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UNIQUENESS AND ASYMPTOTIC BEHAVIOUR FOR SOLUTIONS OF SEMILINEAR PROBLEMS WITH BOUNDARY BLOW-UP

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Abstract. In this paper we prove uniqueness of positive solutions to logistic singular problems $-\Delta u = \lambda(x)u - a(x)u^p$, $u_{|\partial\Omega} = +\infty$, $p > 1$, $a > 0$ in Ω, where the main feature is the fact that $a_{|\partial\Omega} = 0$. More importantly, we provide exact asymptotic estimates describing, in the form of a two-term expansion, the blow-up rate for the solutions near $\partial\Omega$. This expansion involves both the distance function $d(x) = \mathrm{dist}(x, \partial\Omega)$ and the mean curvature H of $\partial\Omega$.

1. Introduction and results

This research originated with the recent paper [10] which contains an exhaustive study of positive solutions u to the logistic problem:

$$\text{(L)} \qquad \begin{cases} -\Delta u = \lambda u - a(x)u^p, & x \in \Omega', \\ \quad u = 0, & x \in \partial\Omega', \end{cases}$$

$p > 1$, $\lambda > 0$ a parameter, $\Omega' \subset \mathbb{R}^n$ a bounded domain and where $a \in C(\overline{\Omega}')$ satisfies $a > 0$ in a proper subdomain $\Omega \subset \overline{\Omega} \subset \Omega'$ while $a = 0$ in $\overline{\Omega}' \setminus \Omega$. It turns out that positive solutions u can only exist when $\lambda_1(\Omega') < \lambda < \lambda_1(\Omega)$ (λ_1 being the first Dirichlet eigenvalue in the corresponding domain) being unique in that range. Moreover, the solution u_λ to (L) satisfies $u_\lambda \to \infty$ uniformly on of $\Omega' \setminus \Omega$ when $\lambda \uparrow \lambda_1(\Omega)$ while $u_\lambda \to \underline{u}$ in $C^{2,\alpha}(\Omega)$ as $\lambda \uparrow \lambda_1(\Omega)$, for certain $0 < \alpha < 1$, where $\underline{u}$ is the minimal solution to the singular boundary value problem,

$$\text{(P)} \qquad \begin{cases} -\Delta u = \lambda(x)u - a(x)u^p, & x \in \Omega, \\ \quad u = +\infty, & x \in \partial\Omega, \end{cases}$$

being $\lambda(x) = \lambda_1(\Omega)$ in this precise case. The boundary condition is understood as $u(x) \to +\infty$ when $d(x) := \mathrm{dist}(x, \partial\Omega) \to 0+$. Observe that $a > 0$ in Ω while $a = 0$ on $\partial\Omega$.

Our main objectives here are to show the uniqueness of positive solutions to (P) together with producing an exact two-term asymptotic expansion of the solutions u

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to (P) near the boundary, involving the distance d together with the mean curvature H of $\partial\Omega$. We also analyze a class of perturbations of (P).

Singular boundary value problems such as (P) go back to the pioneering work [5] on automorphic functions and the equation $-\Delta u = -e^u$ in the plane, and were later studied under the general form $-\Delta u = -f(u)$ in n-dimensional domains in [12], [18]. The more subtle questions of blow-up rates near $\partial\Omega$ and uniqueness of solutions are the goals of more recent literature. The problem $-\Delta u = a(x)e^u$ plus $u_{|\partial\Omega} = +\infty$ is shown to exhibit a unique solution in a smooth domain Ω together with an estimate of the form $u = \log d^{-2} + v(d)$ in [14] (where $a(x) \geq a_0 > 0$ and $v = O(1)$ as $d \to 0$) and in [1] (where $a \equiv 1$, $v = o(1)$ as $d \to 0$).

The equation $-\Delta u = -a(x)u^p$ is first considered in [16] ($a \equiv 1$, $p = \dfrac{n+2}{n-2}$), [13], [2], [3], [20] ($a \geq a_0 > 0$, $p \geq 3$ in [13]) and [8], [9] where the p-Laplacian extension $-\Delta_p u = -a(x)u^q$, $q > p-1$, $a \geq a_0 > 0$, is studied. In all of them, uniqueness is achieved together with a blow-up estimate under the form $u \sim Ad^{-\alpha}$ as $d \to 0+$, with $\alpha = \dfrac{2}{p-1}$ and $A = (2(p+1)/\{a(p-1)^2\})^{1/(p-1)}$.

