A BIFURCATION PROBLEM GOVERNED BY THE BOUNDARY CONDITION II *

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Abstract

In this work we consider the problem $\Delta u = a(x)u^p$ in Ω, $\frac{\partial u}{\partial \nu} = \lambda u$ on $\partial\Omega$, where Ω is a smooth bounded domain, ν is the outward unit normal to $\partial\Omega$, λ is regarded as a parameter and $0 < p < 1$. We consider both cases where $a(x) > 0$ in Ω or $a(x)$ is allowed to vanish in a whole subdomain Ω_0 of Ω. Our main results include existence of nonnegative nontrivial solutions in the range $0 < \lambda < \sigma_1$, where σ_1 is characterized by means of an eigenvalue problem, uniqueness and bifurcation from infinity of such solutions for small λ, and the appearance of dead cores for large enough λ.

1. Introduction

The aim of the present work is to complete the study initiated in [11] of the nonnegative solutions to the following boundary-value problem:

$$\begin{cases} \Delta u = a(x)u^p & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda u & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a $C^{2,\gamma}$ domain of $\mathbb{R}^N$, $N \geq 2$, $a \in C^\gamma(\overline{\Omega})$ is a nonnegative weight function, $p > 0$ and λ is a real parameter. While in [11] the case $p > 1$ was treated, we are focusing our attention here in the complementary range $0 < p < 1$. This range is in principle more complex, since standard techniques employed for $p > 1$ are useless here, namely: sub and supersolutions, global minimization, strong maximum principle and linearization near

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$u = 0$. For some reasons which will become clear later on, this regime is sometimes termed as "degenerate".

On the other hand, in reaction diffusion theory, the nonlinear term u^p represents the degradation rate of a certain reference reactant whose density is given by u. The exponent p is known as the *order* of the reaction. In this context, the case $0 < p < 1$ is interesting in its own right (see [3]).

The main novelty in problem (1.1) is that the parameter appears precisely in the boundary condition. With respect to this parameter, we will perform a complete analysis of the bifurcation diagram of nonnegative solutions to (1.1), which will be shown to be entirely different from the case $p > 1$ (see Figures 1, 2 and 4).

In the course of the exposition, we are considering in first place the case of a positive weight $a(x) > 0$ in Ω, and then we will deal with the situation in which $a(x)$ vanishes in a smooth nonempty subdomain Ω_0 of Ω (see [18], [6], [8], [17], [10], [7] for problems with Dirichlet or Robin boundary conditions, none of them depending on parameters). It should be remarked that dropping the connectedness of Ω_0 does not lead to genuinely new features regarding the material here analyzed and we are therefore omitting its discussion. Also for the sake of brevity, we are restricting our analysis to treat the cases where either Ω_0 is strongly contained in Ω or where Ω_0 "touches" $\partial\Omega$ in a nontrivial way (see hypothesis (H)). On the other hand, we point out that due to the smoothness of the main domains Ω, Ω_0, Ω^+ involved in this work (see details below), their boundaries can only exhibit a finite number of connected components.

An important feature to stress with regard to problem (1.1) is that the strong maximum principle is not applicable and thus nonnegative nontrivial solutions need not be positive in Ω. Indeed, it will be seen that solutions u develop a dead core, that is, the set $\mathcal{O} = \{x \in \Omega : \ u(x) = 0\}$ has nonempty interior for large enough λ (see for instance [3] or [9] for "dead core" phenomenology), and this entails in some domains multiplicity of solutions. Nevertheless, we can still ensure that solutions are unique for small enough λ and moreover that a bifurcation from infinity at $\lambda = 0$ takes place. Additionally, particular properties of problem (1.1) in the ball are also stated (see below).

We come now to give precise statements of our results. First of all we consider the case of positive weights.

Theorem 1. *Assume Ω is a $C^{2,\gamma}$ bounded domain of $\mathbb{R}^N$, $a \in C^\gamma(\overline{\Omega})$, $a(x) > 0$ in Ω and $0 < p < 1$. Then problem (1.1) possesses the following features:*

(i) If $\lambda \le 0$, then problem (1.1) does not have nonnegative nontrivial solutions. For $\lambda > 0$, there is always a nonnegative nontrivial solution $u \in C^{2,\gamma_1}(\overline{\Omega})$, $\gamma_1 = \min\{\gamma, p\}$.

(ii) There exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, problem (1.1) has a unique *classical nonnegative solution u_λ. In addition, u_λ is positive in that range, $u_\lambda \in C^{2,\gamma}(\overline{\Omega})$ while the mapping $\lambda \to u_\lambda$ regarded as attaining its values in $C^{2,\gamma}(\overline{\Omega})$ is real analytic and decreasing. Moreover,*

$$\lim_{\lambda\to 0+} \lambda^{\frac{1}{1-p}} u_\lambda = \left(\frac{1}{|\partial\Omega|} \int_\Omega a(x) \right)^{\frac{1}{1-p}}$$

in $C^{2,\gamma}(\overline{\Omega})$. In particular, $u_\lambda \to +\infty$ uniformly in $\overline{\Omega}$ as $\lambda \to 0+$.

(iii) There exist positive constants C and λ_1 such that for all nonnegative nontrivial solutions $u \in C^{2,\gamma_1}(\overline{\Omega})$ to (1.1) *with $\lambda \geq \lambda_1$ we have*

$$u \leq C\lambda^{-\frac{2}{1-p}}. \tag{1.2}$$

(iv) There exists $\lambda_2 > 0$ such that all nonnegative nontrivial solutions u_λ to (1.1) *for $\lambda \geq \lambda_2$ develop a dead core $\mathcal{O}_\lambda := \{u_\lambda = 0\}$, $\mathcal{O}_\lambda \to \Omega$ uniformly as $\lambda \to +\infty$ in the sense that for λ large $\{d(x) \geq d_\lambda\} \subset \mathcal{O}_\lambda$ with $d_\lambda \to 0$ as $\lambda \to \infty$, $d(x) = \mathrm{dist}(x, \partial\Omega)$. Moreover, d_λ can be chosen as*

$$d_\lambda = \frac{K}{\lambda},$$

for a certain $K > 0$ provided $a > 0$ on $\partial\Omega$.

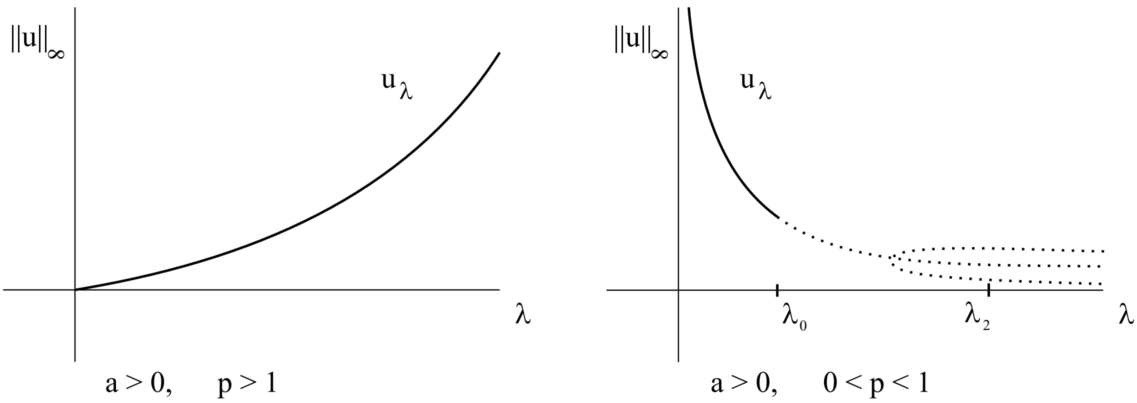

FIGURE 1. Comparison between the bifurcation diagrams of cases $p > 1$ and $0 < p < 1$ for $a > 0$ in Ω. Continuous pieces in the diagrams stand for uniqueness, the corresponding branches being smooth curves. Multiplicity is vaguely depicted by more than one branch in dotted lines.

Part (iv) of Theorem 1 implies that nonnegative solutions are nontrivial only near the boundary of Ω for large λ. By means of a mixed version of problem (1.1) (see (5.1) and Theorem 11) it is possible to show that several solutions can be constructed if we assume that $\partial\Omega$ consists in more than one connected component. We remark that this phenomenon is not present in the equation $\Delta u = \lambda u^p$ if the boundary condition is $u = 1$ on $\partial\Omega$, the problem treated for instance in [3].

Theorem 2. *Assume Ω is a bounded $C^{2,\gamma}$ domain such that $\partial\Omega$ has k connected components. Then problem* (1.1) *has at least $2^k - 1$ nonnegative nontrivial solutions for large enough λ.*

We now turn to consider problem (1.1) in a specific domain: the unit ball B in $\mathbb{R}^N$. In this case one should expect that a radial solution exists provided a is also radial. We show that this is indeed the case, and moreover the radial solution is unique for large λ. It could be thought indeed that there are no nonradial solutions, since in contrast with Theorem 2, ∂B is connected and that result can not be applied. Surprisingly, we prove that this is not the case, by constructing a second solution for large λ which is not radial. For simplicity, we are only considering the case $a(x) = 1$.

Theorem 3. *Assume $\Omega = B$, $a(x) = 1$ and $0 < p < 1$. Then*

(i) For every $\lambda > 0$, there exists at least a radially symmetric nonnegative nontrivial solution to (1.1).

(ii) There exist $\lambda_0, \lambda_3 > 0$ such that problem (1.1) *has a* unique *radially symmetric nonnegative nontrivial solution u_λ both for $0 < \lambda < \lambda_0$ and $\lambda > \lambda_3$.*

(iii) For $\lambda > \lambda_3$ the radial solution u_λ has a dead core $\mathcal{O}_\lambda = \{x \in B : \ |x| \leq r(\lambda)\}$, where

$$r(\lambda) \sim 1 - \frac{\alpha}{\lambda} \qquad \text{as } \lambda \to +\infty$$

and $\alpha = 2/(1-p)$. Moreover, $u_\lambda(1) \sim A\alpha^\alpha \lambda^{-\alpha}$, where $A = [\alpha(\alpha-1)]^{-\frac{1}{1-p}}$.

(iv) There exists $\lambda_4 > 0$ such that problem (1.1) *has at least a nonnegative nontrivial solution v_λ for $\lambda > \lambda_4$, which is not radial.*

Finally, we are treating some features of problem (1.1) when the weight function $a(x)$ is allowed to vanish in some nonempty subdomain Ω_0 of Ω. Observe that the connectedness of Ω together with $a \not\equiv 0$ entail $\partial\Omega_0 \cap \Omega \neq \emptyset$. We are restricting our discussion here to the case where $\Omega_0 \subset \Omega$ is a $C^{2,\gamma}$ smooth domain such that, defining $\Gamma_1 = \partial\Omega_0 \cap \partial\Omega$, $\Gamma_2 = \partial\Omega_0 \cap \Omega$, the following condition holds (cf. [11]):

$$\overline{\Gamma}_2 \subset \Omega. \tag{H}$$

(H) implies that the part of $\partial\Omega_0$ meeting Ω necessarily consists of a closed manifold ($\overline{\Gamma}_2 = \Gamma_2$) entirely contained in Ω, thus lying at positive distance from the remaining part Γ_1 of $\partial\Omega_0$, located on $\partial\Omega$. This is a technical hypothesis which provides a convenient smoothness of the eigenfunctions of the auxiliary problems (1.3) and (1.4) (see the discussion after Remark 3 and Remark 4 (c)). On the other hand, the separation between Γ_1 and Γ_2 is crucial in the dead core analysis carried out in Theorem 11.

It turns out that the first eigenvalue σ_1 of the problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega_0 \\ \dfrac{\partial u}{\partial \nu} = \sigma u & \text{on} \quad \Gamma_1 \\ u = 0 & \text{on} \quad \Gamma_2, \end{cases} \tag{1.3}$$

will be determinant in the existence issue of solutions. The main features concerning σ_1 were studied in [11] (cf. Theorem 6). Some of them are recalled in Remarks 4 and Theorem 8 of the present work. In such statement, and for our purposes here, problem (1.3) is further analyzed in the more ambitious case where (H) fails and $\Gamma_1 \cap \Gamma_2 \neq \emptyset$ (i.e., the "genuine" mixed problem).

To fix the notation we are setting in what follows $\sigma_1 = \infty$ if $\Gamma_1 = \emptyset$, i.e. $\Omega_0 \subset\subset \Omega$. After these preliminaries, we can state the following result.

Theorem 4. *Assume Ω is a $C^{2,\gamma}$ bounded domain of $\mathbb{R}^N$, $a \in C^\gamma(\overline{\Omega})$ and $0 < p < 1$. Then problem* (1.1) *possesses the following features:*

(i) Problem (1.1) *does not have nonnegative nontrivial solutions when* $\lambda \le 0$. *For* $0 < \lambda < \sigma_1$, *there always exists a nonnegative nontrivial solution* $u \in C^{2,\gamma_1}(\overline{\Omega})$, $\gamma_1 = \min\{p,\gamma\}$.

