_2^{2(N-1)} \left(g\left(\frac{1}{R_1^{N-1}} w' \right) + \alpha T_0 w' \right),$$

and this leads, after an integration and a change of variables, to

$$\int_{\frac{\gamma}{R_1^{N-1}}}^{\infty} \frac{ds}{g(s) + \alpha T_0 R_1^{N-1} s} \le \left(\frac{R_2^2}{R_1} \right)^{N-1} T_0.$$

A contradiction is reached when we let $\gamma \to 0$, since the integral then diverges. Thus $\lim_{\gamma \downarrow 0} T(\gamma) = \infty$.

Let us denote $\overline{T} = \lim_{\gamma \to \infty} T(\gamma)$ (which is expected to be zero). If $\overline{T} \le b-a$ we can use the continuity of T to obtain $\gamma > 0$ such that $T(\gamma) = b-a+\varepsilon$ for small positive ε. Taking ε as small as we please we obtain $w_\gamma(b-a)$ as large as we wish, and this provides with a supersolution for large values of c. If, on the contrary, $\overline{T} > b-a$, then all solutions would be defined at least in $[0, b-a]$ and since $w(b-a) \ge \gamma(b-a)$, we obtain that $w_\gamma(b-a)$ is as large as we please by taking large values of γ.

To conclude the proof, we now consider the case when

$$\int_1^\infty \frac{s}{g(s)} ds < \infty$$

and c is small enough. Observe that in this case all solutions blow up in finite time. Indeed, let $T < T(\gamma)$. Since

$$w'' \ge R_2^{2(N-1)} g\left(\frac{1}{R_1^{N-1}} w' \right)$$

we can integrate in $(0, T)$ and let $T \to T(\gamma)$ to arrive at:

$$R_2^{2(N-1)} T(\gamma) \le R_1^{N-1} \int_\gamma^\infty \frac{ds}{g(s)} < \infty, \tag{2.10}$$

since this last integral also converges.

It also follows from (2.10) that $T(\gamma) \to 0$ as $\gamma \to \infty$ (i. e. $\overline{T} = 0$ in the above proof). Since $T(\gamma)$ is continuous with $T(\gamma) \to \infty$ as $\gamma \to 0$, we can choose γ such that $T(\gamma) > b-a$ and obtain a supersolution for $c \le \alpha w_\gamma(T(\gamma))$. It is worth mentioning that in the present case where (2.7) does not hold we cannot guarantee that the first equality in (2.9) holds, so that the supersolution is not valid in principle for large values of c. □

3. Nonexistence of solutions in balls

We tackle in this section the question of nonexistence of solutions to (1.1). We will see in Section 4 that it suffices to show nonexistence of radial solutions when Ω is a ball of $\mathbb{R}^N$ and h is constant. Thus, under several hypotheses, we will show that the problem

$$\begin{cases} -u'' - \dfrac{N-1}{r}u' = -\alpha u + g(|u'|) + c, & 0 < r < R \\ u'(0) = 0, \ u(R) = 0 \end{cases} \tag{3.1}$$

does not admit positive solutions for large values of c.

Lemma 5. *Assume $g \in C^1(\mathbb{R})$ is increasing with $g(0) = 0$ and $\alpha = 0$. Then if*

$$\int_1^\infty \frac{ds}{g(s)} < \infty,$$

there exists $c_0 > 0$ such that problem (3.1) does not admit positive solutions when $c \geq c_0$.

Proof. Assume u is a solution to (3.1). We first claim that $u'(r) < 0$ for $r \in (0, R)$ and $u''(r) < 0$ in $[0, R)$. Observe that $u''(0) = -c/N < 0$, so that $u'(r) < 0$ for $r > 0$ close enough to zero. If we had $u'(r_0) = 0$ for some $r_0 \in (0, R)$ with $u' < 0$ in $(0, r_0)$, then $u''(r_0) \geq 0$ so that from the equation we obtain $u''(r_0) = -c < 0$, which is impossible. Then $u'(r) < 0$ if $0 < r < R$.

