THE SOLVABILITY OF AN ELLIPTIC SYSTEM UNDER A SINGULAR BOUNDARY CONDITION *

J. García-Melián[1], R. Letelier-Albornoz[2†] and J. Sabina de Lis[1‡]

[1] Departamento de Análisis Matemático, Universidad de La Laguna, SPAIN
[2] Departamento de Matemáticas, Universidad de Concepción, CHILE

Synopsis

In this work we are considering both the one-dimensional and the radially symmetric versions of the elliptic system $\Delta u = v^p$, $\Delta v = u^q$ in Ω, where $p, q > 0$, under the boundary condition $u_{|\partial\Omega} = +\infty, v_{|\partial\Omega} = +\infty$. It is shown that no positive solutions exist when $pq \leq 1$, while we provide a detailed account of the set of (infinitely many) positive solutions if $pq > 1$. The behavior near the boundary of all solutions is also elucidated, and symmetric solutions (u, v) are completely characterized in terms of their minima $(u(0), v(0))$. Nonsymmetric solutions are also deeply studied in the one-dimensional problem.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Boundary blow-up elliptic problems of the form

$$\begin{cases} \Delta u = f(u) & \text{in} \quad \Omega \\ u = +\infty & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

regarding the subjects of existence and uniqueness of positive solutions (sometimes called "large") together with estimates of their rate of divergence to infinity at $\partial\Omega$, have been the focus of a great amount of works. We quote the pioneering papers [5], [33], [32], [29] concerning Riemannian geometry and Riemann surfaces, and [23], [24] where (1.1) arises in a problem in electrohydrodynamics (see also [26], where stochastic control problems lead to large solutions). A brief account of more recent literature on the problem is provided by [25], [2], [36], [3], [10], [27], [28], [34], [1] [31], [4], [37], [9] and [16].

In the specific case of $f(u) = u^p$, closer in some sense to the nonlinearities to be dealt with in this paper, problem (1.1) was considered in [29] for $p = \frac{N+2}{N-2}$ while later generalizations of the form $\Delta u = a(x)u^p$, with a Hölder continuous and positive (up to $\partial\Omega$), were given in

*Supported by MCYT – FEDER project under contract # BFM2001-3894 (Spain), Centro de Modelamiento Matemático (Chile) and FONDECYT # 1000333 (Chile).

†The results in this paper were obtained some time before the death of Professor R. G. Letelier-Albornoz.

‡E-mail addresses: jjgarmel@ull.es, rletelie@gauss.cfm.udec.cl, josabina@ull.es

[25] ($p \geq 3$), [2], [3], [36], [31]. In [10], the extension to equations involving the p-Laplacian was first considered. In [16] the case where a vanishes on $\partial\Omega$ was studied, while a is even allowed to be unbounded near $\partial\Omega$ in [37], [6], [7], [14] (see an updated account in [18] and the references included there on this issue). It is also worth remarking that [15] shows a study case where problem (1.1) arises in population dynamics. Specifically, $-\Delta u = \lambda(x)u - a(x)u^p$ in Ω, $u = +\infty$ on $\partial\Omega$ with $p > 1$, λ, a Hölder continuous in Ω, $a > 0$ in Ω but $a_{|\partial\Omega} = 0$ (see also [16]).

However, the corresponding problem for elliptic systems, namely

$$\begin{cases} \Delta u = f(u,v) & x \in \Omega \\ \Delta v = g(u,v) & x \in \Omega \\ \quad u = v = +\infty & x \in \partial\Omega \end{cases} \tag{1.2}$$

has been hardly treated in the literature, even for the more selective classes of nonlinearities f, g. As one could suspect, the main handicap in dealing with (1.2) is the lack of comparison. Indeed, a natural way to construct solutions in the scalar case (1.1) is solving the finite datum Dirichlet problem and then letting the datum grow to infinity. Comparison principles are then instrumental in showing convergence, say by obtaining suitable estimates. In fact, by means of the method of sub and supersolutions, comparison is a fundamental tool in the cases of (1.2) recently studied in [19], [18]. In [19], $-f = \lambda u - u^2 + buv$, $-g = \mu v - v^2 + cuv$ with $b, c > 0$, so (1.2) falls in the cooperative regime. Regarding [18], $f = u^r v^p$, $g = u^q v^s$, $p, q, r, s > 0$ and now (1.2) is of competitive type, being the analysis restricted to the parameter range $(r-1)(s-1) - pq \geq 0$) (see some few comments below). It should be remarked that the research developed in the present paper was the "starting point" for those works.

On the other hand, problems related to (1.2) arise when studying Lotka–Volterra systems of predator-prey and competitive type, under a zero Dirichlet condition and variable coefficients, some of them vanishing on whole subdomains of Ω. See [8], [11], [12], [30].

In this work we are concentrating efforts in the (at first sight) simplest case where coupling and nonlinearity combine together in (1.2). Namely,

$$\begin{cases} \Delta u = v^p & x \in \Omega \\ \Delta v = u^q & x \in \Omega \\ \quad u = v = +\infty & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $p, q > 0$. As will be seen later, it turns out that the analysis of this kind of systems is fairly complicated. Regarding [18], we are dealing here with the complementary case $pq > 1$ (termed there as "supercritical"), and so no kind of comparison results can be employed. That is why we are restricting ourselves to the study of the one–dimensional version:

$$\begin{cases} u'' = v^p & -L < x < L \\ v'' = u^q & -L < x < L \\ u(\pm L) = v(\pm L) = +\infty \end{cases} \qquad \left(' = \frac{d}{dx} \right), \tag{1.4}$$

$L > 0$, searching for both symmetric and nonsymmetric solutions. We are also analyzing the radially symmetric version of (1.3) where Ω is the ball $B(0,R) = \{x : |x| < R\}$ and

$u(x) = u(r)$, $v(x) = v(r)$, $r = |x|$. In this case (1.2) takes the form:

$$\begin{cases} \dfrac{d^2u}{dr^2} + \dfrac{N-1}{r}\dfrac{du}{dr} = v^p & 0 < r < R, \\[2ex] \dfrac{d^2v}{dr^2} + \dfrac{N-1}{r}\dfrac{dv}{dr} = u^q & 0 < r < R, \\[2ex] \dfrac{du}{dr}\Big|_{r=0} = 0, \quad u(R) = \infty, & \dfrac{dv}{dr}\Big|_{r=0} = 0, \quad v(R) = \infty. \end{cases} \tag{1.5}$$

The case of a general domain Ω of $\mathbb{R}^N$ will be the objective of a future work.

It will be seen below that both problems (1.4), (1.5) admit positive solutions only when $pq > 1$, a translation to our setting of the condition $p > 1$ which appears in a single equation. But, as a first outstanding difference with respect to the situations considered before, systems (1.4), (1.5) have not the property of exhibiting a unique positive solution. More precisely, we can find infinitely many positive symmetric solutions to both problems. We suspect that this nonuniqueness is caused by the fact that our system is of "competitive type". Moreover, (1.4) also exhibits nonsymmetric solutions and a detailed description of them is provided. It turns out that all solutions share the same asymptotic behavior at $x = \pm L$.

We next state our first result concerning (1.4). The proof relies on the study of the blow-up property for solutions of the Cauchy problem associated to (1.4), exploiting the scaling invariance of the system.

Theorem 1.1. *Problem (1.4) admits a positive symmetric solution if and only if*

$$pq > 1. \tag{1.6}$$

Moreover, the following properties hold.

i) [Uniqueness]. *There exists a unique solution* $(u, v) = (U_1(x), V_1(x))$ *to (1.4) (respectively,* $(U_2(x), V_2(x))$*), positive in* $x \neq 0$*, under the restriction,*

$$\inf_{(-L,L)} u = 0 \qquad (\textit{respectively}, \inf_{(-L,L)} v = 0).$$

ii) [Multiplicity of solutions]. *The set of positive symmetric solutions* (u, v) *to (1.4) defines a continuous arc joining* $(U_1(x), V_1(x))$ *to* $(U_2(x), V_2(x))$*. More precisely, the set* $\Gamma = \{(\inf u, \inf v) = (u(0), v(0)) : (u, v)$ *positive, symmetric, solving (1.4)*$\} \subset \overline{\mathbb{R}}^2_+$ *is parameterized by* $g = (g_1, g_2) : [0,1] \to \overline{\mathbb{R}}^2_+$*,* g *continuous,* g_1 *nondecreasing,* g_2 *nonincreasing and* $g(0) = (U_1(0), V_1(0))$*,* $g(1) = (U_2(0), V_2(0))$ *(see Figure 2).*

iii) [Asymptotic profile]. *Every positive symmetric solution* (u, v) *to (1.4) verifies the following asymptotic estimates:*

$$\begin{aligned} u(x) &\sim a\ d(x)^{-\xi} \ \textit{as}\ d(x) \to 0+, \\ v(x) &\sim b\ d(x)^{-\eta} \ \textit{as}\ d(x) \to 0+, \end{aligned} \tag{1.7}$$

where $d(x) = \min\{L - x, L + x\}$ *and* $\xi = 2(p+1)/(pq-1)$*,* $\eta = 2(q+1)/(pq-1)$*, while* $a = [\xi(\xi+1)\eta^p(\eta+1)^p]^{1/(pq-1)}$*,* $b = [\eta(\eta+1)\xi^q(\xi+1)^q]^{1/(pq-1)}$*.*

As for solutions to (1.5) our main result is the following.

Theorem 1.2. *The problem,*

$$\begin{cases} \Delta u = v^p & x \in B(0,R) \\ \Delta v = u^q & x \in B(0,R) \\ u = v = +\infty & |x| = R, \end{cases} \tag{1.8}$$

admits a radially symmetric positive solution $(u(r), v(r))$ if and only if

$$pq > 1.$$

The following properties are in addition satisfied.

i) [Change of scale]. *If (u,v) solves (1.8) in $B(0,R)$ then for every $\lambda > 0$, $(u_\lambda(r), v_\lambda(r)) = (\lambda u(\lambda^\theta r), \lambda v^{q+1/p+1}(\lambda^\theta r))$, $\theta = (pq-1)/2(p+1)$ is a solution of (1.8) in $B(0,\lambda^{-\theta}R)$.*

ii) [Uniqueness]. *There exists a unique radially symmetric solution $(u,v) = (U_1(r), V_1(r))$ (respectively, $(U_2(r), V_2(r))$ to (1.8), positive in $B(0,R) \setminus \{0\}$ under the restriction*

$$\inf_{r \in [0,R)} u = 0 \qquad (\textit{respectively}, \inf_{r \in [0,R)} v = 0).$$

iii) [Multiplicity]. *Problem (1.8) admits infinitely many solutions $(u(r), v(r))$. Furthermore, the set $\Gamma_R = \{(u_0, v_0) : u_0 = \inf u, v_0 = \inf v, (u,v) \text{ a radial solution}\} \subset \overline{\mathbb{R}}^2_+$ is contained in $[0, U_2(0)] \times [0, V_1(0)]$ while u_0 varies nondecreasingly, v_0 nonincreasingly when (u_0, v_0) runs Γ_R.*

Our description of the set of positive solutions to (1.4) is completed with the study of its strictly nonsymmetric positive solutions. A first result in this direction asserts, among other things, that every symmetric solution generates two one-parametric families of nonsymmetric positive solutions to (1.4).

Theorem 1.3. *A necessary and sufficient condition in order that problem (1.4) possesses a positive solution (u,v), regardless its symmetry, is again,*

$$pq > 1.$$

Moreover, such a solution exhibits exactly the same profile at the boundary as a symmetric one. Namely,

$$u(x) \sim a\ d(x)^{-\xi}, \qquad v(x) \sim b\ d(x)^{-\eta}, \tag{1.9}$$

as $d \to 0+$, where $d(x) = \min\{L-x, L+x\}$, ξ, η are the exponents, a, b the coefficients in (1.7).

Furthermore, each symmetric *positive solution (u,v) to (1.4) with*

$$u_0 = \inf u \qquad v_0 = \inf v,$$

gives rise to two noncontinuable families of positive solutions satisfying the following properties.

i) There exists $\sigma_1^* = \sigma_1^*(u_0, v_0)$*, positive and continuous in* (u_0, v_0)*, and a family* $\{(\hat{u}(x,\sigma), \hat{v}(x,\sigma))\}_{|\sigma|\le\sigma_1^*}$ *of positive solutions to (1.4) such that* $(\hat{u}, \hat{v})_{|\sigma=0} = (u, v)$ *and:*

a) $(\hat{u}(x,-\sigma), \hat{v}(x,-\sigma)) = (\hat{u}(-x,\sigma), \hat{v}(-x,\sigma))$ *for each* $|\sigma| \le \sigma_1^*$*,* $-L < x < L$*.*

b) $\inf \hat{v}(\cdot,\sigma) = v_0$ *for* $|\sigma| \le \sigma_1^*$*.*

c) $0 < \inf \hat{u}(\cdot,\sigma) < u_0$ *if* $0 < |\sigma| < \sigma_1^*$ *while* $\inf \hat{u}(\cdot,\sigma) = 0$ *at* $\sigma = \pm\sigma_1^*$*.*

Furthermore, $(\hat{u}(\cdot,\sigma), \hat{v}(\cdot,\sigma))$ *is nonsymmetric for* $\sigma \ne 0$*.*

ii) Similarly, problem (1.4) exhibits a family $\{(\tilde{u}(x,\sigma), \tilde{v}(x,\sigma))\}_{|\sigma|\le\sigma_2^*}$ *of positive solutions where* $\sigma_2^* = \sigma_2^*(u_0, v_0)$ *is a positive continuous function of* (u_0, v_0)*, such that* $(\tilde{u}, \tilde{v})_{|\sigma=0} = (u, v)$ *while* $(\tilde{u}(\cdot,\sigma), \tilde{v}(\cdot,\sigma))$ *is nonsymmetric for* $\sigma \ne 0$*. In addition,*

a) $(\tilde{u}(x,-\sigma), \tilde{v}(x,-\sigma)) = (\tilde{u}(-x,\sigma), \tilde{v}(-x,\sigma))$ *for each* $|\sigma| \le \sigma_2^*$*,* $-L < x < L$*.*

b) $\inf \tilde{u}(\cdot,\sigma) = u_0$ *for* $|\sigma| \le \sigma_2^*$*.*

c) $0 < \inf \tilde{v}(\cdot,\sigma) < v_0$ *if* $0 < |\sigma| < \sigma_2^*$ *while* $\inf \hat{v}(\cdot,\sigma) = 0$ *at* $\sigma = \pm\sigma_2^*$*.*

Remark 1.4. For a nonsymmetric solution (u, v) in the family $\{(\hat{u}, \hat{v})\}_{|\sigma|\le\sigma_1^*}$, the parameter σ has the status, after a convenient rescaling, of the derivative of $\hat{u}$ at a certain reference point $x \in (-L, L)$. σ has the symmetric meaning regarding $(\tilde{u}, \tilde{v})$. See Remark 6.5 for precise details and a global bifurcation diagram for nonsymmetric positive solutions in the family $(\hat{u}, \hat{v})$.

A second more ambitious statement ensures the existence of a bidimensional continuum of nonsymmetric solutions generated from a symmetric one. An additional result (Theorem 1.6) characterizes the set of all possible positive solutions (u, v) to problem (1.4) in terms of their infima $u_0 = \inf u, v_0 = \inf v$ by elucidating the set $\mathcal{C} \in \mathbb{R}^2$ where their derivatives $(u_x(0), v_x(0))$ must lie. This permits finding a broad class of nonsymmetric solutions exhibiting the property of being a continuous deformation of a symmetric positive one.

Theorem 1.5. *Let* (u, v) *be a positive symmetric solution of (1.4) with* $u_0 = \inf u$*,* $v_0 = \inf v$*,* $\sigma_i^* = \sigma_i^*(u_0, v_0)$*,* $i = 1, 2$*, being the associated values introduced in Theorem 1.3. Then there exists a continuous bidimensional and noncontinuable family of positive solutions*

$$\{(\tilde{u}(x,\bar{\sigma}), \tilde{v}(x,\bar{\sigma})) : \bar{\sigma} = (\sigma_1, \sigma_2) \in \mathcal{C}_0\}$$

such that,

a) The $\bar{\sigma}$*-domain* $\mathcal{C}_0 \subset \mathbb{R}^2$ *is an open bounded-connected set, symmetric with respect to* $(0,0) \in \mathcal{C}_0$*,* $(\tilde{u}(\cdot,\bar{\sigma}), \tilde{v}(\cdot,\bar{\sigma}))_{|\bar{\sigma}=(0,0)} = (u, v)$*, while:*

$$\tilde{u}(-x,\bar{\sigma}) = \tilde{u}(x,-\bar{\sigma}) \qquad \tilde{v}(-x,\bar{\sigma}) = \tilde{v}(x,-\bar{\sigma}) \qquad -L < x < L,$$

for all $\bar{\sigma} \in \mathcal{C}_0$*. Moreover,* $(\tilde{u}(\cdot,\bar{\sigma}), \tilde{v}(\cdot,\bar{\sigma}))$ *is nonsymmetric if* $\bar{\sigma} \ne (0,0)$*.*

b) There exist continuous decreasing functions $h_-, h_+ : \mathbb{R} \to \mathbb{R}$ *satisfying:*

i) $h_-(\sigma_1) = -h_+(-\sigma_1)$*.*

ii) $h_+(\mp\infty) = \pm\infty$, $h_+(0) = \sigma_2^*$, $h_+(\sigma_1^*) = 0$ *while* $b > \sigma_1^*$ *exists such that* $h_- = h_+$ *at* $\sigma_1 = \pm b$, *being* $h_- < h_+$ *for* $|\sigma_1| < b$.

Moreover, the set $\mathcal{C}_0$ *can be expressed in terms of such functions as (Figure 1):*

$$\mathcal{C}_0 = \{\bar{\sigma} : |\sigma_1| < b,\ h_-(\sigma_1) < \sigma_2 < h_+(\sigma_1)\}. \tag{1.10}$$

c) The noncontinuable character of the family beyond the boundary $\partial\mathcal{C}_0$ *of* $\mathcal{C}_0$ *is reflected by the fact that, modulus a suitable scaling, solutions* $(\tilde{u}, \tilde{v})$ *satisfy either* $\inf \tilde{u} \to 0$ *or* $\inf \tilde{v} \to 0$ *as* $\bar{\sigma} \to \bar{\sigma}_0$ *for every* $\bar{\sigma}_0 \in \partial\mathcal{C}_0$.