(ii) There exists λ_0, $0 < \lambda_0 < \sigma_1$ *such that for* $0 < \lambda < \lambda_0$, *problem* (1.1) *has a* unique *classical nonnegative solution* u_λ. *The mapping* $\lambda \to u_\lambda$ *exhibits the same properties as in* Theorem 1 (ii). *In particular*

$$\lim_{\lambda\to 0+} \lambda^{\frac{1}{1-p}} u_\lambda = \left(\frac{1}{|\partial\Omega|} \int_\Omega a(x) \right)^{\frac{1}{1-p}}$$

in $C^{2,\gamma}(\overline{\Omega})$ *and so* $u_\lambda \to +\infty$ *uniformly in* $\overline{\Omega}$ *as* $\lambda \to 0+$.

(iii) If $\sigma_1 = +\infty$, *then there exist positive constants* C, K *and* λ_2 *such that for all nonnegative nontrivial solutions* $u_\lambda \in C^{2,\gamma_1}(\overline{\Omega})$ *to* (1.1) *with* $\lambda \ge \lambda_2$ *we have*

$$u_\lambda \le C\lambda^{-\frac{2}{1-p}}.$$

In addition u *develops a dead core* $\mathcal{O}_\lambda$ *with* $\{\operatorname{dist}(x,\partial\Omega) \ge d_\lambda\} \subset \mathcal{O}_\lambda$ *where* $d_\lambda \to 0+$ *as* $\lambda \to +\infty$. *Moreover,* $d_\lambda = K/\lambda$, *for a constant* $K > 0$, *provided* $a > 0$ *on* Γ_1.

An important difference of problem (1.1) in the range $0 < p < 1$ with respect to $p > 1$ arises when $\lambda \ge \sigma_1$. Specifically, it was proved in [11] that no positive solutions of (1.1) with $p > 1$ exist when $\lambda \ge \sigma_1$, provided $\sigma_1 < +\infty$ (see Figure 2). We are showing next that this is indeed the case if $\Omega^+ := \{x \in \Omega :\ a(x) > 0\} \subset\subset \Omega$, but things are quite different otherwise.

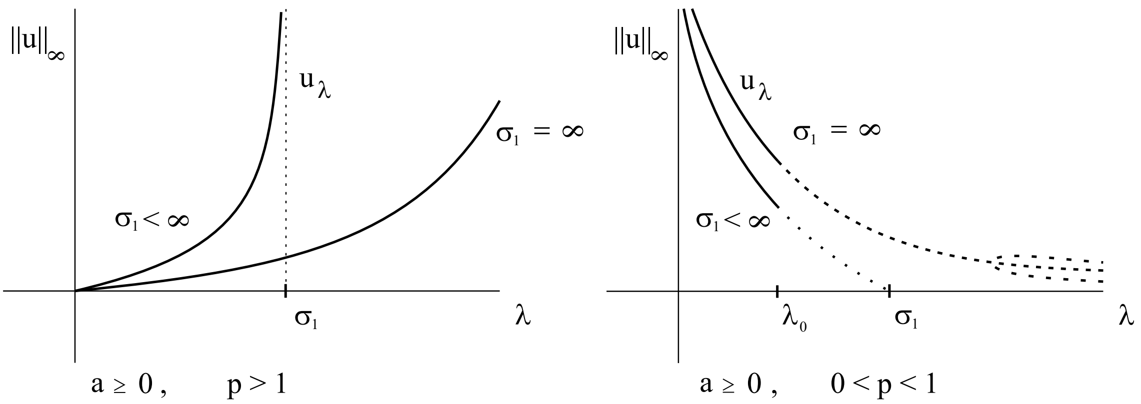

FIGURE 2. Bifurcation diagrams $p > 1$ versus $0 < p < 1$. In both regimes drawings for cases $\sigma_1 = \infty$ and $\sigma_1 < \infty$ but $\tilde{\sigma}_1 = \infty$ are superposed. The meaning of continuous and dotted or dashed arcs is the same as in Figure 1.

In explaining the discrepancy when $\partial\Omega^+ \cap \partial\Omega \ne \emptyset$ the eigenvalue problem (1.3) –now observed in Ω^+– has again an important rôle. Let $\{\Omega_i^+\}$ be the set of (finitely many) connected pieces of Ω^+. Observe that $\partial\Omega^+ \cap \Omega = \partial\Omega_0 \cap \Omega = \Gamma_2$ (as defined before) and so $\Gamma_2 = \cup_i(\partial\Omega_i^+ \cap \Omega)$. Notice again that, due to $a \not\equiv 0$ and the connectedness of Ω each $\Gamma_{2,i} := \partial\Omega_i^+ \cap \Omega$ is nonempty. In addition $\Gamma^+ := \partial\Omega^+ \cap \partial\Omega = \partial\Omega \setminus \Gamma_1 = \cup_i(\partial\Omega_i^+ \cap \partial\Omega)$. If

$\partial\Omega^+ \cap \partial\Omega \neq \emptyset$ some $\partial\Omega_i^+$ meets $\partial\Omega$. Precisely for all those components Ω_i^+ define $\sigma = \tilde{\sigma}_{1,i}$ as the first eigenvalue to the problem,

$$\begin{cases} \Delta u = 0 & x \in \Omega_i^+ \\ \dfrac{\partial u}{\partial \nu} = \sigma u & x \in \partial\Omega_i^+ \cap \partial\Omega \\ u = 0 & x \in \Gamma_{2,i} = \partial\Omega_i^+ \cap \Omega. \end{cases} \tag{1.4}$$

Designate $\tilde{\sigma}_1 = \min \tilde{\sigma}_{1,i}$ if $\partial\Omega^+ \cap \partial\Omega \neq \emptyset$ setting $\tilde{\sigma}_1 = +\infty$ otherwise (i.e. the case $\partial\Omega = \Gamma_1$).

The new features concerning (1.1) in the case a vanishing in Ω are next described.

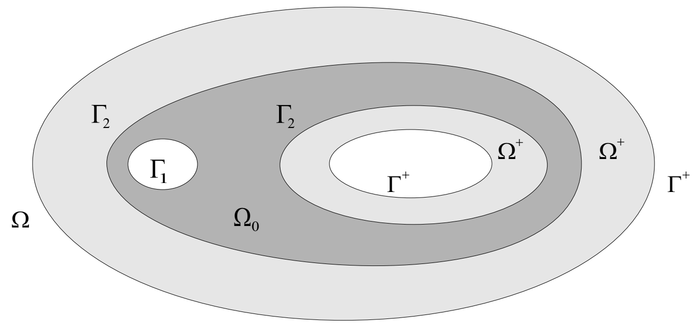

FIGURE 3. A possible configuration for the domains Ω, Ω_0, Ω^+. In this example, Ω^+ possesses two connected pieces

Theorem 5. *Assume $\sigma_1 < +\infty$. Then:*

(i) If $\tilde{\sigma}_1 = +\infty$, then there are no nonnegative nontrivial solutions to (1.1) *with $\lambda \geq \sigma_1$.*

(ii) If $\tilde{\sigma}_1 < +\infty$ and there exists a nonnegative nontrivial solution to (1.1) *with $\lambda \geq \sigma_1$, then $\lambda > \tilde{\sigma}_1$. In particular, if $\tilde{\sigma}_1 \geq \sigma_1$, then there are no solutions for $\lambda \in [\sigma_1, \tilde{\sigma}_1]$.*

(iii) When $\tilde{\sigma}_1 < +\infty$, there exists $\lambda_2 > 0$ such that for $\lambda \geq \lambda_2$ problem (1.1) *has at least a nonnegative nontrivial solution, which develops a dead core $\mathcal{O}_\lambda$. In this case $\{\text{dist}(x, \Gamma^+) \geq d_\lambda\} \subset \mathcal{O}_\lambda$ with $d_\lambda \to 0+$ as $\lambda \to +\infty$, $\Gamma^+ = \partial\Omega^+ \cap \partial\Omega$. Similarly, the choice $d_\lambda = K/\lambda$, $K > 0$, is again possible when $a > 0$ on Γ^+.*

Remarks 1.

(a) By means of examples it is possible to show that both options $\sigma_1 \leq \tilde{\sigma}_1$ and $\sigma_1 > \tilde{\sigma}_1$ can occur. Notice that such relative positions depend only on the support of the weight a, not on its size. See Remark 8.

(b) Regarding (ii) it is also possible to produce examples where $\tilde{\sigma}_1 < \sigma_1$ and either no solutions $u \geq 0$ ($u \not\equiv 0$) exist for $\sigma_1 \leq \lambda \leq \sigma_1 + \varepsilon$, or a nonnegative nontrivial solution exists for all $\lambda \geq \sigma_1 - \varepsilon$, $\varepsilon > 0$ small. See also Remark 8.

(c) As already noticed and in contrast with the case $p > 1$, (1.1) can support nonnegative nontrivial solutions for $\lambda > \sigma_1$, provided $\tilde{\sigma}_1 < \sigma_1$. However, such solutions are "degenerate" in the sense that they must vanish in Ω_0 together with all those (if any) connected pieces Ω_i^+ of Ω^+ such that $\Omega_i^+ \subset\subset \Omega$ (cf. Section 7).

(d) According to (ii)-(iii) the set of solutions to (1.1) undergoes a discontinuity at $\lambda = \sigma_1$ to –roughly speaking– arise again at some λ after $\tilde{\sigma}_1$ (suppose $\sigma_1 \leq \tilde{\sigma}_1$). It should be remarked that the latter arising is necessarily "spontaneous" in the sense that when solutions appear are bounded away from zero. In other words, they are not generated by a bifurcation from the trivial solution $u = 0$. See Remark 7.

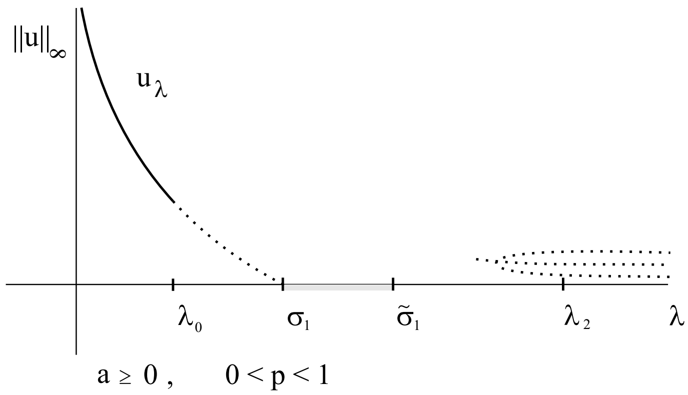

FIGURE 4. A bifurcation diagram for the case $\sigma_1 < \tilde{\sigma}_1 < +\infty$. Solutions disappear converging to zero at σ_1 and spontaneously appear beyond $\lambda = \tilde{\sigma}_1$.

Finally, we conclude our analysis by determining the asymptotic behavior of solutions near $\lambda = \sigma_1$ in the critical situation where solutions cease to exist at that value. We assert that $(\lambda, u) = (\sigma_1, 0)$ defines in that case a bifurcation point for solutions to (1.1).

Theorem 6. *Assume $\sigma_1 < +\infty$, with either $\sigma_1 \leq \tilde{\sigma}_1$ or $\tilde{\sigma}_1 = +\infty$, or even $\tilde{\sigma}_1 < \sigma_1$ but in this case also supposing that (1.1) has only $u = 0$ as solution at $\lambda = \sigma_1$. Then every family of nonnegative solutions $\{u_\lambda\}$ with λ close to σ_1 verifies*

$$u_\lambda \to 0,$$

in $C^{2,\beta}(\overline{\Omega})$, $0 < \beta < \gamma_1$, as $\lambda \to \sigma_1-$. In addition, there exists $\delta > 0$ such that every nonnegative nontrivial solution u_λ to (1.1) for $\sigma_1 - \delta \leq \lambda < \sigma_1$ develops a dead core $\mathcal{O}_\lambda$. More importantly, such dead core must satisfy $\mathcal{O}_\lambda \subset \Omega^+$ together with $\mathcal{O}_\lambda \to \Omega^+$ uniformly as $\lambda \to \sigma_1+$.

Remark 2. Compared with the case $\lambda \geq \sigma_1$ (Remark 1 (c)) it follows from the preceding statement that nonnegative solutions to (1.1) with $\lambda < \sigma_1$, $\lambda \sim \sigma_1$, exhibit the opposite behavior in Ω_0. Namely, all those solutions are strictly positive in Ω_0, aside of converging to zero as $\lambda \to \sigma_1-$.

The paper is organized as follows: in Section 2, we state some preliminaries which will be used in the paper. Section 3 is devoted to existence of nonnegative solutions and uniqueness for small λ, while in Section 4 the asymptotic behavior as $\lambda \to +\infty$ is elucidated. In Sections 5 and 6 we prove Theorems 2 and 3, respectively. A mixed problem, closely related to (1.1) is also studied in Section 5 (Theorem 11). Finally, Section 7 deals with problem (1.1) when the weight $a(x)$ vanishes in a subdomain Ω_0 of Ω (Theorems 4, 5 and 6).