Uniqueness and blow-up rates of $-\Delta u = -f(u)$ are treated in [15] and [4] for selective classes of nonlinearities extending $f = e^u$ and $f = u^p$, where the explosive solution at $x = 0$ of $-u'' = -f(u)$ is involved. For instance in [4] and the case of $-\Delta u = -u^p$ it is shown that $u = Ad^{-\alpha}(1 + O(d))$ as $d \to 0+$ allows the possible presence of a second explosive term in the expansion of u when $\alpha > 1$ ($p < 3$). An explicit expression for this second term has been recently found in [7] as $u = Ad^{-\alpha}(1 + Bd + o(d))$, B given as a function of the mean curvature H, provided that $1 < p < 3$ (in the radial case this term had been already computed in [4] for all $p > 1$).

In this paper we will produce sharper results in two directions. First, uniqueness of solutions to (P) and some of its perturbations is achieved by removing the condition $a > 0$ up to the boundary (uniqueness was obtained in [17] under the assumption $a(x) \geq a_0 > 0$ in $\overline{\Omega}$, however considerably weakening the smoothness of Ω). It should be emphasized that $a = 0$ on $\partial\Omega$ is a *natural* restriction for a inherited from the logistic problem (L)! Second, we are extending the scope of the expansion $u = Ad^{-\alpha}(1 + Bd + o(d))$ both to cover the full range $p > 1$ and the more general nonlinearities in (P) and its perturbations.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^k domain, $k \geq 4$, and λ, $a \in C^\alpha(\overline{\Omega})$ such that $a > 0$ in Ω. Then the singular boundary value problem (P) admits a minimal and a maximal classical solution. If moreover $a = 0$ on $\partial\Omega$ and*

$$a(x) = C_0 d^\gamma + o(d^\gamma) \tag{1}$$

as $d \to 0+$ with $\gamma > 0$ and $C_0 > 0$, then every solution $u \in C^2(\Omega)$ to (P) satisfies

$$\lim_{d\to 0+} \frac{u}{Ad^{-\alpha}} = 1, \tag{2}$$

where $\alpha = (\gamma+2)/(p-1)$ and $A = (\alpha(\alpha+1)/C_0)^{\frac{1}{p-1}}$. As a consequence, (P) admits a unique positive solution. *In addition, if*

$$a(x) = C_0 d^\gamma (1 + C_1 d + o(d)) \tag{3}$$

as $d \to 0+$, a further estimate of the blow-up rate is available, namely

$$u(x) = Ad^{-\alpha}(1 + B(s)d + o(d)) \tag{4}$$

as $d \to 0+$, *where*

$$B(s) = \frac{(n-1)H(s) - (\alpha+1)C_1}{\gamma + p + 3}, \tag{5}$$

with $H(s)$ *standing for the mean curvature of* $\partial\Omega$ *in* s, *and for every* x *near* $\partial\Omega$, $d(x) := \mathrm{dist}(x, \partial\Omega)$ *and* $s = s(x)$ *means the projection of* x *over* $\partial\Omega$.

Remark 1. a) The conclusions remain valid if the constants C_0 and C_1 are allowed to be functions of s, $C_0 = C_0(s), C_1 = C_1(s) \in C^2(\overline{\Omega})$ as long as $C_0(s) > 0$ on $\partial\Omega$. Of course, the special case $\gamma = 0$ in (1) and (3) falls within the scope of Theorem 1.

b) A special case of estimate (2) with a vanishing at $\partial\Omega$ was obtained in [19].

c) Suppose more generally that $\partial\Omega$ splits into $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ connected pieces and consider the problem $-\Delta u = \lambda(x)u - a(x)u^p$ in Ω, $u_{|\Gamma_i} = \sigma_i$, $1 \le i \le m$, with $\sigma_i \ge 0$, some of them $+\infty$. Then the conclusions of Theorem 1 hold, since estimates (2) and (4) are understood near the connected components Γ_i such that $\sigma_i = +\infty$.

Consider now the more general class of problems:

$$\text{(P)}_g \qquad \begin{cases} -\Delta u = \lambda(x)u - a(x)(u^p + g(x,u)), & x \in \Omega, \\ \quad u = +\infty, & x \in \partial\Omega, \end{cases}$$

where g is of lower order than u^p at infinity. Then we have the following results:

Theorem 2. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded* C^k *domain,* $k \ge 4$, *and* λ, $a \in C^\alpha(\overline{\Omega})$ *such that* $a > 0$ *in* Ω. *If* $g \in C(\overline{\Omega} \times \mathbb{R})$ *is locally Lipschitz in* $\overline{\Omega} \times \mathbb{R}$ *and it satisfies the growth conditions*

$$\text{(H)}_g \qquad \lim_{u \to 0} \frac{g(x,u)}{u} = 0, \qquad \lim_{u \to +\infty} \frac{g(x,u)}{u^p} = 0$$

uniformly in $x \in \overline{\Omega}$, *then problem (P)*$_g$ *admits at least a classical solution* $u \in C^{2,\alpha}(\Omega)$. *If* $a(x)$ *vanishes on* $\partial\Omega$ *and verifies (1), then estimate (2) holds for every possible solution to (P)*$_g$. *If, in addition,* g *is continuously differentiable with respect to* u, $g'_u \in C(\overline{\Omega} \times \mathbb{R})$, *while*

$$\text{(H)}_{g'} \qquad \lim_{u \to 0} \frac{g(x,u)}{u} = 0, \qquad \lim_{u \to +\infty} \frac{g'_u(x,u)}{u^{p-1}} = 0$$

uniformly in $\overline{\Omega}$, *then (P)*$_g$ *admits a unique positive solution, provided that* $|a|_\infty$ *is small enough.*