Assume now that for some $\tilde{r}_0 \in (0, R)$ we have $u''(\tilde{r}_0) = 0$. Since $u'''(\tilde{r}_0) \geq 0$ in this case, we obtain by differentiating the equation

$$-u''' - \frac{N-1}{r}u'' + \frac{N-1}{r^2}u' = -g'(-u')u''$$

so that $u'''(\tilde{r}_0) < 0$, a contradiction. Thus $u''(r) < 0$ for $r \in (0, R)$ as well.

Next if we rewrite the equation as $-(r^{N-1}u')' = r^{N-1}(g(-u') + c)$ and integrate in $(0, r)$ we obtain, taking into account that $g(-u')$ is increasing:

$$\begin{aligned} -r^{N-1}u'(r) &= \int_0^r s^{N-1}(g(-u'(s)) + c)ds \\ &\leq (g(-u'(r)) + c) \int_0^r s^{N-1}ds = \frac{r^N}{N}(g(-u'(r)) + c), \end{aligned}$$

so that plugging this in (3.1) we have

$$-u'' \geq \frac{1}{N}(g(-u') + c) \qquad \text{in } (0, R).$$

Integrating in $(0, R)$ we obtain

$$\int_0^\infty \frac{dt}{g(t) + c} > \int_0^{-u'(R)} \frac{dt}{g(t) + c} \geq \frac{1}{N}R.$$

This implies that c cannot be too large in order to have a positive solution to (3.1). $\square$

Lemma 6. *Assume $g \in C^1(\mathbb{R})$ is increasing with $g(0) = 0$ and $\alpha > 0$. Then if*

$$\int_1^\infty \frac{s}{g(s)} ds < \infty, \tag{3.2}$$

there exists $c_0 > 0$ such that problem (3.1) does not admit positive solutions when $c \geq c_0$.

Proof. Let u be a positive solution to (3.1). We first claim that $u < c/\alpha$. Indeed, if we had $u(0) = c/\alpha$ then $u \equiv c/\alpha$ by uniqueness, which is not possible. If $u(0) > c/\alpha$ then $u''(0) > 0$ and u initially increases. According to the boundary condition $u(R) = 0$, there should be a point where u achieves its maximum, but this is in contradiction with the equation. We conclude that $u(0) < c/\alpha$ and again by the equation u initially decreases and cannot reach a minimum, so u is always decreasing. It is seen much as in the previous case that $u'' < 0$ in $[0, R)$ also. Thus arguing as in that proof we obtain

$$-u'' \geq \frac{1}{N}(-\alpha u + g(-u') + c). \tag{3.3}$$

Assume there exists a sequence $c_n \to \infty$ such that a positive solution u_n to (3.1) exists with $c = c_n$ (with no loss of generality we may assume that c_n is increasing). Let $v_n = \dfrac{c_n}{\alpha} - u_n$. Then

$$\begin{cases} v_n'' + \dfrac{N-1}{r} v_n' = \alpha v_n + g(v_n') \\ v_n'(0) = 0, \ v_n(R) = \dfrac{c_n}{\alpha}, \end{cases}$$

with $v_n' > 0$, $v_n'' > 0$. We claim that $v_n(0)$ is bounded as $n \to \infty$. Indeed, since from (3.3) we have

$$v_n'' \geq \frac{1}{N}(\alpha v_n + g(v_n')) \geq \frac{1}{N}(\alpha v_n(0) + g(v_n')),$$

we can integrate to arrive at

$$\frac{1}{N} R \leq \int_0^{v_n'(R)} \frac{ds}{\alpha v_n(0) + g(s)} < \int_0^\infty \frac{ds}{\alpha v_n(0) + g(s)}.$$

Therefore if $v_n(0) \to \infty$ we arrive at a contradiction. Since solutions are increasing in c (thanks to uniqueness), we can guarantee that $v_n(0) \to \bar{v}$ for some $\bar{v} > 0$. It also follows that $v_n \to z$, the unique solution to

$$\begin{cases} z'' + \dfrac{N-1}{r} z' = \alpha z + g(z') \\ z(0) = \bar{v}, \ z'(0) = 0, \end{cases}$$

which is defined in a maximal interval $[0, T)$. When $T < \infty$, we have $\lim_{r \to T} z(r) = \infty$ or $\lim_{r \to T} z'(r) = \infty$. By comparison we also have $v_n \leq z$ in $[0, \min\{T, R\})$.