2. Preliminaries

In this section we collect some results which will be needed for the proofs of our theorems. We begin by proving that weak solutions to (1.1) are indeed classical solutions. It should be stressed that this result is not contained in [1]. However, as will be seen at once, it can be proved by means of their estimates.

Lemma 7. *Let $u \in H^1(\Omega)$ be a weak nonnegative solution to* (1.1)*. Then $u \in C^{2,\gamma_1}(\overline{\Omega})$, $\gamma_1 = \min\{\gamma, p\}$, and thus defines a classical solution.*

Proof. For $\lambda > 0$ fixed and $\mu > 0$ large enough, the problem

$$\begin{cases} \Delta v - \mu v = f & \text{in} \quad \Omega \\ \dfrac{\partial v}{\partial \nu} = \lambda v & \text{on} \quad \partial\Omega \end{cases} \tag{2.1}$$

with $f = 0$ has $u = 0$ as the only weak solution in $H^1(\Omega)$. In fact, any nontrivial solution $u \in H^1(\Omega)$ to (2.1) with $f = 0$ defines a weak eigenfunction corresponding to the eigenvalue $\sigma = \lambda$ of the Steklov-type eigenvalue problem:

$$\begin{cases} \Delta v - \mu v = 0 & \text{in} \quad \Omega \\ \dfrac{\partial v}{\partial \nu} = \sigma v & \text{on} \quad \partial\Omega. \end{cases} \tag{2.2}$$

Techniques in Section 2 of [11] can be used to show that problem (2.2) possesses, for every μ, a *first* eigenvalue $\sigma_1 = \sigma_1(\mu)$ which is the unique associated to a positive eigenfunction in $H^1(\Omega)$ ("a posteriori" in $C^{2,\gamma}(\overline{\Omega})$) and variationally characterized as

$$\sigma_1 = \inf_{v \in H^1(\Omega) \setminus \{0\}} \frac{\int_\Omega \{|\nabla v|^2 + \mu v^2\}}{\int_{\partial\Omega} v^2}. \tag{2.3}$$

Moreover, σ_1 increases with μ. We claim that $\sigma_1 \to \infty$ as $\mu \to \infty$. This implies the desired assertion if μ is so large as to have $\lambda < \sigma_1(\mu)$. To show the claim suppose that $\sigma_1(n) = O(1)$ as $n \to \infty$. If $\phi_n \in H^1(\Omega)$ stands for the positive eigenfunction corresponding to $\sigma_1(n)$ normalized so that $\int_{\partial\Omega} \phi_n^2 = 1$, then ϕ_n remains bounded in $H^1(\Omega)$ under the norm $|u|_{H^1}^2 = \int_\Omega |\nabla u|^2 + \int_{\partial\Omega} u^2$. Modulo a subsequence, $\phi_n \to \phi$ weakly in H^1 and by compactness $\phi_n \to \phi$ in both $L^2(\Omega)$ and $L^2(\partial\Omega)$. In particular $\int_{\partial\Omega} \phi^2 = 1$. However, the boundedness of $\sigma_1(n)$ leads to $\int_\Omega \phi^2 = 0$ which is not possible. Thus the claim is proved.

Since (2.1) can admit at most one classical solution, the results in [14] (Theorem 6.31 and subsequent remark) ensure that such problem has indeed a unique solution $v \in C^{2,\gamma}(\overline{\Omega})$ for every $f \in C^\gamma(\overline{\Omega})$.

Now let $u \in H^1(\Omega)$ be a nonnegative solution to (1.1). By the Sobolev embedding, we have $u, u^p \in L^r(\Omega)$ for every $1 \le r \le 2^* = 2N/(N-2)$ (this is only true if $N \ge 3$; for $N = 2$ the situation is even better). Take a sequence of C^γ functions f_n converging to $a(x)u^p - \mu u$ in $L^r(\Omega)$, and let v_n be the unique solution to (2.1) with $f = f_n$. We now use the estimates of Agmon-Douglis-Nirenberg [1]. There exists a constant $C > 0$ such that

$$|v_n|_{W^{2,r}} \le C |f_n|_{L^r}.$$

It follows that $v_n \to v$ in $W^{2,r}(\Omega)$, where v is the (unique) solution to

$$\begin{cases} \Delta v - \mu v = a(x)u^p - \mu u & \text{in} \quad \Omega \\ \dfrac{\partial v}{\partial \nu} = \lambda v & \text{on} \quad \partial\Omega. \end{cases}$$

Thus $v = u$, and so $u \in W^{2,r}(\Omega)$. Repeating this argument a finite number of times, we arrive at $u \in W^{2,q}(\Omega)$ for some $q > N$, and hence $u \in C^{1,\eta}(\overline{\Omega})$ for a certain $0 < \eta < 1$.

Next we choose a sequence $f_n \in C^{\gamma_1}(\overline{\Omega})$ so that $f_n \to a(x)u^p - \mu u$ in $C^{\gamma_1}(\overline{\Omega})$, and denote by v_n the $C^{2,\gamma_1}(\overline{\Omega})$ solutions to (2.1) corresponding to $f = f_n$. We now use the Schauder estimates provided by Theorem 6.30 in [14] to obtain

$$|v_n|_{C^{2,\gamma_1}} \leq C(|f_n|_{C^{\gamma_1}} + |v_n|_{C^{\gamma_1}}).$$

From which it similarly follows that $v_n \to u$ in $C^{2,\gamma_1}(\overline{\Omega})$, and thus $u \in C^{2,\gamma_1}(\overline{\Omega})$. □

Remark 3. Observe that as a consequence of the last part of the proof of Lemma 7 *positive* solutions u to (1.1) are slightly more regular and directly lie in $C^{2,\gamma}(\overline{\Omega})$, while this is not necessarily the case if such solutions vanish somewhere in Ω.

We are now introducing an eigenvalue problem under mixed boundary conditions of Dirichlet and Steklov type, located on zones Γ and Γ' of the boundary, which will play an important rôle in some of our forthcoming proofs. Theorem 8 in [11] analyzed this problem in full detail when the zones carrying different boundary conditions are *different* –and hence separated away– components of the boundary, providing class $C^{2,\gamma}$ eigenfunctions. This is precisely the more frequently used version in this work. Our next result deals with the more adverse situation in which Γ and Γ' meet on a $N-2$ dimensional closed manifold γ. Such scenario will be required for showing the estimate (1.2) in Section 4. Notice that in this case results in Section 2 of [11] still provide H^1 eigenfunctions (see Remark 8 there). However, their optimal degree of regularity is an issue belonging to the subtle realm of smoothness of weak solutions to mixed problems (see for instance [15, 16]). Accordingly, we only provide next the amount of weak smoothness strictly required for our purposes here. Its proof, an straightforward consequence of Section 2 in [11] and [4], is only sketched.

Theorem 8. *Let $D \subset \mathbb{R}^N$ a bounded domain of class C^3 such that $\partial D = \Gamma \cup \Gamma' \cup \gamma$, Γ, Γ', γ pair-wise disjoint, Γ, Γ' relatively open in ∂D, γ a closed $N-2$ dimensional manifold while $\Gamma \cup \gamma$ and $\Gamma' \cup \gamma$ define $N-1$ dimensional manifolds of class C^3 with common boundary γ. Then the eigenvalue problem*

$$\begin{cases} \Delta u = 0 & \text{in} \quad D \\ \dfrac{\partial u}{\partial \nu} = \sigma u & \text{on} \quad \Gamma \\ u = 0 & \text{on} \quad \Gamma', \end{cases} \tag{2.4}$$

admits a principal eigenvalue σ_1, i. e. an eigenvalue with a one-signed eigenfunction, which is characterized by:

$$\sigma_1 = \inf_{u \in H^1_{\Gamma'}(D)} \frac{\int_D |\nabla u|^2}{\int_\Gamma u^2}, \tag{2.5}$$

where $H^1_{\Gamma'}(D)$ *is the subset of functions of* $H^1(D)$ *which vanish on* Γ'. *Moreover,* σ_1 *is the unique principal eigenvalue, simple and the smallest of all eigenvalues. Moreover, every associated eigenfunction* ϕ *verifies* $\phi \in H^1(D) \cap W^{2,q}(D)$ *for all* $q < 4/3$.

Sketch of the proof. Observe that $H^1_{\Gamma'}(D)$ is well defined even if Γ and Γ' intersect. Consider $\mathcal{M} = \{u \in H^1_{\Gamma'}(D) : \int_\Gamma u^2 = 1\}$ and set $J(u) = \int_D |\nabla u|^2$. Regarding $H^1_{\Gamma'}(D)$ endowed with the equivalent norm $|u|_1 = (\int_D |\nabla u|^2)^{1/2}$ it follows that J is coercive, i. e. $J(u_n) \to +\infty$ whenever $|u_n|_1 \to +\infty$. Since J is also weakly lower semicontinuous and $\mathcal{M}$ is weakly closed the basic result in the calculus of variations (see [20]) provides with a global minimizer $\phi \in H^1_{\Gamma'}(D)$, which is an eigenfunction of (2.4) associated to σ_1.

It can be assumed that $\phi^+ = \max\{\phi, 0\} \not\equiv 0$, and hence it is also a minimizer. Thus ϕ^+ is harmonic in Ω and by the Harnack inequality we have $\phi^+ > 0$, hence $\phi > 0$. By an orthogonality argument it follows that σ_1 is simple, and it is the only eigenvalue associated to a positive eigenfunction. Finally, the extra regularity of the eigenfunction is a consequence of Theorem B in [4].

Remarks 4.

(a) For $\lambda > 0$ designate by D_λ the scaled copy of D, i.e. $D_\lambda = \{\lambda x : x \in D\}$. If $\sigma_1(D_\lambda)$ stands for the principal eigenvalue of (2.4) in D_λ one directly finds that $\sigma_1(D_\lambda) = \dfrac{1}{\lambda}\sigma_1(D)$.

(b) A suitable use of the L^p estimates in [1] proves that every eigenfunction ϕ associated to σ_1 satisfies $\phi \in C^{2,\eta}(D \cup \overline{T})$, $0 < \eta < 1$ arbitrary, where T is any relatively open part of ∂D strongly contained either in Γ or Γ'. Thus, as a consequence of Hopf's maximum principle, every positive eigenfunction satisfies $\dfrac{\partial \phi}{\partial \nu} < 0$ on Γ' where ν is the outward unit normal on Γ'. On the other hand, although not in our framework here, smoothness of such eigenfunctions ϕ is enlarged provided Γ and Γ' meet in a convenient nonsmooth way on γ. For instance, under a suitable –not too large– angle (cf. [15, 16] and their references). In any case, it should be recalled that eigenfunctions achieve the full $C^{2,\gamma}$ regularity assumed that $\partial\Omega$ is *only* $C^{2,\gamma}$ when Γ and Γ' are *disjoint* closed manifolds (see [11]).

(c) The hypothesis that D is of class C^3 is required in [4] for the $W^{2,q}$ regularity of the eigenfunctions. On the other hand, the exponent 4/3 for the integrability of the second derivatives is somehow optimal, as an example in [19] shows.

Finally, we are introducing some results from [12] concerning a singular initial value problem, which will be needed in Section 6. It should be remarked that such results were established there for the harder framework of the p-Laplacian operator. As a very special case of them we consider the Cauchy problem:

$$\begin{cases} ((r+d)^{N-1}u')' = (r+d)^{N-1}u^p & r \in (0,\infty) \\ u(0) = 0, \ u'(0) = 0, \end{cases} \tag{2.6}$$

with $d \geq 0$, which arises after some normalization when one considers the radial version of (1.1) with $a(x) = 1$. As a consequence of Theorems 2.3, 2.5 and 2.6 (see also Corollary 2.4) in [12] and a globalizing argument, we can state the following theorem.

Theorem 9. *For each $d \geq 0$, problem* (2.6) *has a unique* nontrivial *solution $u(r, d)$ defined in $[0, +\infty)$, in the sense that $u(r, d) > 0$ for $r > 0$. Moreover, $u(\cdot, d) \to u_0(\cdot)$ as $d \to \infty$ in $C^1_{\mathrm{loc}}[0, +\infty)$, where $u_0(\cdot)$ is the nontrivial solution to* (2.6) *corresponding to $N = 1$, which is explicitly given by*

$$u_0(r) = Ar^{\alpha},$$

where $\alpha = 2/(1-p)$ and $A = (\alpha(\alpha-1))^{-\frac{1}{1-p}}$. In addition, $u(r, d)$ is differentiable with respect to d and verifies

$$\frac{\partial u}{\partial d}(\cdot, d) \to 0, \qquad \text{as } d \to \infty \tag{2.7}$$

in $C^1_{\mathrm{loc}}[0, +\infty)$.