Remark 2. a) The Lipschitz condition on g can be relaxed to continuity if only strong or weak solutions instead of classical solutions are considered.

b) When $\lambda(x) > 0$ in $\overline{\Omega}$, precise estimates on the size of $|a|_\infty$ in order to get uniqueness can be given solely in terms of g and the geometry of Ω (see section 4).

c) If a verifies (3) and g, together with its derivatives with respect to u up to order three, satisfies suitable growth conditions, then every solution to (P)$_g$ admits in addition the asymptotic expansion (4).

d) The conclusions of Theorem 2 hold for the kind of perturbed equations $-\Delta u = \lambda(x)u - a(x)u^p + a^*(x)g(x,u)$, g as above, $a^* \in C^\alpha(\overline{\Omega})$, but now with $|a^*|_\infty$ small. However, it is required that $a^*(x) = O(a(x))$ at $\partial\Omega$. Otherwise the problem could fall in the superlinear regime where even the existence of solutions may be lost.

The next sections are dedicated to the proof of Theorems 1 and 2. Section 2 is concerned with existence and uniqueness of (P). Estimate (4) is obtained in section 3 while the analysis of the perturbed problem $(P)_g$ is carried out in section 4.

2. Existence and uniqueness

Proof of the existence of solutions. Firstly, it should be remarked that positive solutions to the finite problem $-\Delta u = \lambda(x)u - a(x)u^p$ in Ω, $u_{|\partial\Omega} = \sigma(x)$ are unique as a consequence of the general result in [6]. Now choose $\delta > 0$ such that $p > 1+\delta$, and consider the problem

$$(\mathrm{P})_n \qquad \begin{cases} -\Delta u = \lambda(x)u - (a(x) + \dfrac{1}{n^\delta})u^p, & x \in \Omega, \\ \quad u = n, & x \in \partial\Omega. \end{cases}$$

Since 0 is a subsolution and n is a supersolution for n large, $(\mathrm{P})_n$ admits a solution $u_n \in C^{2,\alpha}(\overline{\Omega})$ with $u_n \le n$, which as quoted before is unique. Using [6], it can be shown that $\{u_n\}_n$ is increasing. Our purpose is to pass to the limit as $n \to \infty$ in $(\mathrm{P})_n$.

The validity of the following lemma (cf. [10]) is the key to guaranteeing the existence of the limit.

Lemma 3. *Let $B \subset \mathbb{R}^n$ be an arbitrary ball and consider the problem*

$$(6) \qquad \begin{cases} -\Delta u = \lambda u - Au^p, & x \in B, \\ \quad u = \sigma, & x \in \partial B, \end{cases}$$

where A, λ are positive constants and $\sigma > (\lambda/A)^{\frac{1}{p-1}}$. Then the unique solution $u_{\sigma,B}$ to (6) is a radial function, $u_{\sigma,B} \in C^\infty(\overline{B})$, while $u_{\sigma,B}$ converges as $\sigma \to +\infty$ to $\underline{u}_B$ in $C^k(B)$ for every $k \in \mathbb{N}$. Moreover, $\underline{u}_B$ defines the minimal solution to (6) with $\sigma = +\infty$. Finally, the following estimate holds:

$$\inf_B \underline{u}_B \ge \frac{B_0^2}{2(p+1)AR^2},$$

where $B_0 = B(\dfrac{p-1}{2(p+1)}, \dfrac{1}{2})$ and B is the Euler Beta function.

Let us finish the proof of the existence assertion. If u_n is the solution to $(\mathrm{P})_n$ we have $-\Delta u_n \le \sup_B \lambda(x)u_n - \inf_B a(x)u_n^p$ in B. Since $u_n \le n$ in $\overline{B}$, we obtain $u_n < u_{n,B} < \underline{u}_B$ in B, where $u_{n,B}$ is as in Lemma 3. This together with the monotonicity of $\{u_n\}$ leads to $u_n \to \underline{u}$, $\underline{u} \in L^\infty_{\mathrm{loc}}(\Omega)$. By elliptic estimates and a standard bootstrapping argument we achieve $u_n \to \underline{u}$ in $C^{2,\alpha}(\Omega)$.