Let us see that $T < R$. Indeed, if $T \geq R$, we would have $v_n(R) = \frac{c_n}{\alpha} \leq z(R)$, and then $T = R$, $z(R) = \infty$ follows. This is impossible, since

$z'' \geq \frac{1}{N} g(z')$, and multiplication by z' and another integration between $\frac{R}{2}$ and $R-\varepsilon$ for some small positive ε yields

$$\frac{1}{N}(z(R-\varepsilon) - z(\frac{R}{2})) \leq \int_{z'(R/2)}^{z'(R-\varepsilon)} \frac{s}{g(s)} ds < \int_{z'(R/2)}^{\infty} \frac{s}{g(s)} ds.$$

Letting $\varepsilon \to 0$ we obtain a contradiction with (3.2). Thus $T < R$.

Now choose a small $\varepsilon > 0$. Since $v_n' \to z'$ uniformly in $[0, T-\varepsilon]$, we have $v_n'(T-\varepsilon) \geq z'(T-\varepsilon) - \varepsilon$ if n is large enough. Therefore

$$\frac{1}{N}(R - T + \varepsilon) \leq \int_{v_n'(T-\varepsilon)}^{v'(R)} \frac{ds}{g(s)} < \int_{z'(T-\varepsilon)-\varepsilon}^{\infty} \frac{ds}{g(s)}.$$

Letting $\varepsilon \to 0$, we arrive at $T \geq R$, a contradiction which shows that no solutions to (3.1) may exist if c is large enough. □

4. Proof of the theorems

This final section will be dedicated to the proof of Theorems 1 and 2. The idea of the proofs of existence comes from [24], and it consists in truncating the function g in order to obtain a solution, and then estimating the gradient of this solution in $\overline{\Omega}$. The essential point is to obtain a suitable supersolution.

Since Ω verifies the uniform exterior sphere condition, there exists $R_1 > 0$ such that for every $x_0 \in \partial\Omega$ there exists $y_0 \in \mathbb{R}^N \setminus \Omega$ with $\overline{B_{R_1}(y_0)} \cap \overline{\Omega} = \{x_0\}$. Choose $R_2 > R_1$ large enough so that $\Omega \subset A := B_{R_2}(y_0) \setminus \overline{B_{R_1}(y_0)}$ for every $x_0 \in \partial\Omega$. Consider the radial problem

$$\text{(4.1)} \qquad \begin{cases} -(r^{N-1}u')' = r^{N-1}(-\alpha u + g(|u'|) + c), & R_1 < r < R_2 \\ u(R_1) = 0, \ u(R_2) \geq 0, \end{cases}$$

where $c > 0$. The first important existence result is the following:

Lemma 7. *Assume $g \in C^1(\mathbb{R})$ is increasing with $g(0) = 0$ and $h \in C^1(\overline{\Omega})$ is nonnegative. If there exists a positive supersolution $\bar{u}$ to problem* (4.1)*, then for every $\lambda \in (0, c/|h|_\infty]$, problem* (1.1) *admits a positive solution.*