Remark 5. Notice that problem (2.6) always has the trivial solution $u \equiv 0$. It follows that this is the only nonnegative solution when $p \geq 1$. However, when $0 < p < 1$ there is a *unique* positive solution and infinitely many nonnegative (nontrivial) solutions, all of them expressible in terms of the positive one (cf. Corollary 2.4 in [12]).

3. Existence of nonnegative solutions. Uniqueness for small λ

In this section we are considering the issue of existence of nonnegative nontrivial solutions to (1.1) when $a(x) > 0$ in Ω. It will be shown that for every $\lambda > 0$ there always exists at least one solution (while for $\lambda \leq 0$ they cease to exist). In addition, we will show that the solution is unique provided λ is sufficiently small. In fact, a bifurcation from infinity at $\lambda = 0$ takes place.

Proof of Theorem 1 (i). According to Lemma 7, all weak solutions in $H^1(\Omega)$ belong to $C^{2,\gamma_1}(\overline{\Omega})$. Assume first $\lambda \leq 0$. Integrating the equation (1.1) in Ω:

$$\int_\Omega a(x)u^p = \int_\Omega \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = \lambda \int_{\partial\Omega} u \leq 0,$$

from which $u \equiv 0$ follows.

Now assume $\lambda > 0$. We use in $H^1(\Omega)$ the norm $|u|^2_{H^1} = \int_\Omega |\nabla u|^2 + \int_{\partial\Omega} u^2$ which is equivalent to the usual one. Define

$$M := \left\{ u \in H^1(\Omega) : \int_\Omega a(x)|u|^{p+1} = 1 \right\}$$

and for $u \in M$ the functional

$$J(u) = \int_\Omega |\nabla u|^2 - \lambda \int_{\partial\Omega} u^2.$$

We claim that J is coercive on M. Indeed, assume on the contrary that there exists a sequence $\{u_n\} \subset M$ such that $|u_n|_{H^1} \to +\infty$ and $J(u_n) \leq C$, that is

$$\int_\Omega |\nabla u_n|^2 \leq \lambda \int_{\partial\Omega} u_n^2 + C. \tag{3.1}$$

Denoting by $t_n = |u_n|_{L^2(\partial\Omega)}$, it follows from (3.1) that $t_n \to +\infty$. Letting $v_n = u_n/t_n$, we obtain again from (3.1) a bound for the H^1 norms of v_n. Passing to a subsequence, we obtain $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$, strongly in $L^2(\Omega)$ and in $L^2(\partial\Omega)$. In particular, $|v|_{L^2(\partial\Omega)} = 1$. On the other hand, since $p < 1$ and $u_n \in M$, we have that $\int_\Omega a(x)|v|^{p+1} = 0$, and so $v = 0$, a contradiction. Thus J is coercive.

Since M is weakly closed, and J is (sequentially) weakly lower semicontinuous, it follows from standard results that J achieves its minimum in M (see [20]). Also, since $J(|u|) = J(u)$ and $|u| \in M$ whenever $u \in M$, we may assume that the minimum is achieved at a nonnegative (nontrivial) function u. By the Lagrange's multipliers rule, there exists $\mu \in \mathbb{R}$ such that

$$\int_\Omega \nabla u \nabla \varphi - \lambda \int_{\partial\Omega} u\varphi = \mu \int_\Omega a(x) u^p \varphi$$

for every $\varphi \in H^1(\Omega)$. Taking in particular $\varphi = u$, we arrive at $J(u) = \mu < 0$, since there is a constant function which belongs to M for which J is negative. Setting $v = |\mu|^{1-p} u$, we obtain a nonnegative nontrivial weak solution to (1.1), which is, as remarked earlier, a classical solution in $C^{2,\gamma_1}(\overline{\Omega})$. □

Remark 6. Nonnegative solutions can also be obtained by applying the Mountain Pass theorem to the functional

$$\tilde{J}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_{\partial\Omega} (u_+)^2 + \frac{1}{p+1}\int_\Omega a(x)|u|^{p+1},$$

in $H^1(\Omega)$, where $u_+ = \max\{u, 0\}$ denotes the positive part of u.

We now proceed to show Theorem 1 (ii). The proof makes use of local bifurcation theory. As a first step in this approach, we begin by characterizing the behavior of all nonnegative solutions for λ approaching zero.

Lemma 10. *Assume $\lambda_n \to 0$, and let u_n be a corresponding sequence of nonnegative nontrivial solutions to* (1.1)*. Then*

$$\begin{cases} \lambda_n = \mu_n t_n \\ u_n = \dfrac{1}{t_n^\theta}(1 + t_n w_n) \end{cases}$$

where $\theta = 1/(1-p)$, $t_n = \left(\frac{1}{|\partial\Omega|}\int_{\partial\Omega} u_n\right)^{p-1} \to 0$, $w_n \in Y := \{u \in C^{2,\gamma}(\overline{\Omega}) : \int_{\partial\Omega} u = 0\}$. Moreover,

$$\mu_n \to \mu_0 = \frac{1}{|\partial\Omega|}\int_\Omega a(x), \tag{3.2}$$

and $w_n \to w_0$ in $C^{2,\beta}(\overline{\Omega})$ for all $0 < \beta < \gamma$, where w_0 is the unique solution in Y of the linear equation

$$\begin{cases} \Delta w = a(x) & \text{in} \quad \Omega \\ \dfrac{\partial w}{\partial \nu} = \mu_0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.3}$$

Proof. First of all, notice that (3.3) has a unique solution in Y, according to the compatibility condition given by the value of μ_0 in (3.2).

We begin by showing that $|u_n|_\infty$ tends to infinity. Assume on the contrary that for a subsequence (still labelled by u_n) there holds $|u_n|_\infty \le C$. Observing that problem (2.1) corresponding to $\mu = 0$ has a unique solution for λ small and by employing the Agmon-Douglis-Nirenberg estimates (see [1]), we have for every $q > 1$

$$|u_n|_{W^{2,q}} \le C_1(|u_n|_{L^q} + |u_n^p|_{L^q}) \le C_2, \tag{3.4}$$

$C_1 > 0$ being certain constant depending on q. Selecting $q > N$, we have by the Sobolev imbedding (passing to a subsequence) that $u_n \to u$ in $C^{1,\eta}(\overline{\Omega})$ for some $0 < \eta < 1$. It follows that u is a nonnegative weak (hence classical by Lemma 7) solution to (1.1) with $\lambda = 0$. Thus, $u \equiv 0$ (in particular, $u_n \to 0$ in $H^1(\Omega)$). We claim that this is impossible. Indeed, let $v_n = u_n/|u_n|_{H^1}$. There exists a subsequence (labelled once more by v_n) such that $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$, strongly in $L^2(\Omega)$, in $L^2(\partial\Omega)$ and almost everywhere in Ω. It follows from (1.1) that

$$\int_\Omega |\nabla v_n|^2 = \lambda_n \int_{\partial\Omega} v_n^2 - |u_n|_{H^1}^{p-1} \int_\Omega a(x) v_n^{p+1} \le \lambda_n \int_{\partial\Omega} v_n^2 \to 0,$$

so by lower semicontinuity we deduce that v is a constant. As a consequence, it follows that $v_n \to v$ *strongly* in $H^1(\Omega)$. On the other hand, again by (1.1), we have that

$$|u_n|_{H^1}^{p-1} \int_\Omega a(x) v_n^{p+1} = \lambda_n \int_{\partial\Omega} v_n^2 - \int_\Omega |\nabla v_n|^2 \to 0,$$

and since $0 < p < 1$, we deduce

$$\int_\Omega a(x) v_n^{p+1} \to 0. \tag{3.5}$$

Now since $v_n \to v$ in $L^2(\Omega)$, we have $v_n \to v$ in $L^{p+1}(\Omega)$, and thus $v = 0$ in Ω, contradicting that $|v_n|_{H^1} = 1$. Thus $|u_n|_\infty \to +\infty$.

Setting again $v_n = u_n/|u_n|_\infty$, and recalling that v_n satisfies

$$\begin{cases} \Delta v_n = a(x) v_n^p |u_n|_\infty^{p-1} & \text{in} \quad \Omega \\ \dfrac{\partial v_n}{\partial \nu} = \lambda_n v_n & \text{on} \quad \partial\Omega, \end{cases}$$

we obtain arguing as before an estimate like (3.4), getting thus a bound in $C^{1,\eta}(\overline{\Omega})$ for the solutions v_n for some $0 < \eta < 1$. Applying Theorem 6.30 in [14] gives $C^{2,\gamma_1}(\overline{\Omega})$ bounds, and thus $v_n \to v$ in $C^{2,\beta}(\overline{\Omega})$, $0 < \beta < \gamma_1$, where v is a unique classical solution to

$$\begin{cases} \Delta v = 0 & \text{in} \quad \Omega \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

with $|v|_\infty = 1$. Hence $v \equiv 1$. Since v_n becomes positive for large n we furthermore find that $v_n \in C^{2,\gamma}(\overline{\Omega})$ (Remark 3) and $v_n \to 1$ in $C^{2,\beta}(\overline{\Omega})$ for all $0 < \beta < \gamma$.

We now split $u_n = c_n + c_n z_n$, where $c_n = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} u_n$ and $z_n \in Y$. It follows that

$$\frac{c_n}{|u_n|_\infty} \to 1 \quad \text{and} \quad z_n \to 0 \text{ in } C^{2,\beta}(\overline{\Omega}).$$

Next, notice that z_n solves

$$\begin{cases} \Delta z_n = a(x) c_n^{p-1}(1+z_n)^p & \text{in} \quad \Omega \\ \dfrac{\partial z_n}{\partial \nu} = \lambda_n (1+z_n) & \text{on} \quad \partial\Omega. \end{cases} \tag{3.6}$$

By integrating in (3.6) we obtain:

$$\int_\Omega a(x)(1+z_n)^p = \lambda_n c_n^{1-p}|\partial\Omega|.$$

Setting $t_n = c_n^{p-1}$ this means that $\lambda_n = t_n \mu_n$ where $\mu_n \to \mu_0$, μ_0 given by (3.2). Next, using that $\lambda_n = O(t_n)$ and from Schauder's estimates we conclude that $z_n = O(t_n)$ in $C^{2,\gamma}(\overline{\Omega})$. Writing $z_n = t_n w_n$ with w_n bounded in $C^{2,\gamma}(\overline{\Omega})$ we conclude that $w_n \to w_0$ in $C^{2,\beta}(\overline{\Omega})$, w_0 being the unique classical solution in Y to (3.3). This proves the lemma. □

Proof of Theorem 1 (ii). According to Lemma 10, all nonnegative nontrivial solutions to (1.1) for small λ are of the form:

$$\begin{cases} \lambda = t\mu \\ u = \dfrac{1}{t^\theta}(1+tw) \end{cases}$$

where $t > 0$ is small and $(\mu, w) \in \mathbb{R} \times Y$ is close to (μ_0, w_0), $\mu_0 \in \mathbb{R}$, $w_0 \in Y$ given by (3.2) and (3.3) respectively. Thus, for small $\lambda > 0$ solving (1.1) is equivalent to solving:

$$\begin{cases} \Delta w = a(x)(1+tw)^p & \text{in} \quad \Omega \\ \dfrac{\partial w}{\partial \nu} = \mu(1+tw) & \text{on} \quad \partial\Omega, \end{cases} \tag{3.7}$$

together with the compatibility condition:

$$\int_\Omega a(x)(1+tw)^p = \mu \int_{\partial\Omega}(1+tw), \tag{3.8}$$

for $|t|$, $|\mu - \mu_0|$ and $|w - w_0|_Y$ small, where $Y = \{u \in C^{2,\gamma}(\overline{\Omega}) : \int_{\partial\Omega} u = 0\}$ is endowed with its natural norm.

We are accordingly showing such uniqueness by means of the Implicit Function theorem as follows. Setting $X = C^\gamma(\overline{\Omega}) \times C^{1,\gamma}(\overline{\Omega})$, define $l \in X^*$ as:

$$l(f,g) = \int_\Omega f - \int_{\partial\Omega} g, \qquad (f,g) \in X.$$

According to Schauder's theory, for each $(f, g) \in \ker(l)$ the problem:

$$\begin{cases} \Delta u = f & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = g & x \in \partial\Omega, \end{cases}$$

has a unique solution $u \in Y$, denoted $u = K(f, g)$. Hence, $K : \ker(l) \to Y$ defines a linear topological isomorphism. Observe in addition that $(f, g) \mapsto (f, g + \dfrac{1}{|\partial\Omega|}\, l(f, g))$ defines a projection from X onto $\ker(l)$. Thus K is always defined at $(f, g + \dfrac{1}{|\partial\Omega|}\, l(f, g))$ for every $(f, g) \in X$.