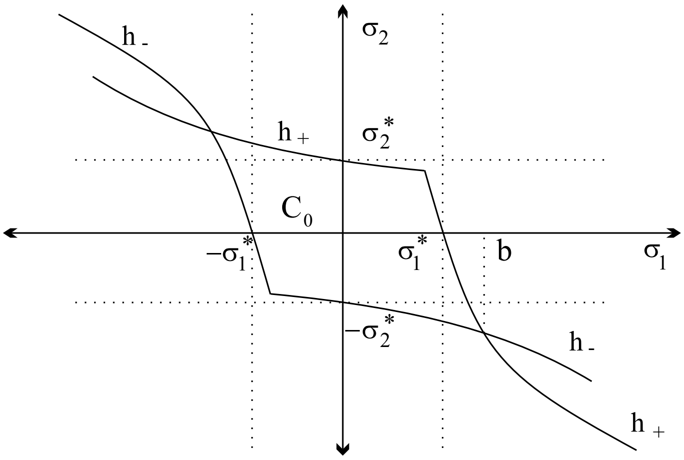

FIGURE 1. The connected piece containing $(0, 0)$, $\mathcal{C}_0$, of the set of values $\bar{\sigma} = (\sigma_1, \sigma_2)$ lying between h_- and h_+.

As will be shown later in Section 7, the functions h_-, h_+ exclusively depend on u_0, v_0. For the purposes of our next statement we introduce the set,

$$\mathcal{C} = \{\bar{\sigma} : h_-(\sigma_1) < \sigma_2 < h_+(\sigma_1)\}. \tag{1.11}$$

Observe that $\mathcal{C}_0$ is nothing else but the connected piece of $\mathcal{C}$ to which $(0, 0)$ belongs. We are now studying conditions ensuring when a positive solution (in general, nonsymmetric) can be regarded as a continuous perturbation of a symmetric one.

Theorem 1.6. *Let* u_0, v_0 *be positive. Then the open set* $\mathcal{C}$ *defined in (1.11) is bounded. Furthermore, suppose that* (u, v) *is an arbitrary positive solution to (1.4) in* $(-L, L)$ *with* $u(0) = u_0$, $v(0) = v_0$, $u_x'(0) = u_0'$, $v_x'(0) = v_0'$. *Then the following properties hold:*

i) $(u_0', v_0') \in \mathcal{C}$.

ii) If u_0', v_0' *have the same sign, i. e.* $u_0'v_0' \geq 0$, *then necessarily* $|u_0'| \leq \sigma_1^*$, $|v_0'| \leq \sigma_2^*$. *Moreover,* (u, v) *can be continuously deformed to a symmetric positive solution.*

iii) If $u_0'v_0' < 0$ *but* $(u_0', v_0') \in \mathcal{C}_0$ *then* (u, v) *comes from a symmetric positive solution by means of a continuous perturbation.*

Remark 1.7. As a consequence of the analysis in Section 7, it follows that (u_0', v_0') in ii) belongs to $\mathcal{C}_0$. Theorems 1.5, 1.6 do not exclude the possible existence of a nonsymmetric solution (u, v) which, in the previous terminology, satisfies $u_0'v_0' < 0$ but $(u_0', v_0') \notin \mathcal{C}_0$ (see Remark 7.5 a)). However, the possibility of deforming it to a symmetric solution can not be ensured in this case.

It should be finally pointed out that similar methods as the ones developed here can be used to analyze (1.2) with nonlinearities $f = e^u$, $g = e^v$. The subsequent results and the analysis of broader classes of competitive nonlinearities $f(u, v), g(u, v)$ will be given elsewhere. On the other hand, the research of this paper was first presented in the "2^{nd} Joint Meeting of the Royal Spanish Mathematical Society" (Tenerife, Spain, February 2002) and in the "18^{th} Spanish Meeting on Differential Equations" (Tarragona, Spain, September 2003), see [17].

The paper is organized as follows: in Section 2, we examine the initial value problem associated to the symmetric solutions to (1.4), providing some preliminary properties. Section 3 is devoted to the statement of the important Theorem 3.1, and the proof of some of its consequences. Section 4 consists in the proof of Theorem 3.1. Radial solutions to (1.5) are studied in Section 5 while a full description of the nonsymmetric solutions to (1.4) is contained in Section 6.

2. AN INITIAL VALUE PROBLEM

In this section we are going to perform a detailed analysis of the positive solutions to the initial value problem,

$$\begin{cases} u'' = v^p \\ v'' = u^q \end{cases} \qquad \begin{cases} u(0) = u_0, & u'(0) = 0 \\ v(0) = v_0, & v'(0) = 0, \end{cases} \tag{2.1}$$

$p > 0, q > 0$, for $u_0, v_0 \geq 0$, where such equations are considered as defined in $u \geq 0, v \geq 0$. We are working with nonnegative solutions so it will not be necessary for the moment to extend u^q, v^p to the whole of $\mathbb{R}$. The main features of (2.1) are examined in the following results.

Lemma 2.1. *Assume that* $u_0, v_0 \geq 0$, $(u_0, v_0) \neq (0, 0)$. *Then (2.1) admits a forward solution* (u, v), *defined in a noncontinuable interval* $[0, \omega)$, $0 < \omega \leq +\infty$, *which is (component wise) positive, increasing, convex and satisfies:*

$$\lim_{x \to \omega -} u = \lim_{x \to \omega -} v = +\infty.$$

Remark 2.2.

a) The assertions of Lemma 2.1 are standard. Notice that full symmetric solutions in the interval $(-\omega, \omega)$ can be obtained by reflection with respect to $x = 0$.

b) It should be stressed that both components u, v blow up at the same points $x = \pm\omega$ regardless ω is finite or not. That will remain true even if (u, v) is nonsymmetric.

Lemma 2.3. (Comparison principle) *Let* (u,v), $(\bar{u},\bar{v})$ *be solutions to*

$$\begin{cases} u'' = v^p \\ v'' = u^q, \end{cases} \tag{2.2}$$

with nonnegative initial data $(u,v,u',v')_{|x=0} = (u_0,v_0,u_0',v_0')$, $(\bar{u},\bar{v},\bar{u}',\bar{v}')_{|x=0} = (\bar{u}_0,\bar{v}_0,\bar{u}_0',\bar{v}_0')$ *such that* $u_0 \le \bar{u}_0$, $v_0 \le \bar{v}_0$, $u_0' \le \bar{u}_0'$, $v_0' \le \bar{v}_0'$ *while* $(u_0,v_0,u_0',v_0') \ne (\bar{u}_0,\bar{v}_0,\bar{u}_0',\bar{v}_0')$. *Then* $u < \bar{u}$ *and* $v < \bar{v}$ *in* $x > 0$ *wherever both solutions are defined.*

Proof. Assume for instance $u_0 < \bar{u}_0$, $v_0 \le \bar{v}_0$, the remaining cases being handled in the same way. Notice that this implies $u < \bar{u}$ in an interval of the form $[0,\varepsilon)$. If ε is such that $u(\varepsilon) = \bar{u}(\varepsilon)$, then

$$v(x) = v(0) + v'(0)x + \int_0^x \int_0^s u(t)^q dt\ ds < \bar{v}(0) + \bar{v}'(0)x + \int_0^x \int_0^s \bar{u}(t)^q dt\ ds = \bar{v}(x)\ ,$$

when $x \in (0,\varepsilon)$. But also

$$u(\varepsilon) = u(0) + \int_0^\varepsilon \int_0^s v(t)^p dt\ ds = \bar{u}(0) + \int_0^\varepsilon \int_0^s \bar{v}(t)^q dt\ ds = \bar{u}(\varepsilon)\ ,$$

which implies $v \equiv \bar{v}$ in $[0,\varepsilon]$, a contradiction. Thus, $u(x) < \bar{u}(x)$, and similarly $v(x) < \bar{v}(x)$. The lemma is proved. □

Lemma 2.4. *Problem (2.1) has a unique solution* (u,v) *for every* $u_0, v_0 \ge 0$, $(u_0,v_0) \ne (0,0)$, *which will be denoted by* $(u(\cdot,u_0,v_0), v(\cdot,u_0,v_0))$. *This solution is increasing in* u_0 *for fixed* v_0, *and in* v_0 *for fixed* u_0. *In the case* $(u_0,v_0) = (0,0)$, $u = 0$, $v = 0$ *is the unique nonnegative solution if and only if* $pq \ge 1$.

Proof. The uniqueness of solutions is a consequence of the standard theory of ode's when $u_0, v_0 > 0$, or when $u_0 = 0$ and $q \ge 1$ or $v_0 = 0$ and $p \ge 1$. Hence we only need to treat the cases $u_0 = 0$, $v_0 > 0$ and $u_0 = v_0 = 0$ (the remaining case $u_0 > 0$, $v_0 = 0$ is similar).

Assume first $u_0 = 0$, $v_0 > 0$ and let (u,v), $(\bar{u},\bar{v})$ be two solutions of (2.1). We adapt an argument in [35]. It is easily shown that $u(x), \bar{u}(x) > 0$, $v(x), \bar{v}(x) > v_0$ for $x > 0$. For $\delta > 0$ small, since $(u(x+\delta), v(x+\delta))$ solves (2.2) with positive initial data $(u(\delta), v(\delta), u'(\delta), v'(\delta))$, we obtain by Lemma 2.3 that $\bar{u}(x) < u(x+\delta)$, $\bar{v}(x) < v(x+\delta)$. Letting $\delta \to 0+$ we arrive at $\bar{u} \le u$, $\bar{v} \le v$, and a symmetric argument proves the uniqueness.

Now consider the case $u_0 = v_0 = 0$, and assume $pq \ge 1$. As before we only prove uniqueness. For $\delta > 0$ denote $|u|_{\infty,\delta} = \sup_{[0,\delta]} u(x)$, $|v|_{\infty,\delta} = \sup_{[0,\delta]} v(x)$. Then since

$$u(x) = \int_0^x \int_0^s v(t)^p dt\ ds\ ,$$

we have $|u|_{\infty,\delta} \le |v|^p_{\infty,\delta} \delta^2/2$, and symmetrically $|v|_{\infty,\delta} \le |u|^q_{\infty,\delta} \delta^2/2$. Combining these two, we arrive at $|u|_{\infty,\delta} \le |u|^{pq}_{\infty,\delta} (\delta^2/2)^{p+1}$, from which $|u|_{\infty,\delta} = 0$ for δ small follows.

The increasing character of the solution with respect to u_0 and v_0 is a direct consequence of Lemma 2.3.

Finally, uniqueness of nonnegative solutions corresponding to initial conditions $u_0 = v_0 = 0$ fails in the complementary range $0 < pq < 1$. Indeed, the pair

$$(\tilde{u}(x), \tilde{v}(x)) = (ax^{|\xi|}, bx^{|\eta|}) ,$$

where (compare with iii) in Theorem 1.1) $|\xi| = 2(p+1)/(1-pq)$, $|\eta| = 2(q+1)/(1-pq)$, $a = \{|\xi|(|\xi|-1)|\eta|^p(|\eta|-1)^p\}^{\frac{1}{pq-1}}$, $b = \{|\eta|(|\eta|-1)|\xi|^q(|\xi|-1)^q\}^{\frac{1}{pq-1}}$, defines a positive solution to (2.1). The previous ideas also permit showing that $(\tilde{u}, \tilde{v})$ is the unique positive solution to (2.1). All other possible nonnegative solutions are defined by $(u(x), v(x)) = (0, 0)$ for $0 \le x \le \tau$, $(u(x), v(x)) = (\tilde{u}(x-\tau), \tilde{v}(x-\tau))$ if $x \ge \tau$, $\tau > 0$ arbitrary (compare with [20]). This concludes the proof. □

Later in Section 6 we are perturbing the initial conditions $(u, v, u', v')_{|x=0}$ of a positive solution (u, v) to (2.2). For our purposes there it is convenient to extend the right hand sides of (2.2) so that the resulting perturbed solutions can vanish or be negative somewhere. The required uniqueness result for the extended equation to be used in this work is stated in our next theorem. We remark that the proof is more involved than that of Lemma 2.4.

Theorem 2.5. *Suppose $pq \ge 1$. Then, for arbitrary initial data $(u_0, v_0, u_0', v_0') \in \mathbb{R}^4$ the Cauchy problem*

$$\left\{\begin{array}{l} u'' = |v|^p \\ v'' = |u|^q \end{array}\right. \qquad \left\{\begin{array}{ll} u(0) = u_0, & u'(0) = u_0' \\ v(0) = v_0, & v'(0) = v_0', \end{array}\right. \tag{2.3}$$

possesses a unique solution (u, v).

Proof. We only need to consider the case $u_0 = 0$ with $0 < q < 1$ and showing local uniqueness. The remaining options correspond either to cases where the standard theory provides local uniqueness or to the symmetric situation $v_0 = 0$ with $0 < p < 1$.

First consider the case $u_0 = 0$, $u_0' \neq 0$ which can be treated according to an idea in [35]. In fact, define $(u, u_1, v, v_1) = (u, u', v, v')$ and perform the change of variable $x \mapsto u$ near zero. This leads to the equivalent four-dimensional system:

$$\frac{dx}{du} = u_1^{-1}, \qquad \frac{du_1}{du} = |v|^p u_1^{-1}, \qquad \frac{dv}{du} = v_1 u_1^{-1}, \qquad \frac{dv_1}{du} = |u|^q u_1^{-1},$$

under initial conditions $x(0) = 0$, $u_1(0) = u_0'$, $v(0) = 0$, $v_1(0) = v_0'$, which falls in the scope of the standard uniqueness theory.

Next, let us deal with the case $u_0 = u_0' = 0$. Since $v_0 = v_0' = 0$ implies that (u, v) is trivial (Lemma 2.4), we can assume $(v_0, v_0') \neq (0, 0)$. So initially suppose $v_0 \neq 0$. If (u, v), $(\bar{u}, \bar{v})$ solve (2.3) set $y(x) = \bar{u}(x) - u(x)$, $z(x) = \bar{v}(x) - v(x)$. Then, (y, z) satisfies,

$$\left\{\begin{array}{l} y'' = c(x)z \\ z'' = d(x)y, \end{array}\right.$$

being $c(x) = \dfrac{|\bar{v}|^p - |v|^p}{\bar{v} - v}$, $d(x) = \dfrac{\bar{u}^q - u^q}{\bar{u} - u}$, and where, under the present assumptions, $p > 1$. Observing that,

$$u(x) = (\frac{|v_0|^p}{2} + o(1))x^2, \qquad \bar{u}(x) = (\frac{|v_0|^p}{2} + o(1))x^2,$$

the representation, for small $|x| > 0$,

$$d(x) = q \int_0^1 \frac{ds}{(u(x) + s(\bar{u}(x) - u(x))^{1-q}},$$

implies that

$$|x|^{2(1-q)}|d(x)| \leq C_1,$$

for $|x| \leq \delta$, $\delta > 0$ small and certain positive C_1. On the other hand,

$$y(x) = x^2 \int_0^1 (1-s)c(sx)z(sx)\, ds.$$

Since $c(x) = p|v_0|^{p-2}v_0 + o(1)$ then,

$$\sup_{0<|x|<\delta} \frac{|y(x)|}{x^2} \leq C_2|z|_{\infty,\delta},$$

where $C_2 > 0$, $|z|_{\infty,\delta} = \sup_{|x|\leq\delta} |z(x)|$. From the identity,

$$z(x) = \int_0^x \int_0^t s^2 d(s) \frac{y(s)}{s^2}\, ds\, dt$$

we achieve,

$$|z|_{\infty,\delta} \leq C_2\delta^2 \sup_{|x|\leq\delta} x^2|d(x)|\, |z|_{\infty,\delta},$$

which implies $z = 0$ in $|x| \leq \delta$ if δ is small. Hence $(u, v) = (\bar{u}, \bar{v})$ near zero.

The case $u_0 = u_0' = v_0 = 0$ with $v_0' \neq 0$ is handled in a similar way. In fact,

$$|x|^{(p+2)(1-q)}|d(x)| = O(1), \qquad c(x) = O(|x|^{p-1}),$$

as $x \to 0$. In the same way,

$$\sup_{0<|x|\leq\delta} \frac{|y(x)|}{|x|^{p+2}} \leq C_3|z'|_{\infty,\delta}.$$

Thus,

$$z'(x) = \int_0^x s^{p+2} d(s) \frac{y(s)}{s^{p+2}}\, ds,$$

leads to,

$$|z'|_{\infty,\delta} \leq C_3\delta \sup_{|x|\leq\delta} |x|^{p+2}|d(x)|\, |z'|_{\infty,\delta},$$

and we have $z' = 0$ in $|x| \leq \delta$. This implies again $(u, v) = (\bar{u}, \bar{v})$ near zero. □

In the sequel, we are mainly interested in determining when a positive solution to (2.1) blows up, i.e. $\omega < +\infty$. Before stating a first partial result in that direction let us introduce some notation. For a positive solution (u, v) to (2.1) defined in a noncontinuable interval $[0, \omega)$ we are fixing the notation:

$$\omega = T(u_0, v_0).$$

At the moment $T(u_0, v_0)$ could be possibly $+\infty$ in some circumstances.

In the next statement we have in mind that (as will be shown in §§3 and 4) blow up is only possible in the regime $pq > 1$.