To prove the minimal character of $\underline{u}$, let u be an arbitrary solution to (P). Since $u \ge n$ near $\partial\Omega$, for every positive integer n, an inner approximation argument leads to $u \ge u_n$, and consequently $u \ge \underline{u}$, as was to be proved.

The existence of a maximal solution follows a similar reasoning. ☐

The following result is especially useful when dealing with the perturbed problem $(\mathrm{P})_g$. The proof, which is omitted for brevity, is carried out by applying the sub- and supersolutions method to problem (P) in domains $\{x \in \Omega : \ d(x) > 1/n\}$, and doing $n \to \infty$ through a diagonal process.

Lemma 4. *Let v, $w \in C^2(\Omega)$ be functions such that $v(x) \le w(x)$ in Ω, $-\Delta v \le \lambda(x)v - a(x)v^p$, $-\Delta w \ge \lambda(x)w - a(x)w^p$ in Ω and $\lim_{d(x)\to 0+} v(x) = \lim_{d(x)\to 0+} w(x) = +\infty$. Then there exists at least a solution $u \in C^2(\Omega)$ of problem (P) satisfying $v(x) \le u(x) \le w(x)$ in Ω.*

Remark 3. The features described in Lemma 4 also hold for the extended problem introduced in Remark 1 c).

Proof of estimate (2). We are only assuming that a verifies hypothesis (1). Fix $\delta_0 > 0$ so that $d(x) := \operatorname{dist}(x, \partial\Omega)$ is a C^k function in $0 < d(x) < \delta_0$. Fixing an arbitrary $\varepsilon > 0$, we have $(C_0 - \varepsilon)d^\gamma \le a(x) \le (C_0 + \varepsilon)d^\gamma$, as long as $0 \le d < \delta_1$. Choosing

$$A^+ = \left\{ \frac{\alpha(\alpha+1)}{C_0 - 2\varepsilon} \right\}^{\frac{1}{p-1}},$$

it is easy to prove that $u^+ = A^+ d^{-\alpha}$, defined for small $d > 0$, is a supersolution. Indeed, it is enough to prove $-\Delta u^+ \ge \lambda(x)u^+ - (C_0 - \varepsilon)d^\gamma (u^+)^p$ for $d \sim 0+$ ($d < \min\{\delta_0, \delta_1\}$). Since $-\Delta u^+ = -A^+\alpha(\alpha+1)d^{-\alpha-2} + O(d^{-\alpha-1})$, this is a consequence of $\alpha(\alpha+1) \le (C_0 - \varepsilon)(A^+)^{p-1} + O(d)$, for d small enough, which is in turn implied by the choice of A^+.

Symmetric calculations show that $u^- = A^- d^{-\alpha}$, $A^- = (\alpha(\alpha+1)/(C_0+2\varepsilon))^{1/(p-1)}$, satisfies $-\Delta u^- \le \lambda(x)u^- - (C_0 + \varepsilon)d^\gamma (u^-)^p$ near $\partial\Omega$, therefore defining a local subsolution to (P) there.

Let us proceed now to estimate the blow-up rate of an arbitrary solution u to (P), under the form (2).

Let us ascertain under which conditions $u^+ + K$ turns out to be a supersolution near $\partial\Omega$. The function $\lambda(x)u - a(x)u^p$ is decreasing in u when $\lambda(x) \le 0$, or if $u > u_c(x) := \left(\frac{|\lambda^+|_\infty}{a(x)} \right)^{1/(p-1)}$, with $\lambda^+ = \max\{\lambda, 0\}$. Taking into account that

$$u_c(x) \sim \left(\frac{\lambda}{C_0} \right)^{1/(p-1)} \left(\frac{1}{d} \right)^{\frac{\gamma}{p-1}}, \quad \text{while} \quad u^+(x) \sim A \left(\frac{1}{d} \right)^{\frac{\gamma+2}{p-1}},$$

we can ensure that for small d (say $0 < d < \delta^*$), $u^+(x) > u_c(x)$ holds. Thus,

$$-\Delta(u^+ + K) \ge \lambda(x)(u^+ + K) - a(x)(u^+ + K)^p,$$

for $0 < d < \delta^*$, $K > 0$.

On the other hand, fix $0 < \tau < \frac{\delta^*}{4}$ and introduce the region $Q_\tau := \{\tau < d < \frac{\delta^*}{2}\}$. Then

$$-\Delta(u^+(d-\tau, s) + K) \ge \lambda(x)(u^+(d-\tau, s) + K) - a(d,s)(u^+(d-\tau, s) + K)^p$$

in Q_τ, where the functions involved have been expressed in local coordinates (d, s). Moreover, this fact is uniformly valid, no matter what the value of τ is.