With regard to nonexistence results, the reference situation is a radial problem in a ball. Observe that, since Ω verifies a uniform interior ball condition and $h \geq 0$, $h \not\equiv 0$ on $\partial\Omega$, we can find $x_0 \in \partial\Omega$, $y_0 \in \Omega$ and $R > 0$ such that $B_R(y_0) \subset \Omega$, $\overline{B_R(y_0)} \cap \partial\Omega = \{x_0\}$ and $h \geq h_0 > 0$ in $\overline{B_R(y_0)}$. Consider the problem

$$\text{(4.2)} \qquad \begin{cases} -(r^{N-1}u')' = r^{N-1}(-\alpha u + g(|u'|) + c), & 0 < r < R \\ u'(0) = 0, \ u(R) = 0, \end{cases}$$

where $c > 0$. Then:

Lemma 8. *Assume $g \in C^1(\mathbb{R})$ is increasing with $g(0) = 0$ and $h \in C^1(\overline{\Omega})$ is nonnegative with $h \geq h_0 > 0$ in $B_R(y_0)$. If problem* (4.2) *does not admit a positive solution for some $c > 0$, then problem* (1.1) *does not have solutions for $\lambda \geq c/h_0$.*

As we have quoted in the introduction, the method we follow for proving existence (and indeed also nonexistence) relies in obtaining good estimates for the gradient of the solutions. This last part is achieved by means of a kind of maximum principle for the gradient of solutions to (1.1). The proof is inspired in the classical method of Bernstein (see for instance [29] or [24]).

Lemma 9. *Let* $u \in C^2(\Omega)\cap C^1(\overline{\Omega})$ *be a solution to* (1.1)*. Assume* $g \in C^1(\mathbb{R})$ *is increasing with* $g(0) = 0$ *and* $h \in C^1(\overline{\Omega})$*. Then there exists a constant* C *which depends on* $\sup_{\overline{\Omega}} |u|$, $\sup_{\partial\Omega} |\nabla u|$, $\sup_{\Omega} |\nabla h|$ *and* λ *such that*

$$|\nabla u| \le C \quad \text{in } \overline{\Omega}.$$

Proof. Let u be a solution to (1.1), and define $w = |\nabla u|^2+u^2$. For simplicity, let us denote $g(|\xi|) = \tilde{g}(|\xi|^2)$. By standard regularity, it follows that $u \in C^3(\Omega_\rho)$, where $\Omega_\rho = \{x \in \Omega : |\nabla u|^2 > \rho\}$ for some $0 < \rho < \|u\|_\infty^2$, and hence $w \in C^2(\Omega_\rho) \cap C(\overline{\Omega}_\rho)$. Then, it is not hard to check that in Ω_ρ one has

$$\Delta w = \frac{2}{N}|D^2u|^2-2\tilde{g}'(|\nabla u|^2)\nabla u\nabla(w-u^2)-2\lambda\nabla h\nabla u+2u\Delta u+(2+2\alpha)|\nabla u|^2.$$

On the other hand,

$$(\Delta u)^2 = \left(\sum_{i=1}^N \partial_{ii}u\right)^2 \le N\sum_{i=1}^N(\partial_{ii}u)^2 \le N|D^2u|^2,$$

and since $\tilde{g}$ is nondecreasing and $u \ge 0$, so that $2\tilde{g}'(|\nabla u|^2)\nabla u\nabla(u^2) \ge 0$, we have

$$\Delta w \ge \frac{2}{N}(\Delta u)^2 - 2\tilde{g}'(|\nabla u|^2)\nabla u\nabla w - 2\lambda\nabla h\nabla u + 2u\Delta u + (2 + 2\alpha)|\nabla u|^2$$

in Ω_ρ. An application of Cauchy-Schwarz's inequality leads to

$$\Delta w \ge \frac{1}{N}(\Delta u)^2 - 2\tilde{g}'(|\nabla u|^2)\nabla u\nabla w - \lambda^2|\nabla h|^2 - Nu^2 + |\nabla u|^2$$

in Ω_ρ. Fix

$$M > \sup_{\partial\Omega}|\nabla u|^2 + 2N\|u\|_\infty^2 + \lambda^2\|\nabla h\|_\infty,$$

and assume that the open set $\Omega_0 = \{x \in \Omega : w > M\}$ is nonempty. It clearly follows that $\Omega_0 \subset\subset \Omega_\rho$ since

$$|\nabla u|^2 \ge \|u\|_\infty^2 > \rho \quad \text{in } \Omega_0.$$

Hence $\mathcal{L}w \le 0$ in Ω_0, where $\mathcal{L}w := -\Delta w-2g'(|\nabla u|)\nabla u\cdot\nabla w$, and the strong maximum principle implies $w < \sup_{\partial\Omega_0} w = M$ in Ω_0, a contradiction. Hence $w \le M$ in Ω. □