Lemma 2.6. *Assume $pq > 1$ and suppose that a certain solution to (2.1) corresponding to positive initial data (u_0^*, v_0^*), blows up in finite time, i.e. $T(u_0^*, v_0^*) < \infty$. Then for every positive initial data (u_0, v_0) the solution (u, v) to (2.1) also blows up in finite time. In other words,*

$$T(u_0, v_0) < \infty$$

for every $u_0, v_0 > 0$. Moreover, the following scaling property holds:

$$T(\lambda u_0, \lambda^{(q+1)/(p+1)} v_0) = \lambda^{-(pq-1)/2(p+1)} T(u_0, v_0), \qquad \lambda > 0. \tag{2.4}$$

Proof. To begin with, it is easily seen that regardless the values of p and q,

$$(u_\lambda, v_\lambda) = (\lambda u(kx), \lambda^{(q+1)/(p+1)} v(kx)), \qquad k = \lambda^{(pq-1)/2(p+1)}, \tag{2.5}$$

solves (2.1) with data $(\lambda u_0, \lambda^{(q+1)/(p+1)} v_0)$, $\lambda > 0$ arbitrary, provided that (u, v) solves (2.1). In this way, $(u(\cdot, u_0^*, v_0^*), v(\cdot, u_0^*, v_0^*))$ gives rise to a family of solutions blowing up at times $x = \lambda^{-(pq-1)/2(p+1)} T(u_0^*, v_0^*)$. Thus, given any positive solution (u, v) to (2.1) with data (u_0, v_0) it is possible to find λ so small as to have,

$$\lambda u_0^* < u_0 \qquad \lambda^{(q+1)/(p+1)} v_0^* < v_0,$$

and Lemma 2.3 will imply that (u, v) blows up at $x = T(u_0, v_0)$ satisfying:

$$0 < T(u_0, v_0) \le \lambda^{-(pq-1)/2(p+1)} T(u_0^*, v_0^*).$$

This proves the lemma. □

Remarks 2.7.

a) As a consequence of the analysis in §§3 and 4 it will follow that $pq > 1$ is a necessary and sufficient condition for the blow up of all positive solutions to (2.1) including those corresponding to semitrivial data of the form $(u_0^*, 0)$ or $(0, v_0^*)$. In fact, in either of the latter cases those solutions remain *positive* for $x > 0$, a regime where such analysis holds valid.

b) An important consequence of the proof of Lemma 2.3 is the fact that $T(\bar{u}_0, \bar{v}_0) \le T(u_0^*, v_0^*)$ provided $u_0^* \le \bar{u}_0, v_0^* \le \bar{v}_0$. We are now further proving that $T(\bar{u}_0, \bar{v}_0) < T(u_0^*, v_0^*)$ if $u_0^* < \bar{u}_0$, $v_0^* < \bar{v}_0$. Indeed, if in that case $T(\bar{u}_0, \bar{v}_0) = T(u_0^*, v_0^*) := L$ then $T(u_0, v_0) = L$ in the rectangle $[u_0^*, \bar{u}_0] \times [v_0^*, \bar{v}_0]$. However, the curve

$$\frac{v_0}{v_0^*} = \left(\frac{u_0}{u_0^*} \right)^{(q+1)/(p+1)}$$

gets into the rectangle while (2.4) says that T strictly decreases along it, which is not possible. Thus, the assertion follows.

c) It is well known from the theory of ordinary differential equations that the upper and lower limits of the maximal interval of existence of a solution are only semicontinuous functions of their initial data (see [21]). In our specific case, $\liminf_{(u_0', v_0') \to (u_0, v_0)} T(u_0', v_0') \ge T(u_0, v_0)$. However, it is already known that $T(u_0', v_0') \le T(u_0, v_0)$ when $u_0 \le u_0', v_0 \le v_0'$. Therefore,

$$\liminf_{(u_0', v_0') \to (u_0, v_0)} T(u_0', v_0') = T(u_0, v_0),$$

for every (u_0, v_0). Nevertheless, it is pointed out here that the continuity of $T(u_0, v_0)$ will follow from the analysis in §3 (Corollary 3.5).

3. Blow-up for solutions to (2.1). Proof of Theorem 1.1.

Our main purpose in this section is stating a basic result (Theorem 3.1 below), which is crucial to elucidate the blow up property for the nonnegative (nontrivial) solutions to (2.1). We are also drawing some important conclusions from it.

Let $(u(x), v(x))$, $0 \le x < \omega$ be any noncontinuable solution to (2.1) corresponding to nonnegative initial conditions $(u_0, v_0) \neq (0,0)$ (for the moment $\omega = +\infty$ is still possible). Writing $u_1 = u'(x)$, $v_1 = v'(x)$ where $' = d/dx$, (u, u_1, v, v_1) solves the equivalent problem,

$$\begin{cases} u' = u_1 & u(0) = u_0 \\ u_1' = v^p & u_1(0) = 0 \\ v' = v_1 & v(0) = v_0 \\ v_1' = u^q & v_1(0) = 0. \end{cases} \tag{3.1}$$

According to Lemma 2.1, u, u_1, v, v_1 and their derivatives up to the second order are all positive. Since (u_0, v_0) is nontrivial and rôles played by u and v are symmetric there is no loss of generality if $v_0 \neq 0$ is assumed in the sequel. In particular we can express $x = x(u_1)$ (the inverse of $x \mapsto u_1$) where $0 \le x < \omega$ as $0 < u_1 < +\infty$. Then $u(u_1) = u(x(u_1))$, $v(u_1) = v(x(u_1))$, $v_1(u_1) = v_1(x(u_1))$ define the solution to the problem,

$$\begin{cases} \dfrac{du}{du_1} = \dfrac{u_1}{v^p} & u(0) = u_0 \\ \dfrac{dv}{du_1} = \dfrac{v_1}{v^p} & v(0) = v_0 \\ \dfrac{dv_1}{du_1} = \dfrac{u^q}{v^p} & v_1(0) = 0. \end{cases} \tag{3.2}$$

It also follows from Lemma 2.1 that $u(u_1), v(u_1), v_1(u_1)$ all diverge to $+\infty$ as $u_1 \to +\infty$. We state next our main result in this section. For the sake of clarity, its proof will be postponed until Section 4.

Theorem 3.1. *For $u_0 \ge 0$, $v_0 > 0$ let (u, v, v_1) be the solution to (3.2). Then $u(u_1), v(u_1)$, $v_1(u_1)$ satisfy the following asymptotic estimates,*

$$\lim_{u_1 \to +\infty} \frac{u}{A u_1^{\alpha}} = 1, \qquad \lim_{u_1 \to +\infty} \frac{v}{B u_1^{\beta}} = 1, \qquad \lim_{u_1 \to +\infty} \frac{v_1}{C u_1^{\gamma}} = 1, \tag{3.3}$$

where the exponents α, β, γ are given by,

$$\alpha = \frac{2(p+1)}{pq + 2p + 1}, \qquad \beta = \frac{2(q+1)}{pq + 2p + 1}, \qquad \gamma = \frac{pq + 2q + 1}{pq + 2p + 1} \tag{3.4}$$

and (A, B, C) is the unique positive solution to,

$$\begin{cases} \alpha A B^p &= 1 \\ \beta B^{p+1} &= C \\ \gamma B^p C &= A^q. \end{cases} \tag{3.5}$$

Remark 3.2. We quote that $A = (\beta\gamma/\alpha^{\frac{2p+1}{p}})^{p/(pq+2p+1)}$, while an explicit expression for B and C will not be strictly required in what follows.

As a consequence of Theorem 3.1, we can completely answer the blow-up question for positive solutions to (2.1). This permits in turn to determine the nature of the set of positive symmetric solutions to (1.4), providing in addition exact asymptotic estimates for their profile near the boundary.

Corollary 3.3. *Let $(u, v) \neq (0, 0)$ be any nonnegative solution to (2.1) defined in a non-continuable interval $0 \leq x < \omega$. Then $\omega < +\infty$, and so blow up occurs, if and only if $pq > 1$.*

Proof. The solution $(u(x), v(x))$ gives rise to its orbital version $(u(u_1), v(u_1), v_1(u_1))$ which solves (3.2). Since $u(u_1)$ is increasing with $u \to +\infty$ as $u_1 \to +\infty$, then its inverse function $u_1 = U_1(u)$, $U_1(u_0) = 0$, has the same behavior. Observe in addition that,

$$\frac{du}{dx} = U_1(u),$$

thus,

$$\omega = \int_{u_0}^{+\infty} \frac{ds}{U_1(s)}. \tag{3.6}$$

Therefore $\omega < +\infty$ amounts to the convergence of the integral. Since, in view of Theorem 3.1

$$U_1(u) \sim \left(\frac{u}{A}\right)^{1/\alpha} \qquad u \to +\infty, \tag{3.7}$$

such convergence is equivalent to $\alpha < 1$ which can be rewritten as $pq > 1$. $\square$

Remark 3.4. The general solution $(U(u_1, u_0, v_0), V(u_1, u_0, v_0), V_1(u_1, u_0, v_0))$ to (3.2) is as smooth as the equation when observed as a mapping of (u_1, u_0, v_0). The same holds true with the inverse $U_1(u, u_0, v_0)$ of the function $u_1 \mapsto u$. Thus,

$$T(u_0, v_0) = \int_{u_0}^{+\infty} \frac{ds}{U_1(s, u_0, v_0)} = \int_{0}^{+\infty} \frac{ds}{U_1(s - u_0, u_0, v_0)},$$

and upon this basis the continuity of T with respect to (u_0, v_0) will be shown. We are now stating this fact, and postponing its proof until Section 4.

Corollary 3.5. *The function $T(u, v)$ defined in §2 is continuous.*

An important consequence of the continuity of T and the scale invariance of the equation is that we can completely determine the set of initial data (u_0, v_0) for symmetric solutions to (1.4) in the interval $-L < x < L$.

Corollary 3.6. *Assume that $pq > 1$ and choose $L > 0$ arbitrary. Then the set Γ consisting of all nonnegative (u_0, v_0) such that:*

$$T(u_0, v_0) = L, \tag{3.8}$$

defines a continuous arc $\{(u_0(\sigma), v_0(\sigma)) \in \mathbb{R}^2_+ : 0 \leq \sigma \leq 1\}$ joining a certain point $(u_0, v_0)_{|\sigma=0} = (0, V_L)$, $V_L > 0$, to $(u_0, v_0)_{|\sigma=1} = (U_L, 0)$, $U_L > 0$, so that $u_0(\sigma)$ is nondecreasing while $v_0(\sigma)$ is nonincreasing. In particular $\Gamma \subset [0, U_L] \times [0, V_L]$ (Figure 2).

Proof. By Theorem 3.1 and since $pq > 1$, an arbitrary solution to (2.1) with initial conditions $(u_0, 0)$ blows up. The same holds true with the solution corresponding to $(\lambda u_0, 0)$, $\lambda > 0$. Then the scaling property (2.4) provides a unique $\lambda_1 > 0$ such that:

$$T(\lambda_1 u_0, 0) = L.$$

We set $U_L = \lambda_1 u_0$. By the same reasoning, for fixed $v_0 > 0$ a positive λ_2 can be found so that, putting $V_L = \lambda_2^{(q+1)/(p+1)} v_0$, the equality

$$T(0, V_L) = L$$

holds. Thus, it follows from Lemma 2.3 and Remark 2.7 b) that $T(u_0, v_0) < L$ whenever $u_0 > U_L$ or $v_0 > V_L$, and so $0 \le u_0 \le U_L$, $0 \le v_0 \le V_L$ for every (u_0, v_0) satisfying (3.8). Therefore, Γ is bounded and meets the u_0 and v_0 axes respectively at $(U_L, 0)$, $(0, V_L)$.

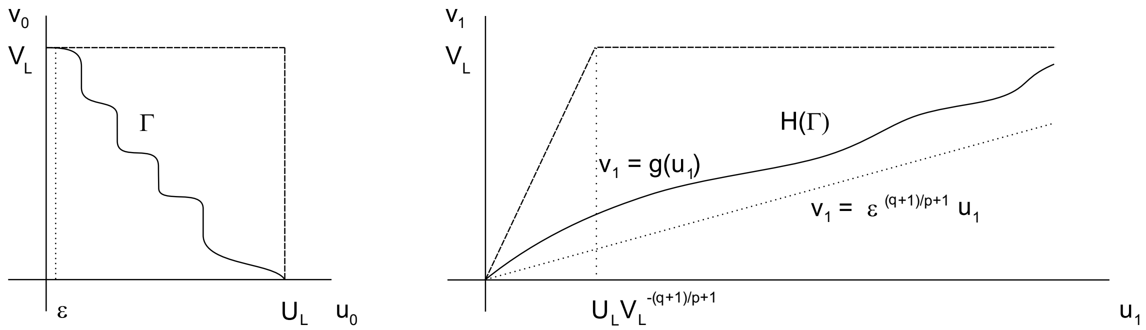

FIGURE 2. The set Γ in the original variables and after the change $u_1 = u_0^{-(q+1)/(p+1)} v_0$, $v_1 = v_0$.

We now claim that for every $0 < u_0 < U_L$ there is at least a (u_0, v_0) such that $T(u_0, v_0) = L$. Indeed, $T(u_0, 0) > L$ if $u_0 < U_L$, $T(u_0, V_L) \le L$ while in addition $T(u_0, \cdot)$ is continuous in $0 \le v_0 \le V_L$. Hence, the claim follows. Unfortunately, it is not possible to ensure the uniqueness of such solution for each u_0.

To study the form of Γ in the set $Q := [0, U_L] \times [0, V_L]$ it is convenient to introduce the coordinates:

$$u_1 = u_0^{-(q+1)/(p+1)} v_0, \qquad v_1 = v_0, \qquad u_0 > 0$$

and define H as the mapping $(u_0, v_0) \mapsto (u_1, v_1)$. Then, the image of $Q_1 = Q \setminus \{u_0 = 0\}$ is,

$$H(Q_1) = \{(u_1, v_1) : u_1 > 0,\ 0 \le v_1 \le \min\{U_L^{(q+1)/(p+1)} u_1, V_L\}\}$$

(see Figure 2). On the other hand, it is implicit in (2.4) that every curve $u_1 = c$, $c > 0$, contains exactly a unique point (u_0, v_0) with $T(u_0, v_0) = L$. Thus, to every $u_1 > 0$ corresponds a unique solution $(u_1, v_1) \in H(Q_1)$ of the transformed equation,

$$T_1(u_1, v_1) = L,$$

where $T_1(u_1, v_1) = T((u_1^{-1} v_1)^{(p+1)/(q+1)}, v_1)$. The continuity of T and a compactness argument then show that the image $H(\Gamma)$ of Γ is the graph of a continuous function $v_1 = g(u_1)$, $u_1 > 0$.

Remark 2.7 b) further implies that g is nondecreasing. Finally, as $(U_L, 0)$ and $(0, V_L)$ are the unique solutions of (3.8) in the u_0, v_0 axes, then it follows from the continuity of T that $\lim_{u_1\to 0+} g(u_1) = 0$ while $\lim_{u_1\to +\infty} g(u_1) = V_L$. Thus, $\Gamma = H^{-1}\{(u_1, g(u_1)) : u_1 > 0\} \cup \{(U_L, 0), (0, V_L)\}$. This proves the lemma. □

A final consequence of Theorem 3.1 is the rate of blow-up of solutions to (1.4):

Corollary 3.7. *Assume $pq > 1$, and let (u, v) be a positive solution to (1.4). Then*

$$\begin{aligned} u(x) &\sim (\xi(\xi+1)\eta^p(\eta+1)^p)^{\frac{1}{pq-1}} d(x)^{-\xi} \text{ as } x \to \pm L, \\ v(x) &\sim (\eta(\eta+1)\xi^q(\xi+1)^q)^{\frac{1}{pq-1}} d(x)^{-\eta} \text{ as } x \to \pm L, \end{aligned} \tag{3.9}$$

where $d(x) = \min\{L - x, L + x\}$, and $\xi = 2(p+1)/(pq-1)$, $\eta = 2(q+1)/(pq-1)$.

Proof. From the proof of Corollary 3.3 (replacing ω by L) it follows that:

$$\int_{u(x)}^{\infty} \frac{ds}{U_1(s)} = L - x.$$

Estimate (3.7) then yields:

$$u(x) \sim \left(\frac{\alpha A^{\frac{1}{\alpha}}}{1-\alpha}\right)^{\frac{\alpha}{1-\alpha}} (L-x)^{-\frac{\alpha}{1-\alpha}}$$

as $x \to L$, and similarly as $x \to -L$. A little algebra and the use of the expression of A provided in Remark 3.2 leads to the first in (3.9). The second is shown in a similar way. This proves the Corollary. □

Proof of Theorem 1.1. Corollaries 3.3 and 3.6 both imply that $pq > 1$ is a necessary and sufficient condition for the solvability of (1.4) in any domain, together with point ii) in Theorem 1.1. The argument at the beginning of the proof of Corollary 3.6 implies the uniqueness assertion in i). In this case observe that $(U_1(x), V_1(x))$, $(U_2(x), V_2(x))$ are the solutions to (2.1) corresponding to initial data $(U_L, 0)$, $(0, V_L)$, respectively. Finally, the asymptotic profile of the symmetric positive solutions near $x = \pm L$ has been obtained in Corollary 3.7. □

4. Proof of Theorem 3.1

Proof of theorem 3.1. Let $(u(x), v(x))$ be the solution to (2.1) corresponding to initial conditions $u_0 \geq 0, v_0 > 0$ while $(u(u_1), v(u_1), v_1(u_1))$, $u_1 \geq 0$, stands for the associated (orbital) solution to (3.2). As pointed out earlier $u(u_1) \to +\infty$, $v(u_1) \to +\infty$, $v_1(u_1) \to +\infty$ as $u_1 \to +\infty$.

Let us introduce now the normalization,

$$u = X(u_1)u_1^{\alpha} \qquad v = Y(u_1)u_1^{\beta} \qquad v_1 = Z(u_1)u_1^{\gamma}, \qquad u_1 > 0, \tag{4.1}$$

where α,β,γ are the exponents introduced in (3.4). Then, the coefficients X,Y,Z satisfy the equation,

$$u_1\frac{dX}{du_1}=\frac{1}{Y^p}-\alpha X,\qquad u_1\frac{dY}{du_1}=\frac{Z}{Y^p}-\beta Y,\qquad u_1\frac{dZ}{du_1}=\frac{X^q}{Y^p}-\gamma Z,\qquad u_1>0,$$

which, after the change $u_1=e^t$, can be written as the autonomous equation,

$$\begin{cases}\dfrac{dX}{dt}=\dfrac{1}{Y^p}-\alpha X\\[2ex] \dfrac{dY}{dt}=\dfrac{Z}{Y^p}-\beta Y & t\in\mathbb{R}.\\[2ex] \dfrac{dZ_1}{dt}=\dfrac{X^q}{Y^p}-\gamma Z\end{cases}\tag{4.2}$$

It should be observed that positive solutions (u,v,v_1) to (3.1) give rise to positive solutions (X,Y,Z) to (4.2) and conversely. On the other hand, the octant $\mathbb{R}^3_+=\{X>0,Y>0,Z>0\}$ is invariant for (4.2). Thus, positive solutions to (4.2) are characterized as those (X,Y,Z) having positive initial data. Finally, notice that $P=(A,B,C)$ (A,B,C given by (3.5)) is the unique positive equilibrium to (4.2).