Now introduce the Nemytskii operators $F(t, w) = a(1+tw)^p$, $G(t, w) = 1+tw$. For small and positive ε, δ, $F : (-\varepsilon, \varepsilon) \times B_Y(w_0, \delta) \to C^\gamma(\overline{\Omega})$, and $G : (-\varepsilon, \varepsilon) \times B_Y(w_0, \delta) \to C^{1,\gamma}(\overline{\Omega})$ define real analytic mappings, $B_Y(w_0, \delta)$ standing for the open ball in Y with radius δ and center w_0. Thus,

$$\begin{array}{rccc} \mathcal{F}: & (-\varepsilon, \varepsilon) \times (\mu_0 - \varepsilon, \mu_0 + \varepsilon) \times B_Y(w_0, \delta) & \longrightarrow & \mathbb{R} \times Y \\ & (t, \mu, w) & \longrightarrow & (\mathcal{F}_1(t, \mu, w), \mathcal{F}_2(t, \mu, w), \end{array}$$

where

$$\mathcal{F}_1(t, \mu, w) = l(F(t, w), \mu G(t, w)),$$

and

$$\mathcal{F}_2(t, \mu, w) = w - K(F(t, w), \mu G(t, w) + \frac{1}{|\partial\Omega|}\mathcal{F}_1(t, \mu, w))$$

is also a real analytic mapping.

On the other hand, solving (3.7), (3.8) for (t, μ, w) close to $(0, \mu_0, w_0)$ in $\mathbb{R}^2 \times Y$ is equivalent to solving:

$$\mathcal{F}(t, \mu, w) = 0, \tag{3.9}$$

with $(t, \mu, w) \in (-\varepsilon, \varepsilon) \times (\mu_0 - \varepsilon, \mu_0 + \varepsilon) \times B(w_0, \delta)$ with ε, δ small enough. However, $\mathcal{F}(0, \mu_0, w_0) = 0$, while

$$\mathcal{F}_1(0, \mu, w) = |\partial\Omega|(\mu_0 - \mu),$$

and,

$$\mathcal{F}_2(0, \mu, w) = w - K(a, \mu_0) = w - w_0,$$

Thus, the Fréchet derivative L acting on $(\hat{\mu}, \hat{w}) \in \mathbb{R} \times Y$ is given by

$$L(\hat{\mu}, \hat{u}) = D_{(\mu,w)}\mathcal{F}(0, \mu_0, w_0)(\hat{\mu}, \hat{w}) = (-|\partial\Omega|\hat{\mu}, \hat{w}).$$

Since L defines an isomorphism from $\mathbb{R}\times Y$ into itself, the real analytic version of the Implicit Function theorem ([5]) ensures that (3.9) is uniquely solvable with $\mu(t)$, $w(t, \cdot)$ real analytic in t and $\lambda = t\mu(t)$, $u = t^{-\theta}(1 + tw(t, \cdot))$ define the unique nonnegative solutions to (1.1) for small $\lambda > 0$. The remaining assertions in (*ii*) follow from this representation. □

4. Behavior for large λ

In this section, we study the properties of the nonnegative nontrivial solutions to (1.1) for large positive λ, assuming $a(x) > 0$ in Ω.

Proof of Theorem 1 (iii). We are using the well-known blow-up technique of Gidas and Spruck (see [13]). Assume that (1.2) does not hold. Let $\lambda_n \to +\infty$ and u_n be a sequence of corresponding nonnegative nontrivial solutions to (1.1), such that

$$\lambda_n^{\frac{2}{1-p}} M_n \to +\infty \tag{4.1}$$

as $n \to +\infty$, where $M_n := \max_{\overline{\Omega}} u_n$. Choose points $\tilde{x}_n \in \partial\Omega$ such that $u_n(\tilde{x}_n) = M_n$ (notice that the functions u_n are subharmonic), and assume (extracting a subsequence if necessary) that $\tilde{x}_n \to \tilde{x}_0 \in \partial\Omega$. By means of a translation and a rotation, it can be assumed that $\tilde{x}_0 = 0$ while $\{x_N = 0\}$ defines the tangent hyperplane to $\partial\Omega$ at such point. By hypothesis, there exist $R > 0$ and $\varphi \in C^{2,\gamma}(B(0,R) \cap \{x_N = 0\})$, $\varphi(0) = 0$, $\nabla\varphi(0) = 0$ such that by writing $x \in \mathbb{R}^n$ as $x = (x', x_N)$, $x' = (x_1, \dots, x_{N-1})$ then $\Omega \cap B(0,R) = \{x : x_N > \varphi(x')\}$ and $\partial\Omega \cap B(0,R) = \{x : x_N = \varphi(x')\}$. Then, the standard $C^{2,\gamma}$ diffeomorphism $y = h(x)$ given by $y' = x'$, $y_N = x_N - \varphi(x')$ maps $B(0,R)$ onto a neighborhood V of $y = 0$ in R^N, $\Omega \cap B(0,R)$ onto $V^+ = V \cap \mathbb{R}^N_+$, $\partial\Omega \cap B(0,R)$ onto $V \cap \partial\mathbb{R}^N_+$. Problem (1.1) in $\Omega \cap B(0,R)$ is transformed in:

$$\begin{cases} \Delta u + \displaystyle\sum_{i=1}^{N-1} a_i(y) \frac{\partial^2 u}{\partial y_i \partial y_N} + |\nabla\varphi(y')|^2 \frac{\partial^2 u}{\partial y_N^2} + b(y) \frac{\partial u}{\partial y_N} = a(y) u^p & y \in V^+ \\ \nabla u \ \cdot \nu_1(y) = \lambda u & y \in V \cap \partial\mathbb{R}^N_+, \end{cases} \tag{4.2}$$

with

$$a_i = -2\frac{\partial\varphi}{\partial x_i}, \qquad b(y) = -\Delta\varphi, \qquad \nu_1 = (\nu', -\nu'\nabla\varphi + \nu_n),$$

$\nu(x) = (\nu'(x), \nu_N(x))$ the outer unit normal on $\partial\Omega$, all such functions being evaluated at $x = h^{-1}(y)$.

On the other hand, setting $\tilde{y}_n = h(\tilde{x}_n)$ then $\tilde{y}_n \to 0$ and a positive R_1, not depending on n, can be found so that the translations $w_n(y) = u_n(y + \tilde{y}_n)$ are all defined in $B(0,R_1) \cap \overline{\mathbb{R}^N_+}$. Setting $U = B(0,R_1) \cap \mathbb{R}^N_+$, $U_n = \lambda_n U$ and performing the change

$$v_n(y) = M_n^{-1} u_n(\lambda_n^{-1} y + \tilde{y}_n),$$

the functions v_n define solutions to:

$$\begin{cases} \Delta v + \displaystyle\sum_{i=1}^{N-1} a_{i,n}(y) \frac{\partial^2 v}{\partial y_i \partial y_N} + a_{N,n}(y) \frac{\partial^2 v}{\partial y_N^2} + \lambda_n^{-1} b_n(y) \frac{\partial v}{\partial y_N} = \lambda_n^{-2} M_n^{p-1} a_n(y) v^p & y \in U_n^+ \\ \nabla v \ \cdot \nu_{1,n}(y) = \lambda v & y \in U_n \cap \partial\mathbb{R}^N_+, \end{cases} \tag{4.3}$$

where $a_{i,n}(y) = a_i(\lambda_n^{-1} y + \tilde{y}_n)$, $1 \le i \le N-1$, $a_{N,n}(y) = |\nabla\varphi(\lambda_n^{-1} y' + \tilde{y}_n')|^2$, $b_n(y) = b(\lambda_n^{-1} y + \tilde{y}_n)$, $a_n(y) = a(\lambda_n^{-1} y + \tilde{y}_n)$, $\nu_{1,n}(y) = \nu_1(\lambda_n^{-1} y + \tilde{y}_n)$.

Observe that $U_n \to \mathbb{R}^N_+$, $\overline{U}_n \to \overline{\mathbb{R}^N_+}$. Since $|v_n|_{\infty,U_n} = 1$ for all n, then the interior version of the L^p and Schauder estimates imply that (modulo a subsequence) $v_n \to v$ in $C^{2,\beta}(\mathbb{R}^N_+)$ for all $0 < \beta < \gamma_1$. Moreover, by employing the "up to the boundary" version of such estimates in the region $\overline{B(0,R_2)} \cap \mathbb{R}^N_+$, $R_2 > 0$ arbitrary and n large, leads to the validity of such convergence in $C^{2,\beta}_{\text{loc}}(\overline{\mathbb{R}^N_+})$. That is why v defines a nonnegative solution to:

$$\begin{cases} \Delta v = 0 & x \in \mathbb{R}^N_+ \\ -\dfrac{\partial v}{\partial y_N} = v & x \in \partial\mathbb{R}^N_+, \end{cases} \tag{4.4}$$

such that $0 \le v \le 1$, $v(0) = 1$. We are now showing that this is impossible. Consider in fact a C^3 bounded subdomain D of $\mathbb{R}^N_+$ such that $\Gamma := \partial D \cap \partial\mathbb{R}^N_+ = \overline{B(0,1)} \cap \partial\mathbb{R}^N_+$, and set $\Gamma' = \partial D \cap \mathbb{R}^N_+$. Define as D_n the magnified version $D_n = nD$ of D, $\Gamma_n = n\Gamma$, $\Gamma'_n = n\Gamma'$. According to Theorem 8 the eigenvalue problem

$$\begin{cases} \Delta u = 0 & x \in D_n \\ -\dfrac{\partial u}{\partial y_N} = \sigma u & x \in \Gamma_n \\ u = 0 & x \in \Gamma'_n, \end{cases} \tag{4.5}$$

admits a first eigenvalue $\sigma = \sigma_{1,n}$ with a positive associated eigenfunction $\phi_n \in H^1(D_n) \cap W^{2,q}(D_n) \cap C^{2,\gamma}(D_n \cup T)$, for all $1 < q < 4/3$ and any closed $T \subset B(0,n) \cap \partial\mathbb{R}^N_+$. Moreover, $\sigma_{1,n} \to 0$ (Remark 4, (a)).

Multiplying the equation in (4.4) by ϕ_n, integrating and taking into account Remark 4, (b) gives

$$0 = (1 - \sigma_n) \int_{\Gamma_n} v\phi_n - \int_{\Gamma'_n} v \frac{\partial \phi_n}{\partial \nu} \ge (1 - \sigma_n) \int_{\Gamma_n} v\phi_n,$$

which is not possible provided n is large.

In conclusion, (4.1) can not hold, and this proves the theorem. □

As a corollary, we obtain that solutions u_λ to (1.1) develop a dead core $\mathcal{O}_\lambda = \{x \in \Omega : u_\lambda(x) = 0\}$ as λ grows. It turns out that this dead core covers Ω with a speed that can be estimated if $a > 0$ on $\partial\Omega$. For the sake of completeness, we are next providing a direct proof of these facts (see also [3] and [9]).

Proof of Theorem 1 (iv). Let us begin with the case $a > 0$ on $\partial\Omega$. Choose $x \in \Omega$ arbitrary and let $a_0 > 0$ be the infimum of $a(x)$ in Ω. For $y \in B$ (the unit ball), define

$$v(y) = d(x)^{-\alpha} a_0^{-\frac{1}{1-p}} u(x + d(x)y),$$

where $\alpha = 2/(1-p)$. Then v satisfies $\Delta v \ge v^p$ in B, $v \le \varepsilon := C a_0^{-\frac{1}{1-p}} (d(x)\lambda)^{-\alpha}$ on ∂B. It follows that $v \le z$, which is the unique solution to the Dirichlet problem:

$$\begin{cases} \Delta z = z^p & \text{in} \quad B \\ z = \varepsilon & \text{on} \quad \partial B. \end{cases} \tag{4.6}$$

On the other hand, it can be directly checked that the radial function

$$\bar{z}(r) = \begin{cases} 0, & r \le \theta \\ A_N(r-\theta)^\alpha & \theta < r < 1 \end{cases}$$

with $r = |x|$, $A_N = [\alpha(\alpha + N - 2)]^{-1/(1-p)}$ (the value A in Theorem 3 (iii) corresponds to A_N with $N = 1$), $\theta = 1 - (\varepsilon/A_N)^{1/\alpha}$, defines a supersolution to (4.6) which equals ε at $r = 1$. Since $\underline{z} = 0$ is a comparable subsolution it follows by the method of sub and supersolutions (cf. [2]) and the uniqueness of solutions to (4.6) that u vanishes in $\{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \ge (\varepsilon/A_N)^{1/\alpha} = K\lambda^{-1}\}$. The last assertion in (iv) is thus proved.

For the general case one repeats the argument by replacing above $d(x)$ with $d(x)/2$ and a_0 with the minimum of a in $\{y \in \Omega : \mathrm{dist}(y, \partial\Omega) \ge d(x)/2\}$ to prove that $\mathcal{O}_\lambda \ne \emptyset$. Then it suffices with observing that $u = 0$ in $\{d(y) \ge \delta\}$ provided $u = 0$ on $\{d(y) = \delta\}$. This finishes the proof. □

5. Multiplicity

The fact that solutions develop a dead core leads to a fail in uniqueness of nonnegative solutions for large λ. Indeed, we are proving in this section that for domains Ω with boundary consisting in more than one connected piece, nonnegative solutions are not unique.

