A BOUNDARY BLOW-UP PROBLEM WITH A NONLOCAL REACTION

JORGE GARCÍA-MELIÁN AND JOSÉ C. SABINA DE LIS

Abstract. In this paper we deal with several kinds of boundary blow-up problems whose main feature is exhibiting a nonlocal reaction term which depends on the integral of some function of the concentration u. As a representative of such problems we consider

$$\begin{cases} \Delta u = \lambda \dfrac{f(u)}{1+\frac{1}{|\Omega|}\int_\Omega g(u)} & \text{in } \Omega \\ u = \infty & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ and the boundary condition is understood as $u(x) \to \infty$ when $\operatorname{dist}(x,\partial\Omega) \to 0$. We find necessary and sufficient conditions for the existence of positive solutions. In some cases, we also obtain uniqueness or multiplicity of solutions. Other classes of non–local dependence of the reaction term are also discussed.

1. Introduction

The purpose of the present work is to analyze the effect of a reaction term which is of nonlocal nature, on several kinds of boundary value problems subject to a singular condition. As a first step in this direction, we consider the nonlinearity

$$\lambda \frac{f(u)}{a + b\displaystyle\int_\Omega g(u)}$$

where $a, b > 0$, f is an increasing positive function and g is positive and monotone for large u. Such reaction term arises when modeling Ohmic heating (see [15] where a Dirichlet problem is studied under the choice $f = g$). Since the parameters a and b will play no essential role in our discussion, we are setting for simplicity $a = 1$ and $b = \dfrac{1}{|\Omega|}$, $|\Omega|$ the Lebesgue measure of Ω. Specifically, our first model will consist in the following nonlocal boundary blow-up problem

$$\begin{cases} \Delta u = \lambda \dfrac{f(u)}{1+\dfrac{1}{|\Omega|}\displaystyle\int_\Omega g(u)} & x \in \Omega \\ u = \infty & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded and smooth enough domain, and the real number λ will be considered as a positive parameter. Regarding problem (1.1), the functions f and g will be assumed to satisfy the following smoothness and monotonicity assumptions:

(Hf_1) $\quad f$ is locally Lipschitz and increasing in $[0,\infty)$, and $f(0) = 0$

and

(Hg) $\quad g$ is continuous in $[0,\infty)$, and there exists t_0 such that g is monotone and positive for $t \geq t_0$.

We mention that condition (Hf_1) is typical when dealing with boundary blow-up problems. As outstanding cases to be examined with more detail in this work we have $f(t) = t^p$ for $p > 1$ and $f(t) = e^t$ (the latter, not falling exactly in the scope of (Hf_1), will be separately studied in Section 6).

Our second example of nonlocal boundary blow-up problem arises in population dynamics and takes the form

$$\begin{cases} \Delta u = \lambda u(\{1 + \dfrac{1}{|\Omega|}\displaystyle\int_\Omega g(u)\}u^{p-1} - 1) & x \in \Omega \\ u = \infty & x \in \Omega, \end{cases} \tag{1.2}$$

with $\Omega \subset \mathbb{R}^N$ as above, $p > 1$ and g is increasing, positive, $g(0) = 0$ and $g(u) \to \infty$ as $u \to \infty$. In fact, consider an animal species confined in a medium Ω whose density is given by $u(x)$ and having a natural growth rate λ. Due to a limitation in the available resources the species is subject to self control which is measured by K (in principle, K may also depend on u). If there is, say, a constant immigration flow through $\partial\Omega$ whose intensity is measured by a parameter μ, then a model problem providing the equilibrium population u could be (see for instance Chapter 11 in [18], Chapter 2 in [19] for a further detailed analysis):

$$\begin{cases} \Delta u = \lambda u(Ku^{p-1} - 1) & x \in \Omega \\ \mathcal{B}(u,\mu) = 0 & x \in \Omega. \end{cases} \tag{1.3}$$

The boundary operator could take either of the options $\mathcal{B}(u,\mu) = u - \mu$ (Dirichlet), $\mathcal{B}(u,\mu) = \dfrac{\partial u}{\partial \nu} - \mu$ (Neumann) or $\mathcal{B}(u,\mu) = \dfrac{\partial u}{\partial \nu} - \mu u$ (Robin). A natural possible way of dependence of the self regulation K on u is through the average population $\dfrac{1}{|\Omega|}\int_\Omega u$ (see [10] for related ideas). The larger the population average is, the smaller the carrying capacity becomes. Therefore, problem (1.2) furnishes the limit profile of the population as the immigration pressure $\mu \to \infty$ (see Remark 6–(ii)). In this case the functional K has been chosen under the general form

$$K(u) = 1 + \frac{1}{|\Omega|}\int_\Omega g(u).$$

There is a great deal of works devoted to the subject of reaction–diffusion equations equipped with nonlocal reaction terms. See for instance [9] (and references therein), [21] (Chapters 2, 3, 8) for elliptic and parabolic equations and systems with nonlocal reactions terms and [22] (Chapter V) for parabolic equations. However, it is our belief that boundary blow-up problems with a nonlocal reaction have never been treated before in the literature.

On the contrary, the local problem

$$\begin{cases} \Delta u = f(u) & \text{in } \Omega \\ u = \infty & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

has been extensively studied in all of its relevant strands: existence and uniqueness of solutions, multiplicity, asymptotic behavior of solutions near the boundary, convexity, symmetry, etc. Among the many available works on the subject, we refer the reader to the pioneering papers [4], [23], [14], [20], [17] and to the review [24]. See also a more updated list of references in [1] or [12].

Assume that f satisfies hypothesis (Hf_1). Then, one of the earliest known features of problem (1.4) is that a necessary and sufficient condition for existence of positive solutions is

$$\text{(KO)} \qquad \int_A^\infty \frac{1}{\sqrt{F(t)}} dt < \infty