In order to prove the estimates (3.3) it must then be shown that *every* positive solution (X,Y,Z) to (4.2) is attracted by the equilibrium P, i. e.,

$$(X(t),Y(t),Z(t))\to(A,B,C)\qquad t\to+\infty.\tag{4.3}$$

Thus, a first step in this direction is checking the local stability of P. The linearization of equation (4.2) at $P=(A,B,C)$ has the coefficients matrix,

$$\mathcal{A}=\begin{pmatrix}-\alpha & -p\dfrac{\beta\gamma}{\alpha}A^{-(q+1)} & 0\\[2ex] 0 & -\beta(p+1) & \alpha A\\[2ex] \alpha qA^q & -p\dfrac{\beta\gamma}{\alpha}A^{-1} & -\gamma\end{pmatrix}.\tag{4.4}$$

Its characteristic polynomial is $p(\lambda)=\lambda^3+a_1\lambda^2+a_2\lambda+a_3$, with coefficients given by,

$$\begin{aligned}a_1&=\alpha+\beta(p+1)+\gamma\\ a_2&=\alpha\beta(p+1)+\alpha\gamma+\beta\gamma(2p+1)\\ a_3&=\alpha\beta\gamma(pq+2p+1).\end{aligned}\tag{4.5}$$

Since a_1,a_3 are positive, the remaining Routh-Hurwitz condition for asymptotic stability $a_1a_2-a_3>0$ reads as,

$$\{\alpha+\beta(p+1)+\gamma\}\{\alpha\beta(p+1)+\alpha\gamma+\beta\gamma(2p+1)\}>\alpha\beta\gamma(pq+2p+1),$$

which is equivalent to

$$\{2(p+1)+2(p+1)(q+1)+pq+2q+1\}\times$$
$$\{4(p+1)^2(q+1)+2(p+1)(pq+2q+1)+2(2p+1)(q+1)(pq+2q+1)\}$$
$$> 4(p+1)(q+1)(pq+2p+1)(pq+2q+1) \qquad\qquad p,q>0.$$

A bit of computation shows that the last inequality holds since it amounts to the positivity of the sixth degree symmetric polynomial in p, q, $4p^3q^3+17p^2q^3+\cdots+34p^2+45p+18$ (omitted remaining eleven terms are positive!) when $p,q>0$. Anyway, the asymptotic stability of P will also be a byproduct of the analysis of the dynamics of (4.2) to be given in a moment.

However, we are proving a stronger property for P, as it is the condition of being a global attractor for all the positive solutions to (4.2). This will be shown in a series of steps. The first one consists in showing that

$$Z=h(X,Y) \qquad X>0,\ Y>0,$$

with

$$h(X,Y)=\frac{X^{q+1}}{q+1}+\frac{Y^{p+1}}{p+1},$$

defines a global and exponentially attracting invariant manifold $\mathcal{M}$ to (4.2). In fact, equations in (4.2) can be written as,

$$\left[\frac{X^{q+1}}{q+1}\right]'+\alpha X^{q+1}=\frac{X^q}{Y^p}, \qquad \left[\frac{Y^{p+1}}{p+1}\right]'+\beta Y^{p+1}=Z, \qquad Z'+\gamma Z=\frac{X^q}{Y^p}.$$

Observing that $\gamma+1=\alpha(q+1)=\beta(p+1)$ it follows that,

$$\left[e^{(\gamma+1)t}\Big(Z-\frac{X^{q+1}}{q+1}+\frac{Y^{p+1}}{p+1}\Big)\right]'=0,$$

which implies,

$$Z(t)=h(X(t),Y(t))-C_0e^{-(\gamma+1)t},$$

where $C_0=h(X(0),Y(0))-Z(0)$. This shows the stated property. Notice that P must lie in $\mathcal{M}$, what anyway is also a direct consequence of the equalities

$$\frac{A^{q+1}}{q+1}+\frac{B^{p+1}}{p+1}=\left\{\frac{\gamma}{\alpha(q+1)}+\frac{1}{\beta(p+1)}\right\}C\ , \qquad \frac{\gamma}{\alpha(q+1)}+\frac{1}{\beta(p+1)}=1.$$

Due to the attracting character of $\mathcal{M}$, a second step towards getting an insight of the global behavior of the dynamics of (4.2) is given by studying its restriction to $\mathcal{M}$, which is governed by the equation,

$$\begin{cases} X'=\dfrac{1}{Y^p}-\alpha X \\[2ex] Y'=\dfrac{1}{q+1}\dfrac{X^{q+1}}{Y^p}-\dfrac{\gamma}{p+1}Y. \end{cases} \tag{4.6}$$

It can be checked that the following assertions hold: a) the first quadrant $\mathbb{R}^2_+ = \{X > 0, Y > 0\}$ is invariant for (4.6), b) $P_{\mathcal{M}} = (A, B)$ is its unique positive equilibrium point, and c) the divergence,

$$\frac{\partial}{\partial X}\left(\frac{1}{Y^p} - \alpha X\right) + \frac{\partial}{\partial Y}\left(\frac{1}{q+1}\frac{X^{q+1}}{Y^p} - \frac{\gamma}{p+1}Y\right),$$

is negative in $X > 0, Y > 0$. This last property ensures that (4.6) can not exhibit closed orbits in $\mathbb{R}^2_+$.

On the other hand $P_{\mathcal{M}} = (A, B)$ is asymptotically stable for (4.6). Indeed the coefficients matrix of the linearized equation around $P_{\mathcal{M}}$ is,

$$\mathcal{A}_{\mathcal{M}} = \begin{pmatrix} -\alpha & -p\alpha\dfrac{A}{B} \\ \gamma\left(\dfrac{q+1}{p+1}\right)\dfrac{B}{A} & -\gamma \end{pmatrix},$$

with characteristic polynomial $p_{\mathcal{M}}(\lambda) = \lambda^2 + (\alpha+\gamma)\lambda + 2\gamma$. Therefore, its eigenvalues λ_1, λ_2 have both negative real parts.

It should be remarked now that, due to the invariant character of $\mathcal{M}$, λ_1, λ_2 are also eigenvalues of $\mathcal{A}$, given by (4.4). Thus, the third eigenvalue λ_3 is negative and given by (see (4.5)),

$$\lambda_3 = \frac{\det \mathcal{A}}{\lambda_1\lambda_2} = -\frac{\alpha\beta(pq + 2p + 1)}{2}.$$

In particular, this shows again the asymptotic stability of $P = (A, B, C)$.

Finally, due to the asymptotic stability of $P_{\mathcal{M}}$ and the fact that (4.6) does not admit closed orbits, a standard application of Poincaré-Bendixon theorem ([21]) implies that $P_{\mathcal{M}}$ attracts all positive solutions to (4.6), i.e., every positive solution (X, Y) satisfies,

$$(X(t), Y(t)) \to (A, B) \qquad \text{as } t \to +\infty.$$

To conclude the proof of the global attractiveness of $P = (A, B, C)$ we are combining in what follows both the attractiveness of $\mathcal{M}$ and the global attractiveness of $P_{\mathcal{M}}$ with respect to the positive solutions to the reduced equation (4.6) .

We begin with some previous technical remarks. For $P_0 = (X_0, Y_0, Z_0)$ set $\phi(t, P_0) = (X(t), Y(t), Z(t))$ the solution to (4.2) with $(X, Y, Z)_{|t=0} = P_0$. Then, performing a local rectification near P of the field associated to (4.2) it is possible to find a small neighborhood Ω of P, with smooth boundary $\partial\Omega$, such that if $\phi(t_0, P_0) \in \partial\Omega$ for a certain t_0, then $\phi(t, P_0) \in \Omega$ for later $t > t_0$ and in addition $\phi(t, P_0) \to P$ as $t \to +\infty$. In particular, the field $(X', Y', Z') = (Y^{-p} - \alpha X, ZY^{-p} - \beta Y, X^qY^{-p} - \gamma Z)$ points inward Ω at every $(X, Y, Z) \in \partial\Omega$.

Put $\Omega_{\mathcal{M}} = \Omega \cap \mathcal{M}$ and for any compact $K \subset \mathbb{R}^2_+$ set $K_{\mathcal{M}} = \{(X, Y, h(X, Y)) : (X, Y) \in K\}$. Since $\phi(t, P_0)$ reaches $\Omega_{\mathcal{M}}$ for every $P_0 \in K_{\mathcal{M}}$ an application (see [22]) of the Implicit Function Theorem leads to the existence of a neighborhood $\mathcal{N} = \{(X, Y, Z) \in \mathbb{R}^3_+ : \text{dist }((X, Y), K) \le \varepsilon,\ |Z - h(X, Y)| \le \delta\}$, $\varepsilon, \delta > 0$, of $K_{\mathcal{M}}$ and of a certain finite time t_K such that $\phi(t, P_0) \in \Omega$ for every $P_0 \in \mathcal{N}$ and $t > t_K$.

A first consequence is the fact that any positive semiorbit $\{\phi(t, P_0) : t \ge 0\}$ to (4.2) with $\{(X(t), Y(t)) : t \ge 0\} \subset K \subset \mathbb{R}^2_+$, for a certain compact K, satisfies $\phi(t, P_0) \to P$ as $t \to +\infty$. Indeed, since $\mathcal{M}$ is attracting, then $\phi(t, P_0) \in \mathcal{N}$ for large t.

As a second conclusion, if a positive semiorbit $\{\phi(t,P_0) : t \geq 0\}$ has $(X(t_n),Y(t_n)) \in K$ for a certain compact $K \subset \mathbb{R}^2_+$ and a sequence $t_n \to +\infty$ then again $\phi(t,P_0) \to P$ as $t \to +\infty$.

Now, let us choose any positive solution (X,Y,Z) to (4.2) coming –via (4.1)– from a solution to the initial value problem (3.1). We claim that,

$$Z(t) < h(X(t),Y(t)), \tag{4.7}$$

for $t < 0$ and $|t|$ large (a proof is delayed to the Remarks 4.1 below). As $\mathcal{M}$ is invariant this means that (4.7) holds for any t and so,

$$Z(t) = h(X(t),Y(t)) - C_0 e^{-(\gamma+1)t} \qquad t \in \mathbb{R},$$

with $C_0 > 0$. Therefore, the component $(X(t),Y(t))$ of any positive solution to (4.2) satisfies the following nonautonomous equation,

$$\begin{cases} X' = \dfrac{1}{Y^p} - \alpha X \\[2ex] Y' = \dfrac{1}{Y^p}\left(\dfrac{X^{q+1}}{q+1} - C_0 e^{-(\gamma+1)t} - \dfrac{\gamma}{p+1} Y^{p+1}\right). \end{cases} \tag{4.8}$$

For immediate use we are setting the t-variable sectors $S_{1,t} = \{X' > 0, Y' > 0\}$, $S_{2,t} = \{X' < 0, Y' > 0\}$, $S_{3,t} = \{X' < 0, Y' < 0\}$, $S_{4,t} = \{X' > 0, Y' < 0\}$, denoting by C_t the mobile null cline $Y' = 0$, namely,

$$Y = \left(\frac{p+1}{\gamma}\right)^{1/p+1} \left(\frac{X^{q+1}}{q+1} - C_0 e^{-(\gamma+1)t}\right)_+^{1/p+1} \qquad X > 0,$$

while (A_t, B_t) will designate the unique positive point where $X' = 0, Y' = 0$ at time t. Observe that $(A_t,B_t) \to (A,B)$ while the curve $C_t \to C$ as $t \to +\infty$, where C is the positive null cline $Y' = 0$ corresponding to (4.6), i. e. $X^{q+1}/(q+1) - (\gamma/p+1)Y^{p+1} = 0$.

Our immediate and final objective will be showing that $(X(t),Y(t)) \to (A,B)$ as $t \to +\infty$, whatever the positive solution (X,Y,Z) to (4.2) is. In what follows, several different possibilities will be considered, all of them leading to this conclusion. Our first remark is that,

$$(X(t),Y(t)) \in S_{3,t} \qquad \text{or} \qquad (X(t),Y(t)) \in S_{4,t}, \tag{4.9}$$

for $t < 0$ and $|t|$ large, provided, respectively, that in the initial conditions (u_0, v_0), $u_0 > 0$ or $u_0 = 0$ ($v_0 > 0$ in both cases) (see Remarks 4.1 below). We are assuming the former option in what follows since, in any case, the reasoning is unaffected by this initial assumption.

Next observe that if a solution (X,Y) to (4.8) satisfies $(X(t),Y(t)) \in S_{i,t}$ for t greater than a certain t_0 and some $1 \leq i \leq 4$, then (X,Y) becomes bounded and by monotonicity $(X(t),Y(t)) \to (A,B)$ as $t \to +\infty$. Since $Z = h(X,Y)$ then $(X,Y,Z) \to P$ as $t \to +\infty$.

Thus, the remaining option is that $(X(t),Y(t))$ leaves every $S_{i,t}$ after finite a time t and even so we are also showing in what follows that $(X,Y) \to (A,B)$ as $t \to +\infty$. Since (X,Y) starts at $S_{3,t}$ and we are supposing that it leaves $S_{3,t}$ at a finite time then only the following options are possible,

a) (X,Y) exits $S_{3,t}$ after reaching $X'=0$ at the component $0<X\le A$,

b) (X,Y) leaves $S_{3,t}$ at some time t after crossing $X'=0$ at $A<X\le A_t$ (Figure 3),

c) (X,Y) leaves $S_{3,t}$ after meeting C_t into the region $X'\le 0$.

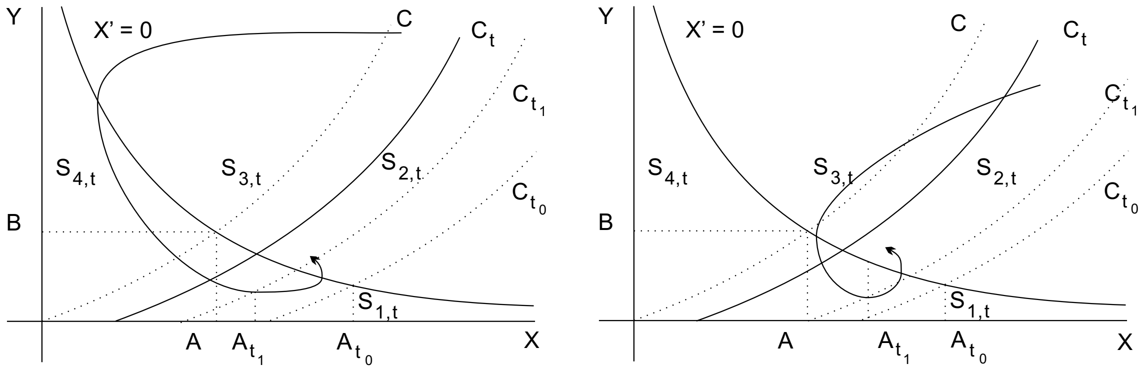

FIGURE 3. Options a) (left) and b) (right).

In case a), (X,Y) strictly enters $S_{4,t}$ and then exits this region at finite time after reaching the mobile null cline C_t at a first time $t=t_1$. It should be remarked that $(X(t_1),Y(t_1))\neq(A_{t_1},B_{t_1})$. Indeed, if on the contrary equality holds then $X'(t_1)=Y'(t_1)=X''(t_1)=0$ ($'=d/dt$) while,

$$Y''(t_1)=B_{t_1}^{-p}C_0(\gamma+1)e^{-(\gamma+1)t_1}>0 \qquad X'''(t_1)=-pB_{t_1}^{-p-1}Y''(t_1)<0.$$

Since $X(t)=X'''(t_1)(t-t_1)^3/6+o((t-t_1)^3)$ this means that $X'<0$ for $t\neq t_1$ being $Y'<0$ for $t<t_1$ and $Y'>0$ for $t>t_1$ ($0<|t-t_1|$ small). Thus the only way of reaching (A_t,B_t) at $t=t_1$ is having $X'<0, Y'<0$ (i. e., $(X(t),Y(t))\in S_{3,t}$) for $t<t_1$, $|t-t_1|$ small, and this is not the situation in the present case. Therefore, (X,Y) enters $S_{1,t}$ after $t=t_1$ and finally reaches $S_{2,t}$ at finite time. Exactly in the same way, the solution (X,Y) also reaches $S_{2,t}$ in case b) after crossing $\{X'=0,\ A\le X\le A_{t_0}\}$, $S_{4,t}$ and $S_{1,t}$, respectively.

In case c) define $t_1=\sup\{t:(X(\tau),Y(\tau))\in\overline{S_{3,t}}$ for each $\tau\le t\}$ and assume $t_1<+\infty$ since the opposite implies that $(X,Y)\to(A,B)$ as $t\to+\infty$. Then, there exists $t_2>t_1$ such that $(X,Y)_{|t=t_2}\in S_{2,t_2}$. Once (X,Y) reaches $S_{2,t}$ at $t=t_2$, the only possible options are the following,

i) $(X(t),Y(t))\in\overline{S_{2,t}\cup S_{3,t}}=\{X'\le 0\}$ for $t\ge t_2$ and possibly undergoing infinitely many transitions between $S_{2,t}$ and $S_{3,t}$ as $t\to+\infty$ (due to analyticity, only a finite number of transitions is possible for bounded t!),

ii) $(X(t),Y(t))\in S_{3,t}$ at some $t=t_3$, $t_3>t_2$ and (X,Y) follows option a) in $t\ge t_3$,

iii) $(X(t),Y(t))\in S_{3,t}$ for some $t=t_3'$, $t_3'>t_2$ and (X,Y) follows option b) in $t\ge t_3'$.

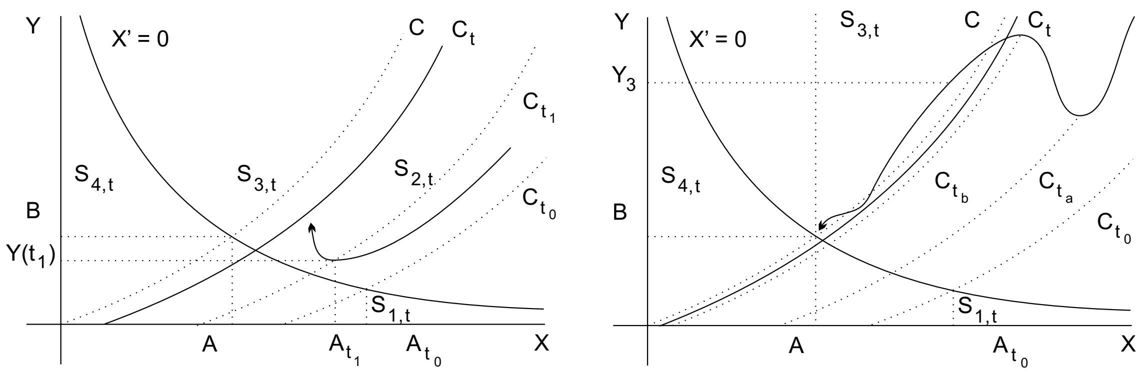

FIGURE 4. A transition $S_{3,t}$ to $S_{2,t}$ (left) and a possible behavior in case i) after transitions $S_{3,t}$ to $S_{2,t}$ and $S_{2,t}$ to $S_{3,t}$ at, respectively, $t = t_a$, $t = t_b$ (right).