The proof of this fact relies in constructing solutions whose support for large λ is concentrated near a prefixed connected piece of the boundary. To succeed in this proposal we need to study the following auxiliary version of problem (1.1).

Theorem 11. *Let $\Omega \subset \mathbb{R}^N$ be a class $C^{2,\gamma}$ bounded domain with $\partial\Omega = \Gamma \cup \Gamma'$, Γ, Γ' nonempty and verifying $\overline{\Gamma} \cap \overline{\Gamma}' = \emptyset$, while $a \in C^\gamma(\overline{\Omega})$, $a(x) > 0$ for each $x \in \Omega$. Then, the boundary value problem,*

$$\begin{cases} \Delta u = a(x)u^p & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda u & x \in \Gamma \\ u = 0 & x \in \Gamma' \end{cases} \tag{5.1}$$

satisfies the following properties:

- *(i) Nontrivial and nonnegative solutions to (5.1) are only possible if $\lambda > \sigma_1$, σ_1 the principal eigenvalue of (2.4). In addition, (5.1) admits a nonnegative nontrivial solution $u \in C^{2,\gamma_1}(\overline{\Omega})$, $\gamma_1 = \min\{\gamma, p\}$, for each $\lambda > \sigma_1$.*

- *(ii) There exists $\lambda_1 > \sigma_1$ such that (5.1) admits a* unique *positive solution $u_\lambda \in C^{2,\gamma_1}(\overline{\Omega})$ for each $\sigma_1 < \lambda < \lambda_1$ where the mapping $\lambda \to u_\lambda$ as attaining values in $C^{2,\gamma_1}(\overline{\Omega})$ is real analytic and,*

$$u_\lambda = \left(\frac{\tilde{\mu}_0}{\lambda - \sigma_1}\right)^{\frac{1}{1-p}} \left(\phi_1 + \frac{1}{\tilde{\mu}_0} w(\cdot, \lambda)(\lambda - \sigma_1)\right), \tag{5.2}$$

where ϕ_1 is a positive normalized eigenfunction associated to σ_1, $w \in C^{2,\gamma_1}(\overline{\Omega})$, $w_{|\Gamma'} = 0$, $\int_\Omega w\phi_1 = 0$, $w = w_0$ at $\lambda = \sigma_1$, with $\tilde{\mu}_0 = \dfrac{\int_\Omega a(x)\phi_1^{p+1}}{\int_{\Gamma'} \phi_1^2}$ and $w_0 \in C^{2,\gamma_1}(\overline{\Omega})$ is the unique solution to the problem

$$\begin{cases} \Delta w = a(x)\phi_1^p & x \in \Omega \\ \dfrac{\partial w}{\partial \nu} = \sigma_1 w + \tilde{\mu}_0 \phi_1 & x \in \Gamma \\ w = 0 & x \in \Gamma'. \end{cases}$$

There exist positive constants λ_2, C such that every nonnegative solution u satisfies

$$u \leq C\lambda^{-\frac{2}{1-p}}, \tag{5.3}$$

for $\lambda \geq \lambda_2$. In addition every nonnegative nontrivial solution u_λ to (5.1) *for $\lambda \geq \lambda_2$ exhibits a dead core $\mathcal{O}_\lambda = \{u_\lambda(x) = 0\}$. Moreover, its support concentrates* near Γ *as $\lambda \to \infty$ in the sense that $\{x : \text{ dist } (x, \Gamma) \geq d_\lambda\} \subset \mathcal{O}_\lambda$ where $d_\lambda \to 0+$ as $\lambda \to +\infty$. Again, $d_\lambda = K/\lambda$, $K > 0$, if $a > 0$ on Γ.*

Proof. Concerning (i) we first observe that the conclusions of Lemma 7 also hold true for weak solutions $u \in H_{\Gamma'}(\Omega) = \{u \in H^1(\Omega) : u_{|\Gamma'} = 0\}$ to (5.1). Accordingly, any nonnegative solution $u \neq 0$ does not vanish identically on Γ. Since such solution satisfies $\int_\Omega |\nabla u|^2 < \lambda \int_\Gamma u^2$ one gets from (2.5) that $\lambda > \sigma_1$.

The existence of a nonnegative nontrivial solution for $\lambda > \sigma_1$ follows as in Theorem 1 by minimizing the functional

$$J(u) = \int_\Omega |\nabla u|^2 - \lambda \int_{\partial\Omega} u^2$$

in the set $M = \{u \in H^1_{\Gamma'}(\Omega) : \int_\Omega a(x)|u|^{p+1} = 1\}$. The crucial point now is that J is negative at the infimum only when $\lambda > \sigma_1$ (otherwise, $J \geq 0$ and the infimum is attained at $u = 0$).

As for (iii) observe that the proof of the estimate $u_\lambda \leq C\lambda^{-\frac{2}{1-p}}$ does not require any change regarding the corresponding one in the case of Theorem 1 since the maximum of u_λ must necessarily be attained on Γ. Therefore and by the same arguments as in Section 4, the support of u_λ for large λ is contained in the set $\{\text{dist } (x, \partial\Omega) < d_\lambda\}$ with $d_\lambda \to 0+$. However, we further assert that the support of u_λ is indeed contained in $\{\text{dist } (x, \Gamma) < d_\lambda\}$. In fact, notice that u_λ verifies $\Delta u \geq 0$ in $D = \{0 < \text{dist } (x, \Gamma') < d_\lambda\}$, together with $u_\lambda = 0$ on ∂D and so $u_\lambda = 0$ in D. Hence, the support of u_λ has to be contained in $\{\text{dist } (x, \Gamma) < d_\lambda\}$.

To finish we sketch the proof of (ii). To begin with, it can be shown by a similar reasoning as in Lemma 10 that any possible sequence (λ_n, u_n) of nonnegative nontrivial solutions to (5.1) with $\lambda_n \to \sigma_1+$ satisfies $|u_n|_\infty \to \infty$ and can be written as:

$$\lambda_n = \sigma_1 + \tilde{\mu}_n t_n, \qquad u_n = t_n^{\frac{1}{1-p}}(\phi_1 + t_n \tilde{w}_n)$$

where $\tilde{\mu}_n \to \tilde{\mu}_0$, $t_n \sim |u_n|_\infty^{p-1} \to 0+$, $\tilde{w}_n \to \tilde{w}_0$ in $C^{2,\beta}(\overline{\Omega})$ for every $0 < \beta < \gamma_1$ where $\tilde{\mu}_0$, $\tilde{w}_0 \in C^{2,\gamma_1}(\overline{\Omega})$ are given in the statement of the theorem.

On the other hand, for λ close to σ_1 and by writing solutions (λ, u) as $\lambda = \sigma_1 + \mu t$, $u = t^{-1/(1-p)}(\phi_1 + tw)$ with $t \sim 0$, $\mu \sim \tilde{\mu}_0$ and w close to $\tilde{w}_0$ in the space $Z := \{u \in C^{2,\gamma_1}(\overline{\Omega}) : u_{|\Gamma'} = 0,\ \int_\Omega u\phi_1 = 0\}$, problem (5.1) is transformed into,

$$\begin{cases} \Delta w = a(x)(\phi_1 + tw)^p & x \in \Omega \\ \dfrac{\partial w}{\partial \nu} - \sigma_1 w = \mu(\phi_1 + tw) & x \in \Gamma \\ w = 0 & x \in \Gamma'. \end{cases} \tag{5.4}$$

Taking into account that the linear problem $\Delta u = f$ in Ω, $\dfrac{\partial u}{\partial \nu} - \sigma_1 u = g$ on Γ, $u_{|\Gamma'} = 0$ is uniquely solvable in Z for $(f, g) \in X := C^{\gamma_1}(\overline{\Omega}) \times C^{1,\gamma_1}(\overline{\Omega})$ provided $\tilde{l}(f, g) := \int_\Omega \phi_1 - \int_{\Gamma'} g\phi_1 = 0$ (cf. the Proof of Lemma 10) then the corresponding solution operator $u = K_1(f, g)$ defines an isomorphism $K_1 : \ker \tilde{l} \to Z$. By observing that $(f, g) \to (f, g + \dfrac{\tilde{l}(f, g)}{\int_\Gamma \phi_1})$ defines a projection from X onto $\ker \tilde{l}$, solving (5.4) in (t, μ, w) close to $(0, \tilde{\mu}_0, \tilde{w}_0)$ in $\mathbb{R}^2 \times Z$ amounts to solving the equation,

$$\mathcal{H}(t, \mu, w) = 0, \tag{5.5}$$

with $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2)$, $\mathcal{H}_1(t, \mu, w) = \tilde{l}(F(t, \mu, w), \mu G(t, w))$, $F = a(\phi_1 + tw)^p$, $G = \phi_1 + tw$, and

$$\mathcal{H}_2(t, \mu, w) = w - K_1\left(F(t, \mu, w), \mu G(t, w) + \frac{(\mathcal{H}_1(t, \mu, w))}{\int_\Gamma \phi_1}\right).$$

That (5.5) is uniquely solved in $(-\varepsilon, \varepsilon) \times (\tilde{\mu}_0 - \varepsilon, \tilde{\mu}_0 + \varepsilon) \times B_Z(\tilde{w}_0, \delta)$ for $\varepsilon, \delta > 0$ small follows again from the Implicit Function theorem. As a preliminary checking of regularity observe that F defines a real analytic Nemytskii operator near $(t, \tilde{\mu}_0, \tilde{w}_0)$ in $\mathbb{R}^2 \times Z$ with values in $C^{\gamma_1}(\overline{\Omega})$. In this regard F must be written as

$$F(t, \mu, w) = a(x)\phi_1^p \left(1 + t\frac{w}{\phi_1}\right)^p,$$

and keep t small for $w \in B_Z(\tilde{w}_0, \delta)$. By using that both ϕ_1, w vanish at Γ' and the fact that $\dfrac{\partial \phi_1}{\partial \nu} < 0$ on Γ' it follows that the mapping $w \to \dfrac{w}{\phi_1}$ from Z to $C^1(\overline{\Omega})$ defines a bounded linear operator. This finishes the proof. □

Remark 7. By an argument similar to the one used in the beginning of the proof of Lemma 10 it can be shown that any sequence of solutions $(\lambda_n, u_{\lambda_n})$ to (5.1) is bounded in L^∞ provided λ_n does not accumulate at $\lambda = \tilde{\sigma}_1$. It is also shown by the same means that no sequence can exhibit a subsequence $(\lambda_{n'}, u_{\lambda_{n'}})$ with $\lambda_{n'} \to \lambda \in (\tilde{\sigma}_1, +\infty)$ and $u_{\lambda_{n'}} \to 0$ in $H^1_{\Gamma'}(\Omega^+)$. In particular, (5.1) does not admit $(\lambda, u) = (\lambda, 0)$ as a bifurcation point to nonnegative solutions for any $\lambda > \tilde{\sigma}_1$.

The next statement deals with a fixed component Γ_i of $\partial\Omega$. It is an immediate consequence of applying Theorem 11 to Ω under the choice $\Gamma = \Gamma_i$, $\Gamma' = \partial\Omega \setminus \Gamma_i$ for the boundary conditions.

Corollary 12. *Let Γ_i be any fixed connected component of $\partial\Omega$. Then for large λ problem* (1.1) *has at least a nonnegative nontrivial solution u such that $u = 0$ in $\{\text{dist}(x, \Gamma_i) \geq d_\lambda\}$, $d_\lambda \to 0$ as $\lambda \to +\infty$. The rate of convergence d_λ can be chosen as K/λ for certain $K > 0$ if $a > 0$ on Γ_i.*

We can now proceed to the proof of Theorem 2.