Thus, for every $\tau \in (0, \frac{\delta^*}{4})$, $v^+ = u^+(d-\tau, s) + K$ is a supersolution to

$$\begin{cases} -\Delta v = \lambda(x)v - a(x)v^p, & x \in Q_\tau, \\ \ \ v = u, & x \in \partial Q_\tau, \end{cases} \tag{7}$$

with u an arbitrary fixed solution to (P), provided that $K > 0$ is chosen so that $u^+(\delta^*/2 - \tau, s) + K \ge u(\delta^*/2, s)$, for every $0 < \tau < \frac{\delta^*}{4}$, $s \in \partial\Omega$. In addition, the auxiliary problem (7) has $v = u$ as its unique solution. Since $v_- = 0$ is a subsolution, we conclude

$$u(x) = u(d, s) \leq u^+(d - \tau, s) + K,$$

for every $s \in \partial\Omega$, $0 < \tau < d < \delta^*/2$ and $0 < \tau < \delta^*/4$. Letting $\tau \to 0+$ we arrive at $u(x) = u(d, s) \leq u^+(x) + K$ in $d(x) < \delta^*/2$, and we obtain

$$\overline{\lim_{d \to 0+}} \frac{u(x)}{Ad^{-\alpha}} \leq 1. \tag{8}$$

In what follows we will obtain the complementary estimate to (8). With an argument similar to the previous one, we conclude $u \geq \theta u^-$ if $0 < d(x) < \delta^*/4$, for some $0 < \theta < 1$.

Our next objective is finding $\delta > 0$ small enough, $\tau_0 = \tau_0(\delta)$ and a constant $K = K(\delta)$ in such a way that $u^-(d + \tau, s) - K(\delta)$ defines a subsolution to $-\Delta u = \lambda(x)u - a(x)u^p$ in the domain $\Omega_\delta := \{0 < d(x) < \delta\}$ for every $\tau \in (0, \tau_0(\delta))$, also satisfying

$$u^-(d + \tau, s) - K(\delta) \leq u(d, s), \tag{9}$$

for $d = \delta$ and every $0 < \tau < \tau_0(\delta)$. Inequality (9) is implied by $u^-(\delta + \tau, s) - \theta u^-(\delta, s) \leq K(\delta)$, where $0 < \tau < \tau_0(\delta)$. By virtue of the decreasing character of u^- with d, this last inequality is a consequence of

$$K(\delta) \sim (1 - \theta)A\delta^{-\alpha},$$

as $\delta \to 0+$.

On the other hand, $u^-(d + \tau, s) - K(\delta)$ will be a subsolution if

$$u^-(d + \tau, s) - K(\delta) \geq \left(\frac{\lambda}{p(C_0 + \varepsilon)(d + \tau)^\gamma} \right)^{\frac{1}{p-1}}, \tag{10}$$

for $0 < d + \tau < \delta + \tau_0(\delta) = (1 + \zeta)\delta$ if $\tau_0(\delta) = \zeta\delta$ is chosen for some fixed $\zeta \in (0, 1)$. For the validity of (10) it is enough to have

$$A \left(\frac{1}{d + \tau} \right)^\alpha - (1 - \theta)A \left(\frac{1}{\delta} \right)^\alpha > \left\{ \frac{\lambda}{pC_0} \right\}^{\frac{1}{p-1}} \left(\frac{1}{d + \tau} \right)^{\frac{\gamma}{p-1}},$$

for $0 < d < (1 + \zeta)\delta$, which in turn is true if ζ, δ are small enough.

We therefore obtain that

$$u^-(d + \tau, s) - K(\delta)$$

defines for $0 < \tau < (1 + \zeta)\delta$ a family of finite subsolutions to problem $-\Delta v = \lambda(x)v - a(x)v^p$ in Ω_δ, with $v = u$ in $d = \dfrac{\delta^*}{4}$ and $v = +\infty$ in $d = \delta$. Then $u^-(d + \tau, s) - K(\delta) \leq u(d, s)$ if $0 < d < \delta$, and letting $\tau \to 0+$ we arrive at

$$u^-(d, s) - K(\delta) \leq u(s, d), \qquad 0 < d < \delta,$$

which completes the proof of (2). ∎

Proof of uniqueness. Now let u and v be *positive* solutions to (P). By virtue of (2), u and v satisfy $\lim_{d(x) \to 0+} u(x)/v(x) = 1$. Thus, for every $\varepsilon > 0$, we can find $\delta > 0$ (as small as we please) such that

$$(1 - \varepsilon)v(x) \leq u(x) \leq (1 + \varepsilon)v(x) \tag{11}$$

when $0 < d(x) \le \delta$. On the other hand, $w^- = (1-\varepsilon)v(x)$ and $w^+ = (1+\varepsilon)v(x)$ are sub- and supersolutions to

$$\begin{cases} -\Delta w = \lambda(x)w - a(x)w^p, & x \in \Omega \setminus \Omega_\delta, \\ \quad w = u, & x \in \partial(\Omega \setminus \Omega_\delta), \end{cases}$$

where $\Omega_\delta = \{0 < d(x) < \delta\}$. The unique solution to this problem is $w = u$. Then $(1-\varepsilon)v(x) \le u(x) \le (1+\varepsilon)v(x)$ holds in $\Omega \setminus \Omega_\delta$, and (11) is true in Ω. Letting $\varepsilon \to 0+$ we arrive at $u = v$. □

3. Estimates for the second term

We will now obtain estimate (4), and so we henceforth assume that the function a verifies (3).