Now we come to the proofs of Lemmas 7 and 8.

Proof of Lemma 7. Fix for the moment $x_0 \in \partial\Omega$ and denote $\bar{v}(x) = \bar{u}(|x - y_0|)$. Take $K > 0$ and let $g_K \in C^1(\mathbb{R})$ be a bounded increasing function verifying $g_K(t) = g(t)$ if $0 \le t \le K$.

Let us consider the truncated problem

$$\left\{\begin{array}{ll} -\Delta u + \alpha u = g_K(|\nabla u|) + \lambda h(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{4.3}$$

When $K > \sup|\bar{u}'|$, the function $\bar{v}$ is a supersolution to (4.3) and since $\underline{v} = 0$ is a subsolution, it follows that there exists a solution u to (4.3) (by using the results in [6] or [28]), which verifies $0 \le u \le \bar{v}$.

By the maximum principle we have $u > 0$ in Ω and $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$. Moreover, since $\bar{v}(x_0) = 0$,

$$\frac{\partial u}{\partial \nu}(x_0) \geq \frac{\partial \bar{v}}{\partial \nu}(x_0) = -\bar{u}'(R_1).$$

Let us see that the same inequality holds for every $x_0' \in \partial\Omega$. Indeed, if we take such a x_0' and A' is the corresponding annulus, then since $-\Delta u + \alpha u \leq g_K(|\nabla u|) + \lambda |h|_\infty$ in Ω, the function $\bar{u}(|x - y_0'|)$ is a supersolution to the problem (4.3), considered in A'. By comparison we obtain $u(x) \leq \bar{u}(|x - y_0'|)$ in Ω. Thus

$$\frac{\partial u}{\partial \nu}(x_0') \geq -\bar{u}'(R_1).$$

Hence

$$-\bar{u}'(R_1) \leq \frac{\partial u}{\partial \nu} \leq 0 \qquad \text{on } \partial\Omega.$$

We are in a position to apply Lemma 9 to obtain a constant $M > 0$, which does not depend on K, such that $|\nabla u| < M$ in $\overline{\Omega}$. Taking $K > M$, we have $g_K(|\nabla u|) = g(|\nabla u|)$ in Ω and u is a solution to our original problem. This concludes the proof. □

Proof of Lemma 8. Assume problem (1.1) has a positive solution u for some $\lambda \geq c/h_0$. Then u is a supersolution to the problem

$$\begin{cases} -\Delta v + \alpha v = g(|\nabla v|) + \lambda h_0 & \text{in } B_R(y_0) \\ v = 0 & \text{on } \partial B_R(y_0). \end{cases}$$

A similar procedure as in Lemma 7 yields the existence of a solution to this problem, which is unique, hence radial. This is in contradiction with the hypothesis. It is important to remark that this procedure works since the supersolution vanishes at $x_0 \in \partial B_R(y_0) \cap \partial\Omega$, which allows us to estimate v' on $\partial B_R(y_0)$ in terms of $|\nabla u(x_0)|$. □

Finally, we proceed to prove our main theorems. We notice that, once we have analyzed separately the cases $\alpha = 0$ and $\alpha > 0$ in Sections 2 and 3, the rest of the proof is exactly the same in both cases.

Proof of Theorems 1 and 2. (i) By Lemmas 3 and 4, there exists a supersolution to (4.1) for every $c > 0$. The existence of a positive solution to (1.1) for every $\lambda > 0$ follows thanks to Lemma 7.