In case i) we claim that (X, Y) remains bounded for $t \geq t_2$. In fact, we have on one hand that $Y \geq (\alpha X)^{-1/p}$ (the only possible contact of (X, Y) with the null cline $X' = 0$ is the point (A_t, B_t)). On the other hand $A < X(t) < X(t_2)$ for $t > t_2$ since $X(t) = A$ at a finite time t implies $Y(t) > B$ ($Y(t) = B$ leads to option a)) and this, in turn, forces (X, Y) to enter S_3 and reach $X' = 0$ at a finite later time (option a)) which is not the present case. In addition, either $\gamma Y^{p+1}/p + 1 \leq X^{q+1}/q + 1$ for $t \geq t_2$, in which case (X, Y) remains bounded, or either $\gamma Y^{p+1}(t_3)/p + 1 > X^{q+1}(t_3)/q + 1$ at a later time $t_3 > t_2$ (Figure 4). Since $Y'(t_3) < 0$ and so $Y(t) < Y_3 := Y(t_3)$ for $t > t_3$ (t close to t_3) then Y will never reach again the value Y_3 because $Y(t)$ should be decreasing when approaching Y_3 from below. Thus $Y(t) \leq Y_3$ for $t \geq t_3$ and then (X, Y) also remains bounded. As pointed out above, this entails $(X, Y) \to (A, B)$ as $t \to +\infty$.

In both options ii) or iii), (X, Y) returns to $S_{2,t}$ in finite time. Thus, (X, Y) may either perform a finite number of returnings to $S_{2,t}$ following ii) or iii) until it chooses option i) (and then (X, Y) asymptotically converges to (A, B)) or either (X, Y) returns infinitely many times to $S_{2,t}$ by means of one of the ways ii) or iii). However, in this last assumption, (X, Y) meets the compact segment $K_1 = \{(X, Y) : 0 \leq X \leq A,\ Y = B\}$ (see Remarks 4.1 below) every time it crosses $X' = 0$ in ii) while (X, Y) meets the compact arc $K_2 = \{(X, Y) : A \leq X \leq A_{t_0},\ Y^p = \alpha X\}$ every time it crosses $X' = 0$ in option iii). In both cases, (X, Y) passes through the compact $K_1 \cup K_2$ infinitely many times as $t \to +\infty$ and, as also remarked before, this entails that $(X, Y) \to (A, B)$ as $t \to +\infty$. This completes the proof of Theorem 3.1. $\square$

Proof of Corollary 3.5. Suppose with no loss of generality that $v_0 > 0$ and assume $(u_{0n}, v_{0n}) \to (u_0, v_0)$. To prove the continuity of T write (Corollary 3.3),

$$T(u_{0n}, v_{0n}) = \left\{ \int_0^\delta + \int_\delta^{\bar{u}} + \int_{\bar{u}}^{+\infty} \right\} \frac{ds}{U_1(s - u_{0n}, u_{0n}, v_{0n})}, \tag{4.10}$$

with $0 < \delta < \bar{u}$ to be chosen now. Regarding the first integral, the continuous dependence of the solution $(u, v, v_1) = (U(u_1, u_0, v_0), V(u_1, u_0, v_0), V_1(u_1, u_0, v_0))$ to (3.2) on (u_0, v_0), the fact that $v_0 > 0$ and so the v_{0n} are bounded away from zero together with,

$$\frac{U_1^2}{2} = \int_{u_{0n}}^{u} V(U_1(s, u_{0n}, v_{0n}), u_{0n}, v_{0n})^p \, ds,$$

imply that positive constants δ, k_1 and k_2, not depending on n, can be found so that,

$$2k_1(u-u_{0n}) \le U_1^2(u,u_{0n},v_{0n}) \le 2k_2(u-u_{0n}),$$

for all $|u-u_{0n}|\le\delta$ and $n\in\mathbb{N}$. Lebesgue's dominated convergence theorem then implies the convergence of the first integral in (4.10) to the corresponding one with u_0, v_0 replacing u_{0n}, v_{0n} in the integrand. The convergence of the second integral follows from the uniform convergence of $U_1(u,u_{0n},v_{0n}))$ to $U_1(u,u_0,v_0))$ in any bounded interval $J\subset R^+$. As for the third integral we rather deal with $U(u_1,u_{0n},v_{0n})$ instead of the inverse U_1 and write,

$$U(u_1,u_{0n},v_{0n}) = X(t,u_{0n},v_{0n})u_1^\alpha \qquad t=\log u_1.$$

We now have that,

$$(X(\cdot,u_{0n},v_{0n}),Y(\cdot,u_{0n},v_{0n}),Z(\cdot,u_{0n},v_{0n}))\to(X(\cdot,u_0,v_0),Y(\cdot,u_0,v_0),Z(\cdot,u_0,v_0))$$

in bounded intervals, in particular, for any fixed $t=t_1$. Since the last solution lies on the neighborhood Ω quoted in the proof of Theorem 3.1 for $t\ge\bar t>t_1$, then the same holds with the solutions for n large and $t\ge\bar t$ (by reducing $\bar t$ a little bit if necessary). We will have in particular that $X(t,u_{0n},v_{0n})\le A+\eta$, $\eta>0$ small, for $t\ge\bar t$ and so:

$$0<U(u_1,u_{0n},v_{0n})\le(A+\eta)u_1^\alpha \qquad u_1\ge e^{\bar t},$$

and, after an appropriate choice of $\bar u$ we will have,

$$\frac{1}{U_1(u,u_{0n},v_{0n})}\le\left(\frac{A+\eta}{u}\right)^{1/\alpha} \qquad u\ge\bar u,$$

for n large. This enables us to introduce the limit in the last integral and the continuity of T follows.

Remarks 4.1.

a) The power X^{q+1} in equation (4.6) could be replaced by $|X|^{q+1}$ and all semiorbits to this equation starting at the positive semiaxis $X=0, Y>0$ are still attracted by the point (A,B). This fact allows us including the extreme $(X,Y)=(0,B)$ in the construction of the compact K_1 for the purposes of the proof of Theorem 3.1.

b) We are showing now claims (4.7) and (4.9). First assume that u_0 and v_0 are both positive. By using the u_1 variable instead of t we observe that $X\simeq u_0u_1^{-\alpha}$, $Y\simeq v_0u_1^{-\beta}$ and $Z\simeq u_0^qv_0^{-p}u_1^{1-\gamma}$ as $u_1\to0+$ $(t\to-\infty)$. Then observe that,

$$u_1X'\simeq v_0^{-p}u_1^{p\beta}-\alpha u_0u_1^{-\alpha}\to-\infty\ , \qquad u_1Y'\simeq\{v_0^{-2p}u_0^qu_1^2-\beta v_0\}u_1^{-\beta}\to-\infty,$$

as $u_1\to0+$. This gives the first of (4.9). As for (4.7) observe that $\{X^{q+1}/(q+1)+Y^{p+1}/(p+1)\}\simeq\{u_0^{q+1}/(q+1)+v_0^{p+1}/(p+1)\}u_1^{-(\gamma+1)}$ which is greater than Z since $Z=O(u_1^{-(\gamma-1)})$ as $u_1\to0+$. Thus (4.7) holds.

Next assume that $u_0=0$, $v_0>0$. Now $u\simeq(v_0^{-p}/2)u_1^2$ while $v_1\simeq(v_0^{-p(q+1)}/\{2^q(2q+1)\})u_1^{2q+1}$ as $u_1\to0+$. Hence,

$$u_1X'\simeq\frac{v_0^{-p}}{2}(2-\alpha)u_1^{2-\alpha}>0, \qquad u_1Y'\simeq\left[\frac{v_0^{-p(q+2)}}{2^q(2q+1)}u_1^{2q+2}-\beta v_0\right]u_1^{-\beta}\to-\infty\ ,$$

as $u_1 \to 0+$ (observe that $0 < \alpha < 2$), which gives the second of (4.9). Finally, observe that

$$Z \simeq \frac{v_0^{-p(q+1)}}{2^q(2q+1)} u_1^{2q+1-\gamma},$$

while still $\{X^{q+1}/(q+1) + Y^{p+1}/(p+1)\} = O(u_1^{-(\gamma+1)})$ as $u_1 \to 0+$ and (4.7) holds again.

5. Radial solutions

For the study of radial solutions to problem (1.8) we require some preliminary properties concerning the initial value problem,

$$\begin{cases} u_{rr} + \dfrac{N-1}{r} u_r = v^p \\ v_{rr} + \dfrac{N-1}{r} v_r = u^q \\ u(0) = u_0, \quad u'(0) = 0, \quad v(0) = v_0, \quad v'(0) = 0, \end{cases} \tag{5.1}$$

$p, q > 0$, u_0, v_0 nonnegative. The first fact to be quoted is that such problem admits a noncontinuable solution $(u(r), v(r))$ defined in an interval $[0, \omega)$, $0 < \omega \le +\infty$ provided $(u_0, v_0) \ne (0, 0)$. In addition, both u, v are positive and increasing while $\lim_{r\to\omega} u = \lim_{r\to\omega} v = +\infty$. To show this we only need to solve (5.1) in a small interval $[0, \delta]$. Then the constructed local solution can be continued according to the standard ode's theory and the remaining assertions easily follow. Thus, to find a local solution observe that (5.1) can be equivalently written as,

$$u(r) = u_0 + \int_0^r \int_0^\rho \left(\frac{s}{\rho}\right)^{N-1} v(s)^p \, ds \, d\rho := \mathcal{T}_1(u, v),$$

$$v(r) = v_0 + \int_0^r \int_0^\rho \left(\frac{s}{\rho}\right)^{N-1} u(s)^q \, ds \, d\rho := \mathcal{T}_2(u, v).$$

By choosing $\delta > 0$ small so that $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2)$ maps $\{(u, v) \in (C[0, \delta])^2 : |u - u_0|_{\infty,\delta}, |v - v_0|_{\infty,\delta} \le \delta\}$ continuously into itself, Schauder's fixed point theorem provides a solution in $[0, \delta]$. In the regular cases where both u_0, v_0 are positive or either $p \ge 1$ if $v_0 = 0$ ($u_0 > 0$) or $q \ge 1$ if $u_0 = 0$ ($v_0 > 0$), Banach's fixed point can be used instead to get a unique solution in $[0, \delta]$.

On the other hand, it follows from the fixed point equation $(u, v) = (\mathcal{T}_1(u, v), \mathcal{T}_2(u, v))$ that solutions $(u(r), v(r))$, $(\tilde{u}(r), \tilde{v}(r))$ corresponding to initial data $(\tilde{u}_0, \tilde{v}_0) \ne (u_0, v_0)$ such that $u_0 \le \tilde{u}_0$, $v_0 \le \tilde{v}_0$ satisfy $u(r) < \tilde{u}(r)$, $v(r) < \tilde{v}(r)$ wherever they are both defined.

As another remark, for each $u_0, v_0 \ge 0$, (5.1) admits a unique solution (u, v) provided $pq > 1$. We only need checking this in the cases $(u_0, v_0) = (0, 0)$ or either when $u_0 = 0$, $v_0 > 0$ and $0 < q < 1$ (the complementary case $v_0 = 0$, $u_0 > 0$ with $0 < p < 1$ being identically handled). The proof of the former case is exactly that for the same case in Lemma 2.4. As for the latter case assume that (u, v), $(\tilde{u}, \tilde{v})$ solve (5.1) and set $(u_\delta(r), v_\delta(r)) = (u(r+\delta), v(r+\delta))$, $\delta > 0$ small. Then both $\tilde{u}(r) < u_\delta(r)$, $\tilde{v}(r) < v_\delta(r)$ for $0 \le r \le c$ and a certain $c > 0$. However, in the radial case one finds that $\{r^{N-1}(u_\delta - \tilde{u})'\}' \ge 0$ and $\{r^{N-1}(u_\delta - \tilde{u})'\}' \ge 0$ in

$[0, c]$. Since $u'_\delta(0), v'_\delta(0) > 0$ we conclude that $\tilde{u}(c) < u_\delta(c)$, $\tilde{v}(c) < v_\delta(c)$ and by the same token, those strict inequalities propagate to the common domains of definition of $(\tilde{u}, \tilde{v})$ and (u_δ, v_δ). We get $\tilde{u} \le u, \tilde{v} \le v$ by letting $\delta \to 0+$. The reverse inequality follows in the same way and the proof of uniqueness is concluded.

For immediate use, $T_\rho(u_0, v_0)$ will designate the right extreme ω of the maximal domain of existence of the solution to (5.1).

Proof of Theorem 1.2. We are first proving that $pq > 1$ is a necessary and sufficient condition in order that every nontrivial nonnegative solution to (5.1) blows up at a finite r, i. e. $T_\rho(u_0, v_0) < +\infty$ for every $u_0, v_0 \ge 0$, $(u_0, v_0) \ne (0, 0)$.

For the necessity assume that $(u(r), v(r))$ is positive and solves (5.1) with $pq \le 1$. Since both u, v are increasing,

$$u_{rr} \le u_{rr} + \frac{N-1}{r} u_r = v^p,$$

and similarly $v_{rr} \le u^q$ what implies that $u(r) \le \bar{u}(r)$, $v(r) \le \bar{v}(r)$, $(\bar{u}, \bar{v})$ being the solution of the one-dimensional problem (2.1) with same initial conditions as (u, v). The comparison gives $T(u_0, v_0) \le T_\rho(u_0, v_0)$ while $T(u_0, v_0) = +\infty$ if $pq \le 1$. Hence (u, v) does not blow up at a finite r.

As for the sufficiency suppose on the contrary that $T_\rho(u_0, v_0) = +\infty$ for the solution (u, v) to (5.1). Observe that the radial equations are again invariant under the scale change (2.5):

$$u_\lambda(r) = \lambda u(kr), \qquad v_\lambda(r) = \lambda^{\frac{q+1}{p+1}} v(kr), \qquad k = \lambda^{\frac{pq-1}{2(p+1)}},$$

with $\lambda > 0$, and so T_ρ satisfies the scaling relation (2.4). This means that we can assume that both u_0, v_0 are as large as desired.

On the other hand, observe that (5.1) can be written as $({}' = d/dr)$,

$$\begin{cases} r^{N-1}(r^{N-1}u')' = r^{2(N-1)}v^p, & u(0) = u_0, \quad u'(0) = 0 \\ r^{N-1}(r^{N-1}v')' = r^{2(N-1)}u^q, & v(0) = v_0, \quad v'(0) = 0, \end{cases} \tag{5.2}$$

which, after the change ([13]),

$$\rho = \begin{cases} \frac{1}{N-2}\{1 - \frac{1}{r^{N-2}}\} & N \ge 3 \\ \log r & N = 2, \end{cases}$$

takes the form $(\dot{} = d/d\rho)$,

$$\begin{cases} \ddot{u} = g(\rho)v^p, & u(-\infty) = u_0, \quad \dot{u}(-\infty) = 0 \\ \ddot{v} = g(\rho)u^q, & v(-\infty) = v_0, \quad \dot{v}(-\infty) = 0, \end{cases} \tag{5.3}$$

where $g(\rho) = r^{2(N-1)}$. Assume $N \ge 3$ in the sequel (the case $N = 2$ is slightly simpler). Observe that $-\infty < \rho < c_N := \dfrac{1}{N-2}$ in that case and the solution $(u(\rho), v(\rho))$ is defined in $(-\infty, c_N)$.

Fix $\rho_0 \in (-\infty, c_N)$. By using the scaling ideas in Section 2 it is possible to choose $u_0, v_0 > 0$ large enough so that the solution $(\tilde{u}(\rho), \tilde{v}(\rho))$ to the auxiliary problem,

$$\begin{cases} \ddot{u} = g(\rho_0)v^p, & \rho \ge \rho_0 \\ \ddot{v} = g(\rho_0)u^q, & \rho \ge \rho_0 \\ u(\rho_0) = u_0, \quad \dot{u}(\rho_0) = 0, \quad v(\rho_0) = v_0, \quad \dot{v}(\rho_0) = 0, \end{cases} \tag{5.4}$$

blows up at a time ρ_ω such that $\rho_0 < \rho_\omega < c_N$. Since $g(\rho)$ is increasing, $u(\rho_0) > u_0, v(\rho_0) > v_0$ and $\dot{u}(\rho_0), \dot{v}(\rho_0)$ are positive we conclude $u(\rho) > \tilde{u}(\rho)$ and $v(\rho) > \tilde{v}(\rho)$ for $\rho \geq \rho_0$. Thus (u, v) must blow up before ρ_ω, that is, at a finite r. This is just the opposite of what was assumed, the sufficiency being thus proved.

To conclude the proof of Theorem 1.2 observe that the scaling relation (2.4) implies the uniqueness assertion in i). Giving to $U_R, V_R > 0$ the same status as the given to the values U_L, V_L introduced in Corollary 3.8 then solutions $(U_1(r), V_1(r))$ and $(U_2(r), V_2(r))$ correspond, respectively, to initial data $(u_0, v_0) = (U_R, 0)$, $(0, V_R)$ in (5.1). The proof of ii) follows from the fact that T_ρ decreases if both u_0, v_0 increase. However, it should be observed that the continuity of T_ρ not being available, the corresponding continuity of Γ_R can not be ensured for the moment. □

6. NONSYMMETRIC SOLUTIONS: THE ONE-DIMENSIONAL CASE

Let us begin with some preliminary remarks. Regarding symmetry, suppose $(u(x), v(x))$ is nontrivial, nonnegative (i.e., both u, v nonnegative) and solves (1.4). It follows that u, v are strictly convex and so two unique $x_{\min}, y_{\min} \in (-L, L)$ exist such that $u(x_{\min}) = \inf u$, $v(y_{\min}) = \inf v$. Then, it can be checked that (u, v) is symmetric if and only if $x_{\min} = y_{\min}$. In such case, $x_{\min} = y_{\min} = 0$. Thus $x_{\min} \neq y_{\min}$ provides a test for nonsymmetry.