Proof of Theorem 2. If $\Gamma_1, \ldots, \Gamma_k$ are the connected components of $\partial\Omega$ then Corollary 12 implies the existence of k nonnegative nontrivial solutions $u_{\lambda,i}$ whose support is localized near Γ_i for large λ. The supports will indeed be disjoint for large λ. This allows us to conclude that every sum of the form $u_{\lambda,i_1} + \ldots + u_{\lambda,i_r}$ is again a solution to (1.1). Since there are $2^k - 1$ such sums, we have shown that there exist at least $2^k - 1$ different nonnegative nontrivial solutions to (1.1) for large λ. □

6. The case of the ball

In this section we are considering a special case of problem (1.1) by analyzing its behavior in the unit ball B of $\mathbb{R}^N$. To simplify the exposition we are focussing our interest in the autonomous case represented by the coefficient $a(x) = 1$. We begin our treatment by searching only for nonnegative radially symmetric solutions $u = u(r)$, $r = |x|$. They satisfy

$$\begin{cases} (r^{N-1}u')' = r^{N-1}u^p & 0 \leq r < 1 \\ u'(0) = 0, \ u'(1) = \lambda u(1). \end{cases} \tag{6.1}$$

The scaling $v(r) = \lambda^\alpha u(\lambda^{-1}r)$, $\alpha = 2/(1-p)$, transforms this problem into,

$$\begin{cases} (r^{N-1}v')' = r^{N-1}v^p & 0 \leq r < \lambda \\ v'(0) = 0, \ v'(\lambda) = v(\lambda). \end{cases} \tag{6.2}$$

Modifying the proof of point (i) in Theorem 1 by both restricting J and M to the class of radial functions in $H^1(B)$, one can show the existence of a nontrivial nonnegative radial solution to (1.1) for all $\lambda > 0$, hence the equivalent assertion for problem (6.1), which is point (i) in Theorem 3. On the other hand, it has been already shown (Theorem 1 (iv)) that all nonnegative solutions exhibit a dead core for large λ which completely fills B as $\lambda \to +\infty$. This implies the existence of $d = d(\lambda) > 0$, $\lambda - K \leq d < \lambda$ and so $d \to \infty$ as $\lambda \to \infty$, such that any nonnegative and nontrivial solution v to (6.2) satisfies $v = 0$ in $[0, d]$, $v > 0$ in $(d, \lambda]$. Therefore, setting $w(r) = v(d + r)$, w defines a solution to

$$\begin{cases} ((r+d)^{N-1}w')' = (r+d)^{N-1}w^p & 0 \leq r < \lambda - d \\ w(0) = 0, \ w'(0) = 0, \ w'(\lambda - d) = w(\lambda - d). \end{cases} \tag{6.3}$$

Such solution is nontrivial in the sense that $w(r) > 0$ for $r \in (0, \lambda - d)$ (observe that solutions to (2.6) are *nontrivial*, see Section 2).

Exploiting these remarks we are now showing items (ii) and (iii) in Theorem 3 (compare with the proof of Theorem 3.4 in [12]).

Proof of Theorem 3 (ii) and (iii). To prove (ii) notice that the uniqueness assertion in Theorem 1 (ii) furnishes a unique radial positive solution u_λ for $0 < \lambda < \lambda_0$ satisfying the properties stated there.

To show the uniqueness for large λ, consider the Cauchy problem

$$\begin{cases} ((r+d)^{N-1}w')' = (r+d)^{N-1}w^p & \text{in } (0,\delta) \\ w(0) = 0, \ w'(0) = 0 \end{cases} \tag{6.4}$$

which, by Theorem 9, has a unique nontrivial solution $w(r,d)$ in $[0,+\infty)$ for every $d \geq 0$. Moreover, as $d \to +\infty$, w converges in $C^1_{\text{loc}}[0,+\infty)$ to $w_0(r) = Ar^\alpha$, where A and α are as in the statement of the theorem.

By the preceding discussion, showing the uniqueness amounts to prove the existence as $d \to \infty$ of a unique positive zero $r = T$ of the equation,

$$w'(r,d) - w(r,d) = 0, \tag{6.5}$$

such that $T = O(1)$ while $T + d = \lambda$ is uniquely solvable in d as $d \to \infty$.

The solvability with uniqueness of (6.5) for d large follows from the fact that the limit equation $w_0'(r) - w_0(r) = 0$ has $r = \alpha$ as a unique simple zero. In fact observe that $w_0''(\alpha) - w_0'(\alpha) = -A\alpha^{\alpha-1}$. Thus (6.5) admits a root $T(d)$ such that $T(d) \to \alpha$ as $d \to +\infty$.

We now claim that $T(d)$ is a C^1 function for large d and verifies $T'(d) \to 0$ as $d \to +\infty$. This is a consequence of the Implicit Function theorem, by first noticing that $w(r,d)$ is C^1 in both variables (Theorem 9), while in addition

$$T'(d) = \frac{w_d(T(d),d) - w_d'(T(d),d)}{w''(T(d),d) - w'(T(d),d)}$$

(the subscript d denotes derivation with respect to d and $'$ stands for the derivative with respect to r). As already seen, the denominator is negative for large d since it converges to $w_0''(\alpha) - w_0'(\alpha) = -A\alpha^{\alpha-1}$, while the numerator tends to zero thanks to (2.7) in Theorem 9. This proves the claim.

Finally, notice that solutions to (6.3) arise whenever there is a solution to the equation $d + T(d) = \lambda$. Since $d + T(d)$ is increasing for large d, it follows that for large λ there is a unique $d = d(\lambda)$ so that $w(r,d)$ is a solution to (6.3). The unique solution to (6.1) for large λ is then obtained by setting

$$u_\lambda(r) := \begin{cases} 0 & \text{if } 0 \leq r \leq \dfrac{d}{\lambda} \\ \lambda^{-\alpha} w(\lambda r - d) & \text{if } r > \dfrac{d}{\lambda}. \end{cases} \tag{6.6}$$

This proves part (ii).

Part (iii) follows at once by noticing that $w(T(d),d) \to w_0(\alpha) = A\alpha^\alpha$ and $r(\lambda) = d(\lambda)/\lambda = 1 - T(d(\lambda))/\lambda \sim 1 - \alpha/\lambda$. □

Corollary 13. *Let u_λ be the unique nonnegative nontrivial radial solution to* (1.1) *for large λ. Then*

$$\int_B u_\lambda^{p+1} \sim \omega_N A^{p+1} \frac{\alpha^{\alpha(p+1)+1}}{\alpha(p+1)+1} \, \lambda^{-\alpha(p+1)-1}, \quad \text{as } \lambda \to +\infty,$$

where ω_N is the surface measure of the unit ball of $\mathbb{R}^N$, and A, α are given in Theorem 3.

Proof. According to (6.6), we have

$$\begin{aligned}\int_B u_\lambda^{p+1} &= \omega_N \lambda^{-\alpha(p+1)} \int_{\frac{d}{\lambda}}^1 r^{N-1} w(\lambda r - d)^{p+1} dr \\ &= \omega_N \lambda^{-\alpha(p+1)-1} \int_0^{\lambda-d} \left(\frac{s+d}{\lambda}\right)^{N-1} w(s)^{p+1} ds \\ &\sim \omega_N \lambda^{-\alpha(p+1)-1} \int_0^{\alpha} w_0(s)^{p+1} ds = \omega_N A^{p+1} \frac{\alpha^{\alpha(p+1)+1}}{\alpha(p+1)+1} \lambda^{-\alpha(p+1)-1},\end{aligned}$$

as $\lambda \to \infty$. □

Of course one could ask the question whether there are nonradial solutions, or if the radial one is the unique one. Since ∂B is connected we can not obtain multiplicity as in Section 5. However, we prove next that besides the radial solution there is indeed another nontrivial nonnegative and nonradial solution for large λ.

Proof of Theorem 3 (iv). We are showing that the solution constructed in Theorem 1 (i) is not the radial one. Denote, as in that theorem,

$$J(u) = \int_B |\nabla u|^2 - \lambda \int_{\partial B} u^2, \qquad u \in M = \left\{ u \in H^1(B) : \int_B |u|^{p+1} = 1 \right\}.$$

For the radial solution u_λ we have

$$J(u_\lambda) = -\int_B u_\lambda^{p+1},$$

so that $v_\lambda = u_\lambda / |u_\lambda|_{L^{p+1}} \in M$ verifies $J(v_\lambda) = -|u_\lambda|_{L^{p+1}}^{-\frac{1-p}{1+p}}$. Then by Corollary 13, there exists a positive constant C so that

$$J(v_\lambda) \sim -C\lambda^{\frac{p+3}{p+1}}, \quad \text{as } \lambda \to +\infty. \tag{6.7}$$

We are next using (6.7) to prove that $J(v_\lambda)$ is not the minimum of J in M for large λ. For this aim, we construct a family of functions $\psi_\lambda \in M$ so that $J(\psi_\lambda) < J(v_\lambda)$ for large λ.

We first claim that a large radius $R > 0$ and a function $\psi \in C_0^\infty(B_R)$, $B_R = B(0, R)$, can be found so that

$$\int_{B_R^+} |\nabla \psi|^2 - \int_{\Gamma_R} \psi^2 < 0, \tag{6.8}$$

where $B_R^+ = B_R \cap \mathbb{R}_+^N$, $\Gamma_R = B_R \cap \partial \mathbb{R}_+^N$.

Assumed the existence of such ψ set $\psi_\lambda(x) = C(\lambda)\psi(\lambda(x + e_1)) \in C_0^\infty(B(-e_N, \lambda^{-1}R))$, which belongs to M provided $C(\lambda) = K\lambda^{\frac{N}{p+1}}$, where $K^{-1} = |\psi|_{p+1,B_R}$. With this choice of ψ_λ:

$$\begin{aligned}J(\psi_\lambda) &= K^2 \lambda^{\frac{2N}{p+1}+2-N} \left(\int_{B_R \cap B(\lambda e_N, \lambda)} |\nabla \psi|^2 - \int_{B_R \cap \partial B(\lambda e_N, \lambda)} \psi^2 \right) \\ &\sim K^2 \lambda^{\frac{1-p}{1+p}N+2} \left(\int_{B_R^+} |\nabla \psi|^2 - \int_{\Gamma_R} \psi^2 \right).\end{aligned}$$

Since $(1-p)N/(1+p)+2 > (p+3)(p+1)$, we obtain in virtue of (6.7) and (6.8), that $J(\psi_\lambda) < J(v_\lambda)$ for large λ. As remarked before, this entails the existence of a nontrivial nonnegative and nonradial solution to (1.1).

To show the claim let $\phi \in H^1(B_R^+)$ be a positive eigenfunction to,

$$\begin{cases} \Delta u = 0 & x \in B_R^+ \\ -\dfrac{\partial u}{\partial x_N} = \sigma u & x \in \Gamma_R \\ u = 0 & x \in \Gamma_R' \end{cases} \tag{6.9}$$

corresponding to the principal eigenvalue $\sigma = \sigma_1(R)$ (Theorem 6 and Remark 8 in [11]). Provided R is big enough so that $\sigma_1(R) < 1$ (Remark 4 (a)) ϕ satisfies:

$$\int_{B_R^+} |\nabla \phi|^2 - \int_{\Gamma_R} \phi^2 = (\sigma_1(R) - 1) \int_{\Gamma_R} \phi^2 < 0.$$

The searched function ψ can be now obtained from ϕ by regularization. This concludes the proof. □

7. Vanishing weights

In this final section, we give the proof of Theorems 4, 5 and 6, considering in problem (1.1) a weight $a(x)$ that vanishes in a subdomain of Ω.

Proof of Theorem 4. We only sketch the proof of part (i), the remaining points being proved in exactly the same way as in Theorem 1 (ii), (iii) and (iv). If u is a nonnegative solution to (1.1) with $\lambda \le 0$ (which, due to Lemma 7 can be supposed in $C^{2,\gamma_1}(\overline{\Omega})$), we deduce

$$\int_\Omega a(x) u^p = 0,$$

and so $u = 0$ in $\Omega \setminus \Omega_0$. Thus u verifies

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega_0 \\ \dfrac{\partial u}{\partial \nu} = \lambda u & \text{on} \quad \Gamma_1 \\ u = 0 & \text{on} \quad \Gamma_2, \end{cases}$$

and it follows from the maximum principle that $u = 0$ in Ω_0 as well. Hence, $u \equiv 0$.

Assume now $0 < \lambda < \sigma_1$. If the functional

$$J(u) = \int_\Omega |\nabla u|^2 - \lambda \int_{\partial\Omega} u^2$$

were not coercive in $M := \{u \in H^1(\Omega) : \int_\Omega a(x)|u|^{p+1} = 1\}$, then we would obtain (proceeding as in the proof of Theorem 1 (i)) a function $v \in H^1(\Omega)$ with $|v|_{L^2(\partial\Omega)} = 1$ and satisfying the relations

$$\int_\Omega |\nabla v|^2 \le \lambda \,, \qquad \int_\Omega a(x)|v|^{p+1} = 0.$$

The last one implies $v = 0$ in $\Omega \setminus \Omega_0$ (in particular on Γ_2). If $\sigma_1 = +\infty$ it follows $v = 0$ on $\partial\Omega$ what is impossible. If, on the contrary, $\sigma_1 < \infty$ then (2.5) yields $\sigma_1 \le \lambda$, also a contradiction. Hence J is coercive and there exists a nonnegative nontrivial solution to (1.1). □

Lemma 14. *Assume $\sigma_1 < +\infty$, and let $u \in C^{2,\gamma}(\overline{\Omega})$ be a nonnegative solution to* (1.1) *with $\lambda \ge \sigma_1$. Then $u \equiv 0$ in Ω_0.*

Proof. Multiplying (1.1) by the eigenfunction ϕ_1 associated to σ_1 and integrating in Ω_0 (notice that $\phi_1 \in C^{2,\gamma}(\overline{\Omega})$, $u \in C^{2,\gamma_1}(\overline{\Omega})$), we obtain

$$(\lambda - \sigma_1) \int_{\Gamma_1} u\phi_1 = \int_{\Gamma_2} u \frac{\partial \phi_1}{\partial \nu} \, . \tag{7.1}$$

If $\lambda > \sigma_1$, we directly obtain $u = 0$ on $\Gamma_1 \cup \Gamma_2 = \partial\Omega_0$, and since $\Delta u = 0$ in Ω_0, $u \equiv 0$ in Ω_0 follows.

If $\lambda = \sigma_1$, then we have from (7.1) that $u = 0$ on Γ_2. Thus, u is an eigenfunction associated to σ_1, and the simplicity of σ_1 ([11], Theorem 6) implies $u = c\phi$ for some nonnegative constant c. If $c > 0$, then $\frac{\partial u}{\partial \nu} < 0$ on Γ_2, which would imply that u changes sign in Ω. Thus $c = 0$ and we arrive again at $u \equiv 0$ in Ω_0. This completes the proof. □

Proof of Theorem 5. To show (i), recall that $\tilde{\sigma}_1 = +\infty$ is equivalent to $\Gamma_1 = \partial\Omega$. Since by Lemma 14 we have $u = 0$ on Γ_1 for every nonnegative solution u with $\lambda \ge \sigma_1$ and u is subharmonic, we arrive at $u = 0$ in Ω.

As for (ii), suppose u is a nonnegative nontrivial solution corresponding to $\lambda \ge \sigma_1$. By Lemma 14, $u \equiv 0$ in Ω_0 which implies $u = 0$ on $\Gamma_1 \cup \Gamma_2$. Since $\partial\Omega^+ \cap \Omega = \Gamma_2$, then u satisfies:

$$\begin{cases} \Delta u = a(x) u^p & x \in \Omega_i^+ \\ \dfrac{\partial u}{\partial \nu} = \lambda u & x \in \partial\Omega_i^+ \cap \partial\Omega \\ u = 0 & x \in \partial\Omega_i^+ \cap \Omega, \end{cases} \tag{7.2}$$

for each component Ω_i^+ having $\partial\Omega_i^+ \cap \partial\Omega \ne \emptyset$. Observe that $u = 0$ in the remaining components $\Omega_j^+ \subset\subset \Omega$ since $u = 0$ on $\partial\Omega_j^+$ and is subharmonic there. Therefore, provided $u \ne 0$, u must define a nontrivial solution of some of the problems (7.2). By Theorem 11, this means $\lambda > \tilde{\sigma}_{1,i} \ge \tilde{\sigma}_1$.

Finally, according to (iii) of Theorem 11, each of the problems (7.2) defines for $\lambda > \tilde{\sigma}_1$ large a nontrivial solution to (1.1). These solutions and a combination of them (in the spirit of the proof of Theorem 2) ensure the validity of (iii).

Proof of Theorem 6. Observe that under the assumptions of the theorem $u = 0$ is the only solution to (1.1) for $\lambda = \sigma_1$.

Let now $\lambda_n \to \sigma_1-$ with u_n a corresponding nonnegative nontrivial solution to (1.1) at $\lambda = \lambda_n$. We claim that $t_n := |u_n|_\infty$ is bounded. If not, passing to a subsequence we can assume $t_n \to +\infty$. Denoting $v_n = u_n/t_n$, we have

$$\begin{cases} \Delta v_n = a(x) v_n^p |u_n|^{p-1} & \text{in} \quad \Omega \\ \dfrac{\partial v_n}{\partial \nu} = \lambda_n v_n & \text{on} \quad \partial\Omega. \end{cases}$$

It is standard to obtain that a subsequence of $\{v_n\}$ converges in $C^2(\overline{\Omega})$ to a function v which satisfies $|v|_\infty = 1$ and:

$$\begin{cases} \Delta v = 0 & \text{in} \quad \Omega \\ \dfrac{\partial v}{\partial \nu} = \sigma_1 v & \text{on} \quad \partial\Omega. \end{cases} \tag{7.3}$$

By directly integrating (7.3), it follows that $v = 0$, and this contradicts $|v|_\infty = 1$. Thus t_n is bounded, and we can obtain as before that for a subsequence $u_n \to u$ in $C^{2,\beta}(\overline{\Omega})$, $0 < \beta < \gamma_1$, where u is a solution to (1.1) with $\lambda = \sigma_1$. Therefore $u = 0$, and this proves the convergence $u_\lambda \to 0$.

Regarding the dead core formation, it follows from the fact $\sup_\Omega u_\lambda \to 0$ as $\lambda \to \sigma_1-$ that a nonempty interior region $\{u_\lambda = 0\}$ is generated in Ω^+ as $\lambda \to \sigma_1-$ (cf. the proof of Theorem 1 (iv)). However, such region can never reach Γ_2 nor Ω_0 for λ close to σ_1. In fact, $u_\lambda(x_0) = 0$ at $x_0 \in \Gamma_2$ implies $u_\lambda = 0$ in Ω_0. Otherwise, since u_λ is harmonic in Ω_0 we get $\dfrac{\partial u_\lambda}{\partial \nu}(x_0) < 0$, ν the outward unit normal to Ω_0 at x_0, what contradicts the nonnegativeness of u_λ. Hence $u_\lambda = 0$ in Ω_0 (notice that such behavior is immediately achieved if $u_\lambda = 0$ somewhere in Ω_0). This means that we get a family u_λ of nontrivial solutions to the mixed problem (5.1) in Ω^+ such that $u_\lambda \to 0$ as $\lambda \to \sigma_1-$. Observe that the existence of such family is forbidden if $\sigma_1 \le \tilde{\sigma}_1$ since such behavior is not possible if $\tilde{\sigma}_1 < \sigma_1$, as observed in Remark 7. This shows that $\mathcal{O}_\lambda \subset \Omega^+$ for $\lambda \to \sigma_1-$. The remaining assertions follow in the same way as in the case of Theorem 1. □

Remark 8. In order to illustrate with an example the relative values of σ_1 and $\tilde{\sigma}_1$ consider the annulus $\Omega = \{R_1 < |x| < R_2\}$, $0 < R_1 < R_2$ fixed. For $R_1 < R < R_2$ variable, set $\Omega^+ = \{R_1 < |x| < R\}$, $\Omega_0 = \{R < |x| < R_2\}$ and so $\Gamma_1 = \{|x| = R_2\}$, $\Gamma_2 = \{|x| = R\}$, $\Gamma^+ = \partial\Omega \setminus \Gamma_1 = \{|x| = R_1\}$. The principal eigenvalues σ_1, $\tilde{\sigma}_1$ (see (1.3), (1.4)) must in this case be associated to radial (harmonic) eigenfunctions and hence are explicitly given by:

$$\sigma_1 = \frac{(N-2)R^{N-2}}{R_2(R_2^{N-2} - R^{N-2})}, \qquad \tilde{\sigma}_1 = \frac{(N-2)R^{N-2}}{R_1(R^{N-2} - R_1^{N-2})}. \tag{7.4}$$

Observe that $\sigma_1 = \tilde{\sigma}_1$ at $R = R^*$, $R^*/R_1 = (\zeta^{N-1} + 1)/(\zeta + 1)$, $\zeta = R_2/R_1$, $\sigma_1 < \tilde{\sigma}_1$ for $R_1 < R < R^*$, $\sigma_1 > \tilde{\sigma}_1$ if $R^* < R < R_2$. This shows that it is possible to have both situations $\sigma_1 \le \tilde{\sigma}_1$ and $\sigma_1 > \tilde{\sigma}_1$.

Let us now construct examples showing the behaviors announced in Remark 1 (b), namely a situation where $\tilde{\sigma}_1 < \sigma_1$ and either no solutions exist for $\lambda > \sigma_1$, $\lambda \sim \sigma_1$, or another one where solutions exist for $\sigma_1 - \varepsilon < \lambda < \sigma_1$ for a certain $\varepsilon > 0$. In what follows, we are having in mind that R is fixed, while R_2 is going to vary. Choose $a \in C^\gamma(\mathbb{R}^N)$ such that $a > 0$ for $|x| < R$, $a = 0$ in $|x| \ge R$, and consider the problem

$$\begin{cases} \Delta u = a(x) u^p & x \in \Omega^+ \\ \dfrac{\partial u}{\partial \nu} = \lambda u & x \in \Gamma^+ \\ u = 0 & x \in \Gamma_2. \end{cases} \tag{7.5}$$

Theorem 11 provides the existence of $\tilde{\sigma}_1 < \lambda^* \le \lambda_1$ (see (ii)) such that the unique positive solution u_λ to (7.5) satisfies $\dfrac{\partial u_\lambda}{\partial \nu} < 0$ on Γ_2 (ν the outward unit normal to Ω^+ on Γ_2) for each $\tilde{\sigma}_1 < \lambda < \lambda^*$.

Now in order to obtain an example of the first situation notice that (7.4) allows to choose R_2 such that $\tilde{\sigma}_1 < \sigma_1 < \lambda^*$. This means that problem (1.1) can not exhibit a nonnegative nontrivial solution u at least for any $\sigma_1 \le \lambda \le \lambda^*$. In fact, such possible solution u should vanish in $\overline{\Omega}_0$ (Lemma 14) thus providing a nonnegative nontrivial solution to (7.5) with $\dfrac{\partial u_\lambda}{\partial \nu} = 0$ on Γ_2 which is not possible in such range for λ.

To achieve an example of the second situation it suffices with choosing R_2 so that $\sigma_1 > \lambda_2$ with λ_2 as in (ii) of Theorem 11. This completes the announced constructions.

References

[1] S. Agmon, A. Douglis, L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I*, Comm. Pure Appl. Math. **12** (1959), 623–727.

[2] H. Amann, *On the existence of positive solutions of nonlinear elliptic boundary value problems*, Indiana Univ. Math. J. **21** (1971), 125–146.

[3] C. Bandle, R. P. Sperb, I. Stakgold, *Diffusion and reaction with monotone kinetics*, Nonlinear Anal. T.M.A. **8** (1984), 321–333.

[4] H. Beirão da Veiga, *On the $W^{2,p}$-regularity for solutions of mixed problems*, J. Math. Pures Appl. **53** (1974), 279–290.

[5] K. Deimling, *Nonlinear Functional Analysis*, Springer Verlag, Berlin, 1985.

[6] M. Del Pino, *Positive solutions of a semilinear elliptic equation on a compact manifold*, Nonlinear Analysis TMA **22** (1994), 1423–1430.

[7] Y. Du, Q. Huang, *Blow-up solutions for a class of semilinear elliptic and parabolic equations*, SIAM J. Math. Anal. **31** (1999), 1–18.

[8] J. M. Fraile, P. Koch-Medina, J. López-Gómez, S. Merino, *Elliptic eigenvalue problems and unbounded continua of positive solutions of a semilinear elliptic equations*, J. Differential Equations **127** (1996), 295–319.

[9] A. Friedman, D. Phillips, *The free boundary of a semilinear elliptic equation*, Trans. Amer. Math. Soc. **282** (1984), 153–182.

[10] J. García-Melián, R. Gómez-Reñasco, J. López-Gómez, J. Sabina de Lis, *Point-wise growth and uniqueness of positive solutions for a class of sublinear elliptic problems where bifurcation from infinity occurs*, Arch. Rat. Mech. Anal. **145** (1998), 261–289.

[11] J. García-Melián, J. D. Rossi, J. Sabina de Lis, *A bifurcation problem governed by the boundary condition I*, to appear in Nonlinear Differential Equations Appl. NoDEA.

[12] J. García-Melián, J. Sabina de Lis, *Uniqueness to quasilinear problems for the p-Laplacian in radially symmetric domains*, Nonlinear Anal. T.M.A. **43** (7) (2001), 803–835.

[13] B. Gidas, J. Spruck, *A priori bounds for positive solutions of nonlinear elliptic equations*, Comm. P. D. E. **6** (1981), 883–901.

[14] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer–Verlag, 1983.

[15] G. Lieberman, *Mixed boundary value problems for elliptic and parabolic differential equations of second order*, J. Math. Anal. Appl. **113** (2) (1986), 422–440.

[16] G. Lieberman, *Optimal Hölder regularity for mixed boundary value problems*, J. Math. Anal. Appl. **143** (2) (1989), 572–586.

[17] J. López-Gómez, J. Sabina de Lis, *First variations of principal eigenvalues with respect to the domain and point-wise growth of positive solutions for problems where bifurcation from infinity occurs*, J. Differential Equations **148** (1998), 47-64.

[18] T. Ouyang, *On the positive solutions of semilinear equations* $\Delta u + \lambda u - hu^p = 0$ *on the compact manifolds*, Trans. Amer. Math. Soc. **331** (1992), 503–527.

[19] E. Shamir, *Regularization of mixed second-order elliptic problems*, Israel J. Math. **6** (1968), 150–168.

[20] M. Struwe, *Variational Methods*, Springer Verlag, Heidelberg-New York, 1989.