(ii) Again by lemmas 3, 4 and 7, there exists a solution for small values of λ. On the other hand, using Lemma 8 in conjunction with Lemmas 5 and 6 we also have that no solutions to (1.1) exist for large values of λ. Hence, we can define

$$\Lambda = \sup\{\lambda > 0 : \text{ there exists a solution to (1.1)}\}$$

and Λ is finite and positive. By its very definition, there are no solutions to (1.1) for $\lambda > \Lambda$. Now, choose an arbitrary $\lambda \in (0, \Lambda)$. Then there exists $\mu \in (\lambda, \Lambda)$ such that (1.1) with λ substituted by μ admits a positive solution v. Since this solution is a supersolution to (1.1), the existence of a positive solution follows as in Lemma 7. □

Acknowledgements. S. A. was partially supported by USM Grant No. 121002 and Fondecyt grant 11110482, J. G-M was partially supported by Ministerio de Ciencia e Innovación under grant MTM2011-27998 (Spain) and A. Q. was partially supported by Fondecyt Grant No. 1110210 and CAPDE, Anillo ACT-125. All three authors were partially supported by Programa Basal CMM, U. de Chile.

References

[1] H. Abdel Hamid, M. F. Bidaut-Veron, *Correlation between two quasilinear elliptic problems with a source term involving the function or its gradient*, C. R. Acad. Sci. Paris Ser. I Math. **346** (2008), 1251–1256.

[2] B. Abdellaoui, A. Dall'Aglio, I. Peral, *Some remarks on elliptic problems with critical growth in the gradient*, J. Diff. Equations **222**, (2006), 21–62.

[3] N. Alaa, M. Pierre, *Weak solutions of some quasilinear elliptic equations with data measures*, SIAM J. Math. Anal. **24** (1993), 23–35.

[4] S. Alarcón, J. García-Melián, A. Quaas, *Keller-Osserman type conditions for some elliptic problems with gradient terms*, J. Diff. Eqns. **252** (2012), 886–914.

[5] A. Alvino, G. Trombetti, P.-L. Lions, *Comparison results for elliptic and parabolic equations via Schwarz symmetrization*, Ann. Inst. H. Poincaré Anal. Non Linaire **7** (1990), 37–65.

[6] H. Amann, M. G. Crandall, *On some existence theorems for semi-linear elliptic equations*, Indiana Univ. Math. J. **27** (1978), 779–790.

[7] G. Barles, *A short proof of the $C^{0,\alpha}$-regularity of viscosity subsolutions for superquadratic viscous Hamilton-Jacobi equations and applications*, Nonlinear Anal. **73** (2010), 31–47.

[8] G. Barles, A. P. Blanc, C. Georgelin, M. Kobylanski, *Remarks on the maximum principle for nonlinear elliptic PDE with quadratic growth conditions*, Ann. Scuola Norm. Sup. Pisa **28** (1999), 381–404.

[9] G. Barles, F. Murat, *Uniqueness and the maximum principle for quasilinear elliptic equations with quadratic growth conditions*, Arch. Rational. Mech Anal. **133** (1995), 77–101.

[10] G. Barles, A. Porretta, *Uniqueness for unbounded solutions to stationary viscous Hamilton-Jacobi equations*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **5** (2006), 107–136.

[11] L. Boccardo, T. Gallouët, F. Murat, *A unified presentation of two existence results for problems with natural growth*, in "Progress in partial differential equations: the Metz surveys", 2 (1992), 127–137, Pitman Res. Notes Math. Ser. 296, Longman Sci. Tech., Harlow, 1993.

[12] L. Boccardo, F. Murat, J. P. Puel, *Resultats d'existence pour certains problemes elliptiques quasilineaires*, Ann. Scuola Norm. Sup. Pisa **11**, (1984), 213–235.