A second remark concerns the extended problem (2.3). In the range $pq > 1$, denote by $(u(x), u_1(x), v(x), v_1(x))$ the noncontinuable solution to the initial value problem (see Theorem 2.5),

$$\begin{cases} u' = u_1 & u(0) = u_0 \\ u_1' = |v|^p & u_1(0) = u_0' \\ v' = v_1 & v(0) = v_0 \\ v_1' = |u|^q & v_1(0) = v_0', \end{cases} \qquad \left(' = \frac{d}{dx} \right) \tag{6.1}$$

defined in (ω_1, ω_2) with (u_0, u_0', v_0, v_0') arbitrary (specially, regarding sign). Suppose that u, v become positive with u', v' positive (respectively, negative) at some $x = x_1$. Then, it follows from the proof of Theorem 3.1 that ω_2 (ω_1) is finite and hence such solution blows up there. On the contrary, no solution to (6.1) can exhibit this behavior when $pq \leq 1$.

A further feature has also to do with characterizing the finiteness of both ω_i, $i = 1, 2$. Notice that the group $u_1 v_1 - h(u, v)$, $h(u, v) = |u|^q u/(q+1) + |v|^p v/(p+1)$, keeps constant on solutions of (6.1).

Lemma 6.1. *Suppose $pq > 1$ and let (u, v) be a noncontinuable solution to (2.3) defined on (ω_1, ω_2). If*

$$u'v' - h(u, v) \neq 0 \tag{6.2}$$

at some x, then both ω_1, ω_2 *are finite. Moreover, this property is equivalent to (6.2) if (u, v) is nonnegative and nontrivial.*

Proof. It suffices with studying the behavior at ω_2 (otherwise, perform the change $x \mapsto -x$). Two alternative options will be considered in turn: a) u', v' negative in (ω_1, ω_2) and b) there exists x_0 where either $u' \geq 0$ or $v' \geq 0$.

Case a) is not compatible with (6.2) since fixing x_1 we get by convexity $u'(x_1) < u'(x) < 0$, $v'(x_1) < v'(x) < 0$ together with $u(x_1) + u'(x_1)(x - x_1) < u(x) < u(x_1)$, $v(x_1) + v'(x_1)(x -$

$x_1) < v(x) < v(x_1)$ for $x_1 < x < \omega_2$. Thus, $\omega_2 = \infty$. The existence of $\lim u'$, $\lim v'$ as $x \to \infty$ and the identities $u'(x) = u'(x_1) + \int_{x_1}^x |v|^p$, $v'(x) = v'(x_1) + \int_{x_1}^x |u|^q$ imply the convergence of $\int_{x_1}^\infty |v|^p$, $\int_{x_1}^\infty |u|^q$ and hence $u \to 0$, $v \to 0$ as $x \to \infty$. Therefore, $\lim u' = \lim v' = 0$ as $x \to \infty$, $u'v' - h(u,v) \to 0$ as $x \to \infty$ and (6.2) can not hold, as we wanted to prove.

As for b), suppose for instance $u'(x_0) \geq 0$. Then $u' > 0$ for $x_0 < x < \omega_2$ since due to (6.2) (u,v) is nontrivial. If in addition $v' \geq 0$ at some $x_2 \geq x_0$ then u', v' are positive in $x > x_2$ and by convexity one finds that both u, v must become nonnegative at some $x > x_2$. In view of the preliminary remark $\omega_2 < \infty$. The opposite option, $v' < 0$ for $x_0 \leq x < \omega_2$ can not occur. In fact, one has that v remains bounded for x bounded and, by means of (6.1), the same happens to u, u', u''. In conclusion $\omega_2 = \infty$ and as before $\int_{x_0}^\infty |u|^q < \infty$ implying $u \to 0$ as $x \to \infty$. However, strict convexity of u is not compatible with that behavior.

We finally show that (6.2) is necessary to have both ω_1, ω_2 finite in case of nonnegative solutions. In fact, if $(u,v) \neq (0,0)$ violates (6.2) somewhere then $u'v' = h(u,v)$ for all x and $u'v'$ is positive. It can be assumed without loss of generality that both u', v' are negative in (ω_1, ω_2) and hence, as shown in a), $\omega_2 = \infty$ (observe that ω_1 must be finite in this case). □

Remarks 6.2.

a) The necessity of (6.2) for the finiteness of ω_1, ω_2 fails since symmetric two-signed solutions to (2.3) can be constructed so that $u'v' = h(u,v)$ for all x while ω_1 and ω_2 are finite.

b) The functions

$$u(x) = \frac{a}{x^\xi} \qquad v(x) = \frac{b}{x^\eta} \qquad x > 0,$$

with ξ, η the exponents and a, b the constants in (1.7), provide an explicit example of solution behaving as in the last part of the proof of the lemma. By the way, they define the "prototype" positive solution to problem (1.4) in an unbounded interval, say $(0,\infty)$.

The proof of Theorem 1.3 is a consequence of the next perturbation result.

Theorem 6.3. *Suppose $pq > 1$ and $u_0 > 0$, $v_0 \geq 0$. Consider the problem,*

$$\begin{cases} u'' = |v|^p \\ u(0) = u_0 \\ u'(0) = \sigma \end{cases} \qquad \begin{cases} v'' = |u|^q \\ v(0) = v_0 \\ v'(0) = 0 \end{cases} \tag{6.3}$$

which can be regarded as a perturbation of problem (2.3) controlled by the parameter σ. Then, there exist $\sigma_1^* > 0$ and continuous functions $\omega^+, \omega^- : [-\sigma_1^*, \sigma_1^*] \to \mathbb{R}_+$, ω^+ nonincreasing, ω^- nondecreasing such that,

i) the noncontinuable solution $(u(x,\sigma), v(x,\sigma))$ to (6.3) is positive for $x \neq 0$ if and only if $|\sigma| < \sigma_1^*$ being $(-\omega^-(\sigma), \omega^+(\sigma))$ its definition domain in that range for σ.

ii) $(u(x,-\sigma), v(x,-\sigma)) = (u(-x,\sigma), v(-x,\sigma))$ for all $\sigma \in \mathbb{R}$.

iii) If $x_{\min}(\sigma)$ is defined as $u(x_{\min}(\sigma),\sigma) = \inf u(\cdot,\sigma) := h(\sigma)$ then $x_{\min}$ and h are continuous in $|\sigma| \leq \sigma_1^*$, C^1 in $\sigma \neq 0$, $x_{\min}(-\sigma) = -x_{\min}(\sigma)$, $h(-\sigma) = h(\sigma)$, $h(\pm\sigma_1^*) = 0$ while both $x_{\min}$ and h decrease in $0 \leq \sigma \leq \sigma_1^*$. Moreover, $\inf v(\cdot,\sigma) = v_0$ for all σ.

iv) $(u(\cdot,\sigma), v(\cdot,\sigma))$ *blows up at* $x = -\omega^-, \omega^+$ *for every* $|\sigma| \leq \sigma_1^*$. *Moreover,*

$$u(x,\sigma) \sim \frac{a}{d(x)^\xi} \qquad v(x,\sigma) \sim \frac{b}{d(x)^\eta},$$

as $d(x) = \min\{\omega^+ - x, x + \omega^-\} \to 0+$, *where* a, b, ξ, η *are the coefficients and exponents involved in Theorems 1.1 and 1.3.*

v) As a function of (u_0, v_0) *in* $u_0, v_0 \geq 0$, σ_1^* *is continuous and separately increasing in* u_0 *and* v_0. *Moreover,* σ_1^* *vanishes at* $u_0 = 0$.

Proof. Since ii) follows from uniqueness (Theorem 2.5) and changing x by $-x$ in (6.3), only the case $\sigma \geq 0$ needs to be studied. In addition, as condition (6.2) in Lemma 6.1 holds (check at $x = 0$) the noncontinuable solution (u, v) to (6.3) has both ω_1, ω_2 finite, regardless $\sigma \in \mathbb{R}$. This fact and continuous dependence on σ assure the positivity of the number,

$$\sigma_1^* = \sup\{\sigma \geq 0 : \inf u(\cdot,\sigma) > 0\}, \tag{6.4}$$

where we remark that, whenever possible, writing the dependence of (u, v) on σ will be avoided in the sequel. By convexity one finds $x_{\min} < 0$ for $\sigma > 0$ while the implicit function theorem gives $x_{\min}(\sigma)$ is C^1 in $\sigma > 0$.

On the other hand, it follows by comparison (Lemma 2.3) that both $u(x,\sigma)$, $v(x,\sigma)$ decrease (respectively, increase) when σ increases in each $x < 0$ ($x > 0$) provided $\inf u \geq 0$. We next use this fact to show that $\sigma_1^* < \infty$. Otherwise, for a fixed $x_0 < 0$ we have $u > 0$ in $x_0 \leq x \leq 0$ for all $\sigma > 0$. The estimate $v_0 < v(x,\sigma) < v(x,0)$ in $(x_0, 0]$ together with,

$$u_x(x_0,\sigma) = \sigma - \int_{x_0}^0 |v|^p,$$

imply that $u_{x|x=x_0} \sim \sigma$ as $\sigma \to \infty$. That is not possible since $u_0 > -u_x(x_0,\sigma)x_0$. Thus, $\sigma_1^* < \infty$. Remark that Lemma 6.1 implies that $x_{\min} > -\infty$ at $\sigma = \sigma_1^*$. Furthermore, σ_1^* is the unique $\sigma > 0$ such that $\inf u = 0$. In fact, for $\sigma = \sigma_1^*$,

$$-x_{\min} = \frac{1}{\sigma}\{u_0 + \int_{x_{\min}}^0 \int_t^0 v^p\}.$$

If $\inf u = 0$ for $\sigma > \sigma_1^*$ then $x_{\min}(\sigma) = x_{\min}(\sigma_1^*)$ while both $u(\cdot,\sigma)$, $u(\cdot,\sigma_1^*)$ are positive in $(x_{\min}(\sigma_1^*), 0]$. That contradicts the fact that the above expression for $x_{\min}$ decreases with σ.

It is clear that $h(\sigma) = \inf u(\cdot,\sigma)$ decreases in $0 \leq \sigma \leq \sigma_1^*$ while for $0 \leq \sigma_1 < \sigma_2 \leq \sigma_1^*$ the option $x_{\min}(\sigma_2) \geq x_{\min}(\sigma_1)$ can not happen since:

$$\sigma_2 = \int_{x_{\min}(\sigma_2)}^0 v(\cdot,\sigma_2) \leq \int_{x_{\min}(\sigma_1)}^0 v(\cdot,\sigma_2) < \int_{x_{\min}(\sigma_1)}^0 v(\cdot,\sigma_1) = \sigma_1,$$

against the initial assumption. Thus $x_{\min}$ decreases in $[0, \sigma_1^*]$.

The remaining assertions in i)-iv) can be shown by setting $\omega^- = -\omega_1, \omega^+ = \omega_2$ and observing that, from the proof of Theorem 3.1, blow-up rates in iv) are common for all positive solutions (u, v) existing only up to some finite value $x = \omega$. As in Corollary 3.5, these estimates yield the continuity of $\omega^\pm$.

Regarding the continuity of $\sigma_1^*(u_0,v_0)$ it firstly follows from Lemma 2.3 that σ_1^* increases separately in u_0,v_0. Thus if $P_n=(u_{0n},v_{0n})\to P_0=(u_0,v_0)$, then $\sigma_{1n}^*:=\sigma_1^*(P_n)$ remains bounded. We are showing that any limit point of σ_{1n}^* must be σ_1^*, hence $\sigma_1^*(P_n)\to\sigma_1^*$ as desired. Assume that for a certain subsequence $\sigma_{1n'}^*\to\sigma_\infty$. Redefining n' as n set $(u_n(x),v_n(x))$ the solution to (6.3) corresponding to initial data u_{0n},v_{0n} with $\sigma=\sigma_{1n}^*$, $(u(x),v(x))$ the solution corresponding to $\sigma=\sigma_\infty$ with noncontinuable interval (ω_1,ω_2). Supposing $u_0>0$, both ω_1,ω_2 are finite (Lemma 6.1) and so $u'<0$ at some $b<0$. Being $(u_n,v_n)\to(u,v)$ in $C^2[b,0]$ ([21]) then $x_n^*:=x_{\min}(\sigma_{1n}^*)\ge b$ for large n and has a limit point $x^*\in[b,0]$. One finds that $u\ge0$ together with $u_{|x=x^*}=0$. Therefore $\inf u(\cdot,\sigma_\infty)=0$ what necessarily implies $\sigma_\infty=\sigma_1^*$. The proof is easily adapted to achieve the case $u_0=0$. □

Remark 6.4. Theorem 6.3 admits the corresponding version in which $v'(0)$ varies instead of $u'(0)$, which is kept as zero. More precisely, consider the full σ_1,σ_2 perturbation problem,

$$\begin{cases} u''=|v|^p \\ u(0)=u_0 \\ u'(0)=\sigma_1 \end{cases} \qquad \begin{cases} v''=|u|^q \\ v(0)=v_0 \\ v'(0)=\sigma_2, \end{cases} \tag{6.5}$$

whose unique (Theorem 2.5) noncontinuable solution $(u(x,\bar\sigma),v(x,\bar\sigma))$, $\bar\sigma=(\sigma_1,\sigma_2)$, is defined in $(\omega_1,\omega_2)=(-\omega^-(\bar\sigma),\omega^+(\bar\sigma))$. By setting $\sigma_1=0$ and assuming $u_0\ge0$, $v_0>0$ one shows the existence of $\sigma_2^*>0$ such that $(u(x,(0,\sigma_2)),v(x,(0,\sigma_2)))$ is nonnegative if and only if $|\sigma|\le\sigma_2^*$ with $\inf v(\cdot,(0,\pm\sigma_2^*))=0$, $\omega^\pm(0,\sigma_2)$ behaving as $\omega^\pm(\sigma)$ and the nonnegative solutions blowing-up at $x=\pm\omega^\pm(0,\sigma_2)$ according to the same rates as in iv) of Theorem 6.3.

We are treating in more detail the σ_1,σ_2 perturbation problem (6.5) in a moment. We proceed first to the proof of Theorem 1.3.

Proof of Theorem 1.3. The preliminary facts of the theorem have already been shown. Thus we only need to deal with case i). In order to construct the family $(\hat u,\hat v)$ consider $(u(\cdot,\sigma),v(\cdot,\sigma))$ as in Theorem 6.3 and define (check the alternative notation of Remark 6.4),

$$m=\frac12(\omega^+(\sigma,0)-\omega^-(\sigma,0)),\quad l=\frac12(\omega^+(\sigma,0)+\omega^-(\sigma,0)),\quad \lambda=\left(\frac{l}{L}\right)^{1/\theta}, \tag{6.6}$$

$\theta=(pq-1)/2(p+1)$. The searched family of positive solutions to (1.4) can be constructed by using the scaling properties in Lemma 2.6. Namely,

$$(\hat u(x,\sigma),\hat v(x,\sigma))=(\lambda u(\lambda^\theta x+m,(\sigma,0)),\lambda^{\frac{q+1}{p+1}}v(\lambda^\theta x+m,(\sigma,0))). \tag{6.7}$$

with $|\sigma|\le\sigma_1^*$. Observe that $(\hat u,\hat v)$ is nonsymmetric for $\sigma\ne0$ since $x_{\min}\ne y_{\min}$ in that case.
According to Remark 6.4 the proof of ii) is identical. □

Remark 6.5. A bifurcation surface for the family $(\hat u,\hat v)$ of nonsymmetric solutions with respect to the generating symmetric solutions is provided in Figure 5 (a corresponding diagram for $(\tilde u,\tilde v)$ is entirely similar). The parameters involved are u_0,v_0 corresponding to the minima in symmetric solutions and σ. Observe that for $|\sigma|\le\sigma_1^*$ parameter σ has value $\sigma=\lambda^{-\theta-1}\hat u'_{|x=-m\lambda^{-\theta}}$ in the case of the family $(\hat u,\hat v)$, (for $(\tilde u,\tilde v)$ the corresponding parameter is $\sigma=\lambda^{-\theta-(q+1)/p+1}\tilde v'_{|x=-m\lambda^{-\theta}}$ with $|\sigma|\le\sigma_2^*$).

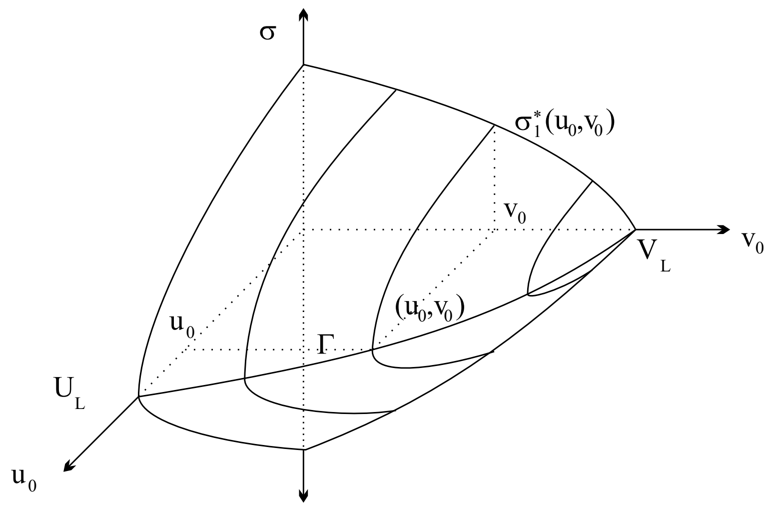

Figure 5. Bifurcation surface for nonsymmetric solutions to (1.4) in the family $(\hat{u}, \hat{v})$. Symmetric solutions correspond to points $(u_0, v_0, 0)$ at curve Γ. The surface is symmetric with respect to the u_0, v_0 plane.

7. Proofs of Theorems 1.5 and 1.6

We are beginning with a basic result. Its proof is a consequence of Lemmas 2.3 and 6.1 and will therefore be omitted.