In this case, with an argument similar to the one used to show estimate (2), we can obtain for every $\varepsilon > 0$ a pair of sub- and supersolutions of the form

$$u^\pm = Ad^{-\alpha}(1 + B^\pm d), \quad 0 < d(x) < \delta,$$

where $B^+ \ge B$, $B^- \le B$, $B^\pm \to B$ as $\varepsilon \to 0+$, B given by (5). It is worthy of mention to say that $\Delta d(x) \to -(n-1)H(s)$ when $d(x) \to 0+$, where $H(s)$ is the mean curvature of $\partial\Omega$ at s (cf. [11]).

Moreover, $u^+ + K$ and $u^- - K(\delta)$ are still sub- and supersolutions with a convenient choice of K and $K(\delta)$. However, this only leads to estimate (4) in the case $\alpha > 1$, that is, for $p < \gamma + 3$. This is why we need an alternative procedure for the complementary range $p \ge \gamma + 3$. The following reasoning is indeed valid if $p \ge 2$.

We are looking for a constant $K > 0$ such that

$$-\Delta(AKd^{2-\alpha}) \ge \lambda(x)AKd^{2-\alpha} + a(x)(u^+)^p - a(x)(u^+ + AKd^{2-\alpha})^p.$$

This inequality, together with $-\Delta u^+ \ge \lambda(x)u^+ - a(x)(u^+)^p$, will imply that $u^+ + AKd^{2-\alpha}$ is a supersolution to (P) near $\partial\Omega$.

Since for $t \ge 0$, $1 - (1+t)^p \le -pt$, it is enough to prove $\Delta d^{2-\alpha} \le -\lambda(x)d^{2-\alpha} + p\, a(x)(u^+)^{p-1}d^{2-\alpha}$, which is *independent* of K. Now $\Delta d^{2-\alpha} = (2-\alpha)(1-\alpha)d^{-\alpha} + O(d^{-\alpha+1})$, and $a(x) \ge C_0 d^\gamma(1-\varepsilon)$ for small d, so this is a consequence of

$$(2-\alpha)(1-\alpha) \le p\ \alpha(\alpha+1)(1-\varepsilon) + O(d),$$

which in turn holds if $(2-\alpha)(1-\alpha) < p\ \alpha(\alpha+1)$. This inequality holds no matter what the values of p and γ are, thus we have a supersolution $v^+ = u^+ + AKd^{2-\alpha}$, for every $K > 0$.

If u is an arbitrary solution to (P) and we take K to have $v^+ \ge u$ in $d = \delta$, we obtain, with an argument similar to the one used in section 2, that $v^+ \ge u$ in $0 < d(x) < \delta$. Thus letting $d \to 0+$ and then $\varepsilon \to 0+$, we conclude

$$\overline{\lim_{d \to 0+}}\ \frac{1}{d}\left(\frac{u(x)}{Ad^{-\alpha}} - 1\right) \le B.$$

The lower estimate is *not* obtained in a completely similar way. We are now bound to find K so that

$$-\Delta(-AKd^{2-\alpha}) \le -\lambda(x)AKd^{2-\alpha} + a(x)(u^-)^p - a(x)(u^- - AKd^{2-\alpha})^p,$$

and we will get that $v^- = u^- - AKd^{2-\alpha}$ is a subsolution.

If $p \geq 2$, $t \geq 0$, we have $1-(1-t)^p \geq pt - p(p-1)t^2/2$, and it is then sufficient to show

$$\Delta d^{2-\alpha} \leq p\ a(x)(u^-)^{p-1}(d^{2-\alpha} - \frac{p-1}{2}\frac{AKd^{4-2\alpha}}{u^-}) - \lambda(x)d^{2-\alpha},$$

which is implied by

$$(2-\alpha)(1-\alpha) \leq p\ a(x)\frac{\alpha(\alpha+1)}{C_0 d^\gamma}(1+B^-d)^{p-1}(1-\frac{p-1}{2}\frac{Kd^2}{1+B^-d}) + O(d).$$

Since in addition

$$\frac{Kd^2}{1+B^-d} \leq \frac{K\delta^2}{1-|B^-|_\infty\delta}\ ,$$

if we choose $K\delta^2 = 2\theta(1-|B^-|_\infty\delta)/(p-1)$, $0<\theta<1$, it is enough to have

$$(2-\alpha)(1-\alpha) < p\ \alpha(\alpha+1)(1-\varepsilon)(1-\theta). \tag{12}$$

We can certainly take ε and θ small enough to satisfy (12). With this choice, $v^- = u^- - AKd^{2-\alpha}$ is a subsolution.