[13] L. Boccardo, F. Murat, J. P. Puel, *L^∞ estimate for some nonlinear elliptic partial differential equations and application to an existence result*, SIAM J. Math. Anal. **23** (2) (1992), 326–333.

[14] A. Dall'Aglio, D. Giachetti, J. P. Puel, *Nonlinear elliptic equations with natural growth in general domains*, Ann. Mat. Pura Appl. **181** (2002), 407–426.

[15] T. Del Vecchio, M.M. Porzio, *Existence results for a class of non-coercive Dirichlet problems*, Ricerche Mat. **44** (1995), 421–438.

[16] P. Felmer, A. Quaas, *On the strong maximum principle for quasilinear elliptic equations and systems*, Adv. Differential Equations **7** (2002), no. 1, 25–46.

[17] V. Ferone, F. Murat, *Quasilinear problems having quadratic growth in the gradient: an existence result when the source term is small*, Equations aux derivees partielles et applications, Gauthier-Villars, Ed. Sci. Med. Elsevier, Paris (1998), 497–515.

[18] V. Ferone, F. Murat, *Nonlinear problems having quadratic growth in the gradient: an existence result when the source term is small*, Non. Anal. TMA **42** (2000), 1309–1326.

[19] D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer–Verlag, 1983.

[20] N. Grenon, F. Murat, A. Porretta, *Existence and a priori estimate for elliptic problems with subquadratic gradient dependent terms*, C. R. Math. Acad. Sci. Paris **342** (1), (2006), 23–28.

[21] K. Hansson, V. Mazja, I.E. Verbitsky, *Criteria of solvability for multidimensional Riccati equations*, Ark. Mat. **37** (1999), 87–120.

[22] L. Jeanjean, B. Sirakov, *Existence and multiplicity for elliptic problems with quadratic growth in the gradient*, Comm. Part. Diff. Eqns. **38** (2012), 244–264.

[23] J. M. Lasry, P. L. Lions, *Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints. I. The model problem*, Math. Ann. **283** (1989), 583–630.

[24] P. L. Lions, *Résolution des problèmes elliptiques quasilinéaires*, Arch. Rat. Mech. Anal. **74** (1980), 336–353.

[25] P. L. Lions, *Quelques remarques sur les problemes elliptiques quasilineaires du second ordre*, J. Anal. Math. **45** (1985), 234–254.

[26] A. Porretta, *The ergodic limit for a viscous Hamilton-Jacobi equation with Dirichlet conditions*, Rend. Lincei (9) Mat. Appl. **21** (2010), 59–78.

[27] P. Pucci, J. Serrin, H. Zou, *A strong maximum principle and a compact support principle for singular elliptic equations*, J. Math. Pures Appl. **78** (1999), 769–789.

[28] J. Schoenenberger-Deuel, P. Hess, *A criterion for the existence of solutions of non-linear elliptic boundary value problems*, Proc. Roy. Soc. Edinburgh Sect. A **74** (1974/75), 49–54 (1976).

[29] J. Serrin, *The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables*, Philos. Trans. R. Soc. London A **264** (1969), 413–496.

[30] B. Sirakov, *Solvability of uniformly elliptic fully nonlinear PDE*, Arch. Rat. Mech. Anal. **195** (2010), 579–607.

S. Alarcón and A. Quaas
Departamento de Matemática, Universidad Técnica Federico Santa María
Casilla V-110, Avda. España, 1680 – Valparaíso, CHILE.
E-mail address: salomon.alarcon@usm.cl, alexander.quaas@usm.cl

J. García-Melián
Departamento de Análisis Matemático, Universidad de La Laguna.
C/. Astrofísico Francisco Sánchez s/n, 38271 – La Laguna, SPAIN
and
Instituto Universitario de Estudios Avanzados (IUdEA) en Física Atómica, Molecular y Fotónica, Facultad de Física, Universidad de La Laguna
C/. Astrofísico Francisco Sánchez s/n, 38203 – La Laguna , SPAIN
E-mail address: jjgarmel@ull.es