Lemma 7.1. *Let $(u(\cdot,\bar\sigma), v(\cdot,\bar\sigma))$, $\bar\sigma = (\sigma_1,\sigma_2)$, be the solution of the problem (6.5) with domain of existence (ω_1,ω_2). Assume that both u_0, v_0 are positive while:*

$$\inf u(\cdot,\bar\sigma)_{|\bar\sigma=\bar\sigma_0} \geq 0 \qquad \inf v(\cdot,\bar\sigma)_{|\bar\sigma=\bar\sigma_0} \geq 0, \tag{7.1}$$

for a certain $\bar\sigma_0 = (\sigma_{0,1},\sigma_{0,2}) \in \{\sigma_1 \geq 0, \sigma_2 \geq 0\}$, $\bar\sigma_0 \neq 0$. Then, $\inf u(\cdot,\bar\sigma) > 0$, $\inf v(\cdot,\bar\sigma) > 0$ and both ω_1,ω_2 are finite for every $0 \leq \bar\sigma \leq \bar\sigma_0$, i.e. $0 \leq \sigma_1 \leq \sigma_{01}$, $0 \leq \sigma_2 \leq \sigma_{02}$, and $\bar\sigma \neq \bar\sigma_0$. In particular, if $(u(\cdot,\bar\sigma), v(\cdot,\bar\sigma))$ is positive with $\sigma_i \geq 0$, $i = 1,2$ then necessarily $\sigma_i < \sigma_i^*$ for $i = 1,2$.

Remark 7.2. Observe that the extreme case $\inf u = \inf v = 0$ at $\bar\sigma = \bar\sigma_0$ is allowed in (7.1), being in that case $\omega_1 = -\infty$.

The proof of point ii) in Theorem 1.6 can now be given since an arbitrary positive solution (u,v) to (1.4) with $u'(0)v'(0) \geq 0$ satisfies the conditions of Lemma 7.1 with $\bar\sigma_0 = (u'(0), v'(0))$, if necessary after changing x by $-x$. In fact, notice that Lemma 7.1 ensures that the solution $(u(\cdot,\bar\sigma), v(\cdot,\bar\sigma))$ to (6.5) is positive and has $\omega_1 = -\omega^-(\bar\sigma), \omega_2 = \omega^+(\bar\sigma)$ finite for $0 \leq \bar\sigma \leq \bar\sigma_0$. The finiteness of both ω^-, ω^+ together with estimates (1.9) permit showing the continuity of the functions $\omega^\pm$ in $0 \leq \bar\sigma \leq \bar\sigma_0$ as in the proof of Corollary 3.5. By replacing $(u(\cdot,(\sigma,0)), v(\cdot,(\sigma,0))$ with $(u(\cdot,\bar\sigma), v(\cdot,\bar\sigma))$ in (6.7), substituting also $\omega^\pm(\sigma,0)$ with $\omega^\pm(\bar\sigma)$ in (6.6), $0 \leq \bar\sigma \leq \bar\sigma_0$, one obtains a continuous bidimensional family of positive solutions connecting (u,v) with the symmetric solution attained when $\bar\sigma = 0$, as desired. □

The proofs of Theorem 1.5 and the remaining part of Theorem 1.6 require an study of the following function:

$$g(\sigma_1) = \sup\{\sigma_2 : \inf v(\cdot,(\sigma_1,\sigma_2')) > 0 \text{ for } 0 \le \sigma_2' \le \sigma_2\}, \tag{7.2}$$

and of its dual version:

$$f(\sigma_2) = \sup\{\sigma_1 : \inf u(\cdot,(\sigma_1',\sigma_2)) > 0 \text{ for } 0 \le \sigma_1' \le \sigma_1\}, \tag{7.3}$$

where $(u(\cdot,\bar\sigma), v(\cdot,\bar\sigma))$ stands for the solution of (6.5).

Both the "supremum" and the restriction $0 \le \sigma_2' \le \sigma_2$ can be replaced in the definition of g respectively by "infimum" and the range $\sigma_2 \le \sigma_2' \le 0$ (this is nothing else but the effect of the change $x \mapsto -x$ in problem (1.4)). Then what one finds is just the symmetric function $g_-(\sigma_1) := -g(-\sigma_1)$ of g. The symmetric function $f_-(\sigma_2) := -f(-\sigma_2)$ has exactly the same meaning.

The main features exhibited by g are collected in the next Lemma.

Lemma 7.3. *The function g is well defined and positive in $\mathbb{R}$. In addition,*

a) g is decreasing and continuous in $\sigma_1 \le 0$ with $g(0) = \sigma_2^*$. Moreover,

$$\inf v(\cdot,\bar\sigma) < 0,$$

for $\sigma_1 \le 0$ and $\sigma_2 > g(\sigma_1)$ and hence (6.5) does not admit positive solutions for $\sigma_1 \le 0$ and $\sigma_2 \ge g(\sigma_1)$.

b) $\lim_{\sigma_1\to-\infty} g(\sigma_1) = +\infty$ and more precisely,

$$g(\sigma_1) \sim \frac{1}{(q+1)}[(q+2)v_0]^{\frac{q+1}{q+2}}(-\sigma_1)^{\frac{q}{q+2}} \tag{7.4}$$

as $\sigma_1 \to -\infty$.

c) $g \in L^\infty_{\text{loc}}[0,\infty)$ while g is continuous and decreasing in a neighborhood of any value $\sigma_1 \ge 0$ such that $\inf u(\cdot,\bar\sigma) > 0$ at $\bar\sigma = (\sigma_1, g(\sigma_1))$. In particular, g is continuous and decreasing in $(-\infty,\delta]$ for some positive $0 < \delta < \sigma_1^*$.

Proof. By Theorem 6.3, we have $g(0) = \sigma_2^*$ (see Remark 6.4). Let us show the finiteness of g for $\sigma_1 \ne 0$. If $\sigma_1 > 0$ and there exists $\bar\sigma_n = (\sigma_1, \sigma_{2,n})$ with $\sigma_{2,n} \to \infty$ and $\inf v(\cdot,\bar\sigma_n) > 0$ then Lemma 7.1 implies $\inf u(\cdot,\bar\sigma_n) < 0$ for n large. In particular, $u(\cdot,\bar\sigma_n)$ will be defined in $[-u_0/\sigma_1, 0]$ since, by convexity, $x_{\min} < -u_0/\sigma_1$ for large n, $x_{\min}$ being the point where $u(\cdot,\bar\sigma_n)$ attains the infimum. In addition $u(x,\bar\sigma_n) \le u_0$ in that interval. If we now take $\hat x \in [-u_0/\sigma_1, 0]$ then

$$0 < \inf v(\cdot,\bar\sigma_n) \le v(\hat x,\bar\sigma_n) \le v_0 + \sigma_{2,n}\hat x + \frac{\hat x^2}{2}u_0^q,$$

and so:

$$(-\hat x)\sigma_{2,n} \le v_0 + \frac{\hat x^2}{2}u_0^q.$$

which is not compatible with the divergence of $\sigma_{2,n}$. We have shown in particular that for fixed $\sigma_1 > 0$, the set of those $\sigma_2 > 0$ such that $\inf v(\cdot, \bar{\sigma}) > 0$ is bounded above with a bound that can be chosen independent of σ_1 for σ_1 varying in small intervals in $[0, \infty)$. Hence $g \in L^{\infty}_{\mathrm{loc}}(0, \infty)$ (it will be even shown below that g is continuous in an interval containing $\sigma_1 = 0$).

As for $\sigma_1 < 0$, $u(x, \bar{\sigma}) \geq u_0$ wherever it is defined in $x \leq 0$. Lemma 2.3 implies that $u(x, \bar{\sigma}) < u(x, (\sigma_1, 0))$ for each $-\omega^-(\sigma_1, 0) < x < 0$ and for every $\sigma_2 > 0$ such that $\inf v(\cdot, \bar{\sigma}) > 0$. Fixing now $\tilde{x} \in (-\omega^-(\sigma_1, 0), 0)$ such numbers σ_2 satisfy:

$$(-\tilde{x})\sigma_2 \leq v_0 + \int_0^{\tilde{x}} \int_0^t u(s, (\sigma_1, 0))\, ds\, dt,$$

and so they are bounded above. Thus $g(\sigma_1)$ must be finite. Notice also that in this case, Lemma 2.3 provides that g decreases with σ_1 while necessarily $\inf v(\cdot, \bar{\sigma}) < 0$ for $\sigma_2 > g(\sigma_1)$.

To complete the proof of a) let us show the continuity of g in $\sigma_1 \leq 0$. Firstly observe that for $\sigma_1 \leq 0$ both limits $g(\sigma_1-) \geq g(\sigma_1+) \geq \sigma_2^*$ are finite. Since (6.2) holds at $\bar{\sigma} = (\sigma_1, g(\sigma_1\pm))$ Lemma 6.1 assures that both extremes $\pm\omega^{\pm}(\bar{\sigma})$ of the existence interval of the solution $(u(\cdot, \bar{\sigma}), v(\cdot, \bar{\sigma}))$ to (6.5) corresponding to $\bar{\sigma} = (\sigma_1, g(\sigma_1\pm))$ are finite. This is crucial to conclude –after a careful application of the continuous dependence results in [21]– that $\inf v(\cdot, \bar{\sigma})) = 0$ at $\bar{\sigma} = (\sigma_1, g(\sigma_1\pm))$. Since $\sigma_1 \leq 0$, $g(\sigma_1)$ is just the unique value of σ_2 such that $\inf v = 0$. Hence $g(\sigma_1\pm) = g(\sigma_1)$ as desired.

Let us prove the continuity assertion in c) and so assume that $\inf u(\cdot, \bar{\sigma}) > 0$ at $\bar{\sigma} = (\sigma_1, g(\sigma_1))$. As shown in the proof of Lemma 6.1 the positivity of $\inf u$ implies also the finiteness of the extremes $\pm\omega^{\pm}(\bar{\sigma})$ of the interval of existence for the solution (u, v) to (6.5) corresponding to $\bar{\sigma} = (\sigma_1, g(\sigma_1))$. Continuous dependence provides the existence of a small $\eta > 0$ such that $\inf u(\cdot, \bar{\sigma}') > 0$ for every $\bar{\sigma}'$ satisfying $|\sigma_i' - \sigma_i| \leq \eta$, $i = 1, 2$. Given $\varepsilon > 0$, $0 < \varepsilon_1 < \max\{\varepsilon, \eta\}$ and $0 < \delta < \eta$ can be found –with the help of Lemma 2.3– such that $\inf v(\cdot, \bar{\sigma}') > 0$ in $\sigma_2' = g(\sigma_1) - \varepsilon_1$, $\inf v(\cdot, \bar{\sigma}') < 0$ in $\sigma_2' = g(\sigma_1) + \varepsilon_1$ for each σ_1' such that $|\sigma_1' - \sigma_1| \leq \delta$. Therefore, $g(\sigma_1) - \varepsilon < g(\sigma_1') < g(\sigma_1) + \varepsilon$ in the last interval. Moreover, Lemma 7.1 directly implies that g decreases there. This concludes the proof of c).

We finally show the asymptotic estimate (7.4) for g. To this proposal we are equivalently analyzing the behavior as $\sigma_1 \to \infty$ of the symmetric function g_- of g (see the remarks after (7.3)):

$$g_-(\sigma_1) = \inf\{\sigma_2 < 0 : \inf v(\cdot, \bar{\sigma}) > 0\}.$$

For $\bar{\sigma} = (\sigma_1, g_-(\sigma_1))$ define $y_{\min}$ as the point where $v(x, \bar{\sigma})$ attains its minimum. Notice that $g_-(\sigma_1) < 0$ implies that $y_{\min} > 0$. We first prove that

$$\lim_{\sigma_1 \to \infty} g_-(\sigma_1) = -\infty.$$

If, on the contrary, $\inf_{\sigma_1 \geq 0} g_- = -l > -\infty$ we find by convexity that $v(x, \bar{\sigma})$, $\bar{\sigma} = (\sigma_1, g_-(\sigma_1))$, is defined in $[0, v_0/l]$ wherein $v \leq v_0$. By using that $u(x, \bar{\sigma}) \geq u_0 + \sigma_1 x$ and fixing $\tilde{x} \in [0, v_0/l]$ we arrive at,

$$v_0 \geq v(\tilde{x}, \bar{\sigma}) \geq v_0 - l\tilde{x} + \int_0^{\tilde{x}} \int_0^t (u_0 + \sigma_1 s)^q\, ds\, dt \to \infty,$$

as $\sigma_1 \to \infty$. Since this is not possible, it follows that $l = \infty$.

On the other hand, v, v_x vanish at $x = y_{\min}$. Thus, the following identities hold,

$$v_0 = \int_0^{y_{\min}} su^q \, ds, \qquad -g_-(\sigma_1) = \int_0^{y_{\min}} su^q \, ds, \tag{7.5}$$

where we have used the positivity of $u = u(x, \bar{\sigma})$ in $x \geq 0$. Indeed, $u(x, \bar{\sigma}) \geq u_0 + \sigma_1 x$ in $x \geq 0$. By using the first equality in (7.5) we achieve,

$$v_0 \geq \frac{\sigma_1^q}{q+2} \, y_{\min}^{q+2}.$$

This means that $y_{\min} \to 0+$ as $\sigma_1 \to \infty$. We recall that $0 \leq v(x, \bar{\sigma}) \leq v_0$ in $[0, y_{\min}]$ and conclude, by employing the second equality in (7.5):

$$\int_0^{y_{\min}} (u_0 + \sigma_1 s)^q \, ds \leq -g_-(\sigma_1) \leq \int_0^{y_{\min}} (u_0 + \sigma_1 s + \frac{v_0^p}{2} s^2)^q \, ds. \tag{7.6}$$

Hence:

$$-g_-(\sigma_1) \leq y_{\min} \int_0^1 (u_0 + \sigma_1 y_{\min} t + \frac{v_0^p}{2} y_{\min}^2 t^2)^q \, dt.$$

We immediately achieve that $\sigma_1 y_{\min} \to \infty$ as $\sigma_1 \to \infty$ since $\sigma_1 y_{\min} = O(1)$ on a possible sequence $\sigma_{1,n} \to \infty$ together with the previous estimate imply $-g_-(\sigma_1) = O(1)$ on that sequence, what is not possible.

The first equality in (7.5) leads to,

$$\int_0^{y_{\min}} s(u_0 + \sigma_1 s)^q \, ds \leq v_0 \leq \int_0^{y_{\min}} s(u_0 + \sigma_1 s + \frac{v_0^p}{2} s^2)^q \, ds,$$

and so,

$$\frac{(\sigma_1 y_{\min})^q}{q+2} \leq v_0 y_{\min}^{-2} \leq \int_0^1 t(u_0 + \sigma_1 y_{\min} t + \frac{v_0^p}{2} y_{\min} t^2)^q \, dt.$$

Since,

$$\int_0^1 t(u_0 + \sigma_1 y_{\min} t + \frac{v_0^p}{2} y_{\min} t^2)^q \, dt \sim \frac{(\sigma_1 y_{\min})^q}{q+2}, \qquad \sigma_1 \to \infty$$

then,

$$\frac{1}{y_{\min}} \sim \left\{ \frac{\sigma_1^q}{(q+2) v_0} \right\}^{\frac{1}{q+2}}, \qquad \sigma_1 \to \infty. \tag{7.7}$$

Now, after multiplying (7.6) by σ_1 we get,

$$\sigma_1 y_{\min} \int_0^1 (u_0 + \sigma_1 y_{\min} t)^q \, dt \leq -\sigma_1 g_-(\sigma_1) \leq \sigma_1 y_{\min} \int_0^1 (u_0 + \sigma_1 y_{\min} t + \frac{v_0^p}{2} y_{\min}^2 t^2)^q \, dt.$$

Hence,

$$\sigma_1 g_-(\sigma_1) \sim -\frac{(\sigma_1 y_{\min})^{q+1}}{q+1}, \qquad \sigma_1 \to \infty,$$

which, together with (7.7) leads to the desired estimate (7.4). This finishes the proof of the Lemma. □

Remark 7.4. The dual function f of g defined in (7.3) satisfies – when conveniently transposed – the properties of g in Lemma 7.3. In particular, $f(\sigma_2)$ is continuous and decreasing in the interval $-\infty < \sigma_2 \leq \delta_1$ for certain positive $\delta_1 < \sigma_2^*$, $f(0) = \sigma_1^*$ while $f(\sigma_2) \to \infty$ as $\sigma_2 \to -\infty$ with the asymptotic behavior,

$$f(\sigma_2) \sim C_p(-\sigma_2)^{\frac{p}{p+2}}, \qquad \sigma_2 \to -\infty, \tag{7.8}$$

where $C_p = [(p+2)u_0]^{(p+1)/(p+2)}/(p+1)$. More importantly, for $\sigma_2 \leq 0$, problem (6.5) does not admit positive solutions for $\sigma_1 \geq f(\sigma_2)$. Finally f will be decreasing and continuous in a certain neighborhood of any σ_2 where $\inf v(\cdot, (f(\sigma_2), \sigma_2)) > 0$.

Let us proceed now to complete the proof of Theorem 1.6 and to show Theorem 1.5. Define the function,

$$h_+(\sigma_1) = \begin{cases} g(\sigma_1) & \sigma_1 \leq 0 \\ g \wedge g_1(\sigma_1) & 0 < \sigma_1 < \sigma_1^* \\ f^{-1}(\sigma_1) & \sigma_1 \geq \sigma_1^*, \end{cases}$$

where $g \wedge g_1(\sigma_1) = \min\{g(\sigma_1), g_1(\sigma_1)\}$ and the function g_1 is defined in $0 < \sigma_1 < \sigma_1^*$ as,

$$g_1(\sigma_1) = \sup\{\sigma_2 : \inf u(\cdot, (\sigma_1, \sigma_2')) > 0 \text{ for } 0 \leq \sigma_2' \leq \sigma_2\}.$$

Notice that g_1 may be infinite at some σ_1.