Now let u be an arbitrary solution to (P). We need to choose K so that $v^- \leq u$ if $d=\delta$. Since $u \geq (1-\varepsilon)Ad^{-\alpha}$, it suffices to have

$$A\delta^{-\alpha}(1+B^-\delta) - AK\delta^{2-\alpha} \leq (1-\varepsilon)A\delta^{-\alpha}.$$

This last inequality is a consequence of $\theta > (p-1)\varepsilon/2$, which we can achieve by diminishing ε if necessary.

To summarize, we arrive at $u \geq v^-$ in $0<d<\delta$, which in turn implies

$$\varliminf_{d\to 0+} \frac{1}{d}\left(\frac{u(x)}{Ad^{-\alpha}} - 1\right) \geq B,$$

and estimate (4) is proved. ∎

4. Perturbed problems

This section is devoted to the proof of Theorem 2. To begin with the existence notice that $(\mathrm{P})_g$ admits arbitrarily large supersolutions. Indeed, given $\varepsilon > 0$, we have $g(x,u) \geq -\varepsilon u^p$ if $u \geq u_0(\varepsilon) > 0$. So, denoting by v the unique solution to the problem $-\Delta u = \lambda(x)u - (1-\varepsilon)a(x)u^p$, $u_{\partial\Omega} = +\infty$ and choosing $\theta > 1$ to have $\theta v \geq u_0$, it turns out that $u^+ = \theta v$ is a supersolution to $(\mathrm{P})_g$. On the other hand, it is possible to construct an increasing sequence $\{u_n\}$, $u = u_n$ a solution of:

$$\begin{cases} -\Delta u = \lambda(x)u - a(x)(u^p + g(x,u)), & x \in \Omega, \\ \ \ u = n, & x \in \partial\Omega, \end{cases}$$

as follows: $u_0 = 0$, u_{n+1} obtained using u_n as a subsolution and u^+ as a supersolution. The arguments in section 2 lead to $u_n \to u$, $u \in C^{2,\alpha}(\Omega)$ a solution to $(\mathrm{P})_g$.

To achieve uniqueness we will first prove estimate (2) for all solutions to $(\mathrm{P})_g$. Accordingly, let u be any solution to $(\mathrm{P})_g$ and choose $\varepsilon > 0$, $u_0(\varepsilon)$ as above. Taking δ small to have $u \geq u_0$ in $\Omega_\delta = \{0 < d(x) < \delta\}$, it follows that u is a subsolution to

$$\begin{cases} -\Delta v = \lambda(x)v - a(x)(1-\varepsilon)v^p, & x \in \Omega_\delta, \\ \ \ v = u, & x \in \partial\Omega_\delta. \end{cases} \tag{13}$$

It is easy to prove that θu is a supersolution to (13) if θ is large. Since uniqueness holds for problem (13) (Remark 1 c)), then Lemma 4 and Remark 3 lead to

$$\overline{\lim_{d\to 0+}} \frac{u}{Ad^{-\alpha}} \leq (1-\varepsilon)^{-1/(p-1)}.$$

Letting $\varepsilon \to 0$, we obtain $\overline{\lim}\, u/Ad^{-\alpha} \leq 1$. The lower estimate is performed in a similar way. Thus (2) is proved.

Now let us undertake the uniqueness assertion. Assume g satisfies $(\mathrm{H})_{g'}$. We claim the existence of $c > 0$ such that *every* positive solution u to $(\mathrm{P})_g$ verifies $u \geq c$ in Ω. Thus, defining $M_g = \max\{0, \sup_{u\geq c} \frac{g(x,u)}{u^p}\}$, we conclude that every solution u to $(\mathrm{P})_g$ satisfies $u \geq w$, where w uniquely solves the problem

$$(\mathrm{P})' \qquad \begin{cases} -\Delta w = \lambda(x) w - (1+M_g)|a|_\infty w^p, & x \in \Omega, \\ \quad w = +\infty, & x \in \partial\Omega\ . \end{cases}$$

On the other hand, problem $(\mathrm{P})'$ can be estimated from below by the one-dimensional version of (6) on intervals $(-R, R)$, with $2R \geq L_\Omega$, L_Ω being the minimum distance between parallel hyperplanes enclosing Ω (see [10] for more details). Hence, from Lemma 3 we have the estimate

$$\inf_\Omega w \geq \frac{2B_0^2}{(p+1)(1+M_g)L_\Omega^2 |a|_\infty}. \tag{14}$$

Thus if $|a|_\infty$ is small enough we have that all positive solutions u to $(\mathrm{P})_g$ fall within the range where $u^{p-1} + g(x,u)/u$ is increasing. The proof of uniqueness then follows word-for-word the one given in section 2.

Finally let us show the claim. If $\lambda(x) > 0$ in $\overline{\Omega}$, direct checking reveals that c can be chosen as any positive value such that $u^{p-1} + g(x,u)/u < \inf_\Omega(\lambda/a)$ for $x \in \overline{\Omega}$ and $0 \leq u \leq c$. In this case, (14) shows that uniqueness holds provided that $(1+M_g)L_\Omega^2 |a|_\infty$ is small. In the case where $\lambda(x)$ changes sign, the argument in Lemma 3.2 of [10] leads to the desired lower estimate for solutions to $(\mathrm{P})_g$.

References

1. Bandle C., Essèn M., *On the solutions of quasilinear elliptic problems with boundary blow-up*, Sympos. Math. **35** (1994), 93-111. MR **95f:**35077
2. Bandle C., Marcus M., *Sur les solutions maximales de problèmes elliptiques nonlinéaires: bornes isopérimetriques et comportement asymptotique*, C. R. Acad. Sci. Paris Série I **311** (1990), 91-93. MR **91f:**35096
3. ———, *'Large' solutions of semilinear elliptic equations: Existence, uniqueness, and asymptotic behaviour*, J. Anal. Math. **58** (1992), 9-24. MR **94c:**35081
4. ———, *On second order effects in the boundary behaviour of large solutions of semilinear elliptic problems*, Differential Integral Equations **11** (1) (1998), 23-34. MR **98m:**35049
5. Bieberbach L., *$\Delta u = e^u$ und die automorphen Funktionen*, Math. Ann. **77** (1916), 173-212.
6. Brézis H., Oswald L., *Remarks on sublinear elliptic equations*, Nonlinear Anal. **10** (1986), 55-64. MR **87c:**35057
7. Del Pino M., Letelier R., *The influence of domain geometry in boundary blow-up elliptic problems*, Nonlinear Anal. (to appear) (1999).
8. Díaz G., Letelier R., *Unicidad de soluciones locales en algunas ecuaciones semilineales*, Proceedings of the XIth Congress on Differential Equations and Applications/First Congress on Applied Mathematics, Univ. Málaga, Málaga. (1989), 301-305. MR **91j:**00015
9. Díaz G., Letelier R., *Explosive solutions of quasilinear elliptic equations: Existence and uniqueness*, Nonlinear Anal. **20** (1993), 97-125. MR **94a:**35017
10. García-Melián J., Gómez-Reñasco R., López-Gómez J., Sabina de Lis J., *Point-wise growth and uniqueness of positive solutions for a class of sublinear elliptic problems where bifurcation from infinity occurs*, Arch. Rational Mech. Anal. **145** (1998), 261-289. MR **2000b:**35079

11. D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order,* 2nd. edition, Springer Verlag, Berlin/New York, 1983. MR **86c:**35035
12. Keller J. B., *On solutions of* $\Delta u = f(u)$, Comm. Pure Appl. Math. **10** (1957), 503-510. MR **19:**964c
13. Kondrat'ev V. A., Nikishkin V. A., *Asymptotics, near the boundary, of a solution of a singular boundary value problem for a semilinear elliptic equation*, Differential Equations **26** (1990), 345-348. MR **91g:**35048
14. Lazer A. C., McKenna P. J., *On a problem of Bieberbach and Rademacher*, Nonlinear Anal. **21** (1993), 327-335. MR **95b:**35070
15. ———, *Asymptotic behaviour of solutions of boundary blow-up problems*, Differential Integral Equations **7** (1994), 1001-1019. MR **95c:**35084
16. Loewner C., Nirenberg L., *Partial differential equations invariant under conformal or projective transformations*, Contributions to Analysis (a collection of papers dedicated to Lipman Bers), Academic Press, New York, 1974, p. 245-272. MR **50:**10543
17. Marcus M., Véron L., *Uniqueness and asymptotic behaviour of solutions with boundary blow-up for a class of nonlinear elliptic equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (2) (1997), 237-274. MR **97m:**35068
18. Osserman R., *On the inequality* $\Delta u \geq f(u)$, Pacific J. Math. **7** (1957), 1641-1647. MR **20:**4701
19. Ratto A., Rigoli M., Véron L., *Scalar Curvature and Conformal Deformation of Hyperbolic Space*, J. Funct. Anal. **121** (1994), 15-77. MR **95a:**53062
20. Véron L., *Semilinear elliptic equations with uniform blow-up on the boundary*, J. Anal. Math. **59** (1992), 231-250. MR **94k:**35113

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