We claim that h_+ is continuous and decreasing. In fact, Lemma 7.1 directly implies that $g \wedge g_1$ decreases in the interval $(0, \sigma_1^*)$. On the other hand, $g \wedge g_1(\sigma_1) = g(\sigma_1)$ for $0 < \sigma_1 < \delta$ (see Lemma 7.3), while $g \wedge g_1(\sigma_1) = f^{-1}(\sigma_1)$ for $\sigma_1^* - \delta_2 < \sigma_1 < \sigma_1^*$ and a certain $\delta_2 > 0$ small enough (see Remark 7.4). Moreover, Lemma 7.3-c) ensures that $g \wedge g_1 = g$ in a neighborhood of any σ_1 where $g(\sigma_1) < g_1(\sigma_1)$. Thus $g \wedge g_1$ is continuous in that neighborhood. The corresponding assertion holds true for those σ_1 where $g_1(\sigma_1) < g(\sigma_1)$ being $g \wedge g_1 = f^{-1}$ in a certain neighborhood of σ_1 (Remark 7.4). To complete the proof of the claim we show the continuity of $g \wedge g_1$. Indeed, assume on the contrary that $g \wedge g_1(\sigma_1-) > g \wedge g_1(\sigma_1+)$ and define $\sigma_2' = g \wedge g_1(\sigma_1-)$, $\sigma_2'' = g \wedge g_1(\sigma_1+)$, with $\bar{\sigma}' = (\sigma_1, \sigma_2')$, $\bar{\sigma}'' = (\sigma_1, \sigma_2'')$. A careful use of continuous dependence of the solutions to (6.5) with respect to $\bar{\sigma}$ permits to ensure that $\inf u(\cdot, \bar{\sigma}') \geq 0, \inf v(\cdot, \bar{\sigma}') \geq 0$. Lemma 7.1 then implies that $\inf u(\cdot, \bar{\sigma}'') > 0, \inf v(\cdot, \bar{\sigma}'') > 0$. However, that is not possible due again to continuous dependence.

Observe that the function h_+ satisfies all the properties stated in Theorem 1.5-b), with the exception of the existence of the value $\sigma_1 = b$ (see further details below). Moreover, by its own definition no positive solutions to (6.5) are possible for $0 < \sigma_1 < \sigma_1^*$ provided $\sigma_2 \geq g \wedge g_1(\sigma_1)$. Hence that problem can not exhibit positive solutions for $\sigma_2 \geq h_+(\sigma_1)$ and by introducing the symmetric function $h_-(\sigma_1) := -h_+(-\sigma_1)$, exactly the same assertion holds true for $\sigma_2 \leq h_-(\sigma_1)$. In other words, the set (see (1.11))

$$\mathcal{C} = \{\bar{\sigma} : h_-(\sigma_1) < \sigma_2 < h_+(\sigma_1)\},$$

characterizes the existence of positive solutions to (6.5) regarding the values of $\bar{\sigma}$. This provides the proof of point i) in Theorem 1.6.

We further show the boundedness of $\mathcal{C}$. Since $\mathcal{C} \cap \{\sigma_1 \geq 0, \sigma_2 \geq 0\} \subset [0, \sigma_1^*] \times [0, \sigma_2^*]$, due to the symmetry of $\mathcal{C}$ it suffices with showing that $\mathcal{C} \cap \{\sigma_1 \geq 0, \sigma_2 \leq 0\}$ is bounded. In fact, observe that $h_+ = f^{-1}$ for $\sigma_1 \geq \sigma_1^*$ and so (see Remark 7.4),

$$h_+(\sigma_1) \sim -C_p^{-\frac{p+2}{p}} \sigma_1^{\frac{p+2}{p}} \qquad \sigma_1 \to \infty.$$

Since $h_-(\sigma_1) = g_-(\sigma_1)$ for $\sigma_1 \geq 0$ and $g_-(\sigma_1) \sim -C_q\sigma_1^{q/(q+2)}$ as $\sigma_1 \to \infty$ (C_q is the coefficient in (7.4)), we conclude that $h_+(\sigma_1) < h_-(\sigma_1)$ for large σ_1. Thus $\mathcal{C}$ is bounded. On the other hand, observe that $h_- < -\sigma_2^*$ in $(0, \sigma_1^*]$ while $h_+ > 0$ in $(0, \sigma_1^*)$ (Figure 1). This means that a first value $b > \sigma_1^*$ must exist such that $h_-(b) = h_+(b)$.

Next consider the set $\mathcal{C}_0 = \{\bar\sigma : h_-(\sigma_1) < \sigma_2 < h_+(\sigma_1),\ |\sigma_1| < b\}$ (see (1.10)). By construction, both extremes $\pm\omega^\pm(\bar\sigma)$ of the maximal interval of existence of the solution (u, v) to (6.5) (solution which is in addition positive) are finite and, by the reasons already explained in the proof of Theorem 1.6-ii), they vary continuously as $\bar\sigma \in \mathcal{C}_0$. Therefore, the scale change (6.7):

$$(\tilde u(x,\bar\sigma), \tilde v(x,\bar\sigma)) = (\lambda u(\lambda^\theta x + m, \bar\sigma), \lambda^{\frac{q+1}{p+1}} v(\lambda^\theta x + m, \bar\sigma)) \tag{7.9}$$

with $\omega^\pm(\bar\sigma)$ replacing the values of $\omega^\pm(\sigma, 0)$ in (6.6), defines a continuous bidimensional family of positive solutions to (1.4) which produces a symmetric solution exclusively at $\bar\sigma = 0$. This concludes the proofs of Theorem 1.5 and Theorem 1.6-iii). □

Remarks 7.5.

a) The open set $\mathcal{C}$ could possibly exhibit a connected piece $\mathcal{C}_1$ different from $\mathcal{C}_0$ in $\{\sigma_1 < 0, \sigma_2 < 0\}$ if h_- and h_+ coincide in values σ_1 greater than b. In this case and regarding assertion iii) of Theorem 1.6 it is unclear if a positive solution (u, v) to (1.4) with derivatives $(u_x'(0), v_x'(0)) \in \mathcal{C}_1$ could be deformed, keeping its sign, to produce a symmetric solution.

b) When $\bar\sigma \to \partial\mathcal{C}_0$, different types of nonnegative solutions (u, v) to (1.4) are obtained in the limit, one of whose components (or even both of them) vanishes at a unique point of $(-L, L)$. In the case of $\partial\mathcal{C}_0 \cap \{\sigma_1 \geq 0, \sigma_2 \leq 0\}$ solutions to (6.5) lead, by means of the scaling (7.9), to solutions (u, v) to (1.4) such that $\inf u > 0, \inf v = 0$ for $\sigma_2 = g_-(\sigma_1)$, $0 \leq \sigma_1 < b$, $\inf u = \inf v = 0$ at $\sigma_2 = g_-(b) = f^{-1}(b)$, while $\inf u = 0, \inf v > 0$ for $\sigma_2 = f^{-1}(\sigma_1)$, $\sigma_1^* \leq \sigma_1 < b$ (Figure 6). The situation in the case $\partial\mathcal{C}_0 \cap \{\sigma_1 \geq 0, \sigma_2 \geq 0\}$ is slightly different. In the arc $\sigma_2 = h_+(\sigma_1)$, $0 \leq \sigma_1 \leq \sigma_1^*$, solutions $(u(\cdot,\bar\sigma), v(\cdot,\bar\sigma))$ of (6.5) exhibit two kind of features. In the first one, $\inf u > 0, \inf v = 0$ (as is the case for $\sigma_1 \sim 0$) or $\inf u = 0, \inf v > 0$ (which happens at least near σ_1^*). Both behaviors are observed in the solutions to (1.4) obtained after the change (7.9). See the first and third pictures in Figure 7. A second one, due to the existence of at least some σ_1 such that the solution $(u(\cdot,\bar\sigma), v(\cdot,\bar\sigma))$ to (6.5) satisfies $\inf u = \inf v = 0$. However, a solution to (1.4) with the same behavior is now generated via (7.9) only when $-\omega^-(\bar\sigma) > -\infty$ with $\bar\sigma = (\sigma_1, h_+(\sigma_1))$ (the case in the middle in Figure 7). According to Lemma 6.1, this case is characterized by the fact that $\sigma_1\sigma_2 < h(u_0, v_0)$ at $\sigma_2 = h_+(\sigma_1)$. Indeed, one can obtain conditions on u_0, v_0 and parameters p, q ensuring that $\sigma_1 h_+(\sigma_1) < h(u_0, v_0)$ for all $\sigma_1 \in [0, \sigma_1^*]$ and so all possible solutions (u, v) to (6.5) in the second kind generate solutions to (1.4) (details are omitted for the sake of brevity). If on the contrary, $\sigma_1 h_+(\sigma_1) = h(u_0, v_0)$ then $\omega^-(\bar\sigma_0) = \infty$ at $\bar\sigma_0 = (\sigma_1, h_+(\sigma_1))$ and the change (7.9) plainly has no sense. Moreover, it is unclear which is the limit profile in $x \in (-L, L)$ of the solutions $(\tilde u(x,\bar\sigma), \tilde v(x,\bar\sigma))$ to (1.4) obtained in (7.9) when $\bar\sigma \in \mathcal{C}_0$ and $\bar\sigma \to \bar\sigma_0$.

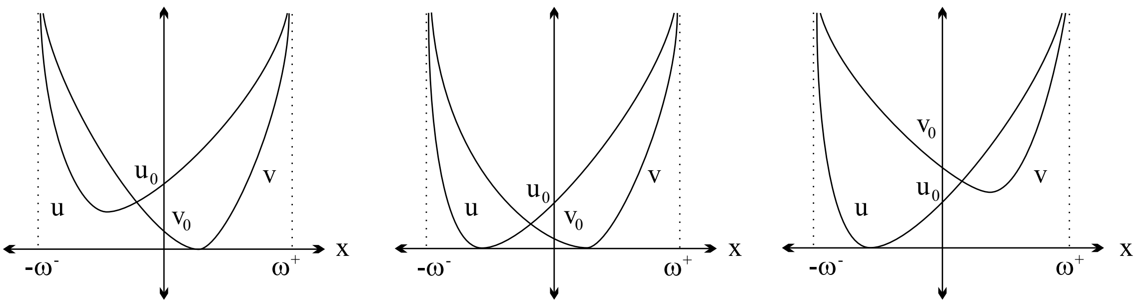

FIGURE 6. Solutions $(u(\cdot,\bar{\sigma}), v(\cdot,\bar{\sigma}))$ of (6.5) for $\bar{\sigma} \in \partial\mathcal{C}_0 \cap \{\sigma_1 \geq 0\}$ corresponding to $\sigma_1\sigma_2 < 0$. The second configuration corresponds to the values $\sigma_1 = b, \sigma_2 = h_+(b)$.

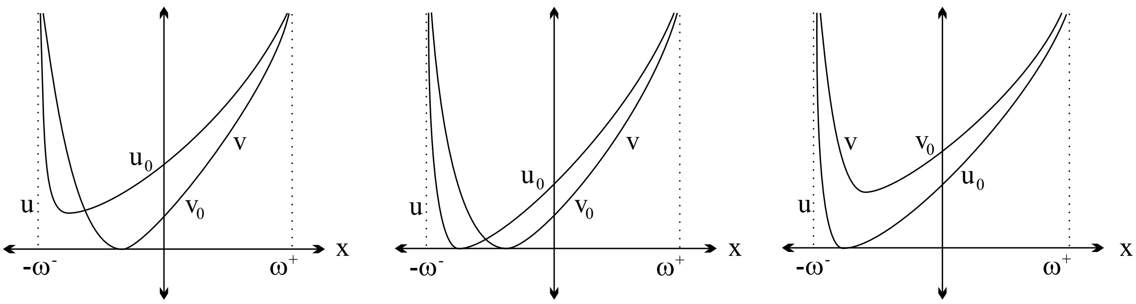

FIGURE 7. Nonnegative solutions associated to values $\bar{\sigma} \in \partial\mathcal{C}_0 \cap \{\sigma_1 \geq 0\}$ in the case where $\sigma_1\sigma_2 > 0$. Occurrence of solutions as in the middle configuration require some additional information.

ACKNOWLEDGEMENTS

Part of this work was done while the first author was a post-doc fellow in Centro de Modelamiento Matemático (Universidad de Chile) in Santiago de Chile.

REFERENCES

[1] BANDLE C., ESSÈN C., *On the solutions of quasilinear elliptic problems with boundary blow-up*, Sympos. Math. **35** (1994), 93–111.

[2] BANDLE C., MARCUS M., *Sur les solutions maximales de problèmes elliptiques non linéaires: bornes isopérimetriques et comportement asymptotique*, C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), 91–93.

[3] BANDLE C., MARCUS M., *'Large' solutions of semilinear elliptic equations: Existence, uniqueness and asymptotic behaviour*, J. Anal. Math. **58** (1992), 9–24.

[4] BANDLE C., MARCUS M., *On second order effects in the boundary behaviour of large solutions of semilinear elliptic problems*, Differential Integral Equations **11** (1) (1998), 23–34.

[5] Bieberbach L., $\Delta u = e^u$ *un die Automorphen Funktionen*, Math. Ann. **77** (1916), 173-212.

[6] Chuaqui M., Cortázar C., Elgueta M., Flores C., García-Melián J., Letelier R., *On an elliptic problem with boundary blow-up and a singular weight: the radial case*, Proc. Royal Soc. Edinburgh **133** (6) (2003), 1283-1297.

[7] Chuaqui M., Cortázar C., Elgueta M., García-Melián, *Uniqueness and boundary behavior of large solutions to elliptic problems with singular weights*, Comm. Pure Appl. Anal. **3** (4) (2004), 653-662.

[8] Dancer N., Du, Y., *Effects of certain degeneracies in the predator-prey model*, SIAM J. Math. Anal. **34** (2002), 292-314.

[9] Del Pino M., Letelier R., *The influence of domain geometry in boundary blow-up elliptic problems*, Nonlinear Anal. **48** (6) (2002), 897–904.

[10] Díaz G., Letelier R., *Explosive solutions of quasilinear elliptic equations: Existence and uniqueness*, Nonlinear Anal. **20** (1993), 97–125.

[11] Du Y., *Effects of a degeneracy in the competition model. Part I: classical and generalized steady-state solutions*, J. Diff. Eqns. **181** (2002), 92–132.

[12] Du Y., *Effects of a degeneracy in the competition model. Part II: perturbation and dynamical behaviour*, J. Diff. Eqns. **181** (2002), 133–164.

[13] Fraile J., López-Gómez J., Sabina J., *On the global structure of the set of positive solutions of some semilinear elliptic boundary value problems* , J. Diff. Eqns. **123** (1995), no. 1,180–212.

[14] García-Melián J., *A remark on the existence of positive large solutions via sub and supersolutions*, Electronic J. of Differential Equations **2003** (2003), n. 110, 1–4.

[15] García-Melián J., Gómez-Reñasco R., López-Gómez J., Sabina de Lis J., *Point-whise growth and uniqueness of positive solutions of sublinear elliptic problems where bifurcation from infinity occurs*, Arch. Rational Mech. Anal. **145** (1998), 261–289.

[16] García-Melián J., Letelier-Albornoz R., Sabina de Lis, J., *Uniqueness and asymptotic behaviour for solutions of semilinear problems with boundary blow-up*, Proc. Amer. Math. Soc. **129** (2001), no. 12, 3593–3602.

[17] García-Melián J., Letelier-Albornoz R., Sabina de Lis, J., *Competitive elliptic systems: blow-up on the boundary* (Spanish), Proceedings of the XVIII-th Spanish Conference on Differential Equations (J. Jornet, J. López, J. Olivé, R. Ramírez Eds.), CD-Rom, ISBN 84-930923-2-0, Tarragona (Spain), 2003, 6 pages.

[18] García-Melián J., Rossi J., *Boundary blow-up solutions to elliptic systems of competitive type*, J. Differential Equations **206** (2004), 156-181.

[19] García-Melián J., Suárez A., *Existence and uniqueness of positive large solutions to some cooperative elliptic systems*, Advanced Nonlinear Studies **3** (2003), 193-206.

[20] García-Melián J., Sabina de Lis J., *Uniqueness to quasilinear problems for the p-Laplacian in radially symmetric domains*, Nonlinear Anal. T.M.A. **43** (7) (2001), 803-835.

[21] Hartman P., *Ordinary Differential Equations*, John Wiley & Sons, N. Jersey, 1964.

[22] Hirsch M. W., Smale S., *Differential Equations, Dynamical Systems and Linear Algebra*, Academic Press, New York, 1974.

[23] Keller J. B., *Electrohydrodynamics I. The equilibrium of a charged gass in a container*, J. Rational Mech. Anal. **5** (1956), 715–724.

[24] Keller J. B., *On solution of* $\Delta u = f(u)$, Comm. Pure Appl. Math. **10** (1957), 503–510.

[25] Kondrat'ev V. A., Nikishkin V. A., *Asymptotics, near the boundary, of a solution of a singular boundary value problem for a semilinear elliptic equation*, Differential Equations **26** (1990), 345–348.

[26] Lasry J. M., Lions P. L., *Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constrains*, Math. Ann. **283** (1989), 583–630.

[27] Lazer A. C., Mckenna P. J., *On a problem of Bieberbach and Rademacher*, Nonlinear Anal. **21** (1993), 327–335.

[28] Lazer A. C., P. J. Mckenna P. J., *Asymptotic behaviour of solutions of boundary blow-up problems*, Differential Integral Equations **7** (1994), 1001–1019.

[29] Loewner C., Nirenberg L., *Partial differential equations invariant under conformal or projective transformations*, Contributions to Analysis (a collection of papers dedicated to Lipman Bers), Academic Press, New York, 1974, p. 245–272.

[30] López-Gómez J., *Existence and metacoexistence states in competing species models*, Houston J. Math. **29** (2003), 483–536.

[31] Marcus M., Véron L., *Uniqueness and asymptotic behaviour of solutions with boundary blow-up for a class of nonlinear elliptic equations* , Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (1997), 237–274.

[32] Osserman R., *On the inequality* $\Delta u \geq f(u)$, Pacific J. Math. **7** (1957), 1641–1647.

[33] Rademacher H., *Einige besondere probleme partieller Differentialgleichungen* in "Die Differential und Integralgleichungen der Mechanik und Physik" (by P. Frank and R. von Mises), vol. I, Rosenberg, New York, 1943, 838–845.

[34] Ratto, A., Rigoli, M., Véron, L., *Scalar curvature and conformal deformation of hyperbolic space*. J. Funct. Anal. **121** (1) (1994), 15–77.

[35] Reichel W., Walter W., *Radial solutions of equations and inequalities involving the p-Laplacian*, J. Inequal. Appl. **1** (1997), 47–71.

[36] Véron L., *Semilinear elliptic equations with uniform blowup on the boundary*, J. Anal. Math. **59** (1992), 231–250.

[37] Zhang Z., *A remark on the existence of explosive solutions for a class of semilinear elliptic equations*, Nonlinear Anal. TMA **41** (2000), 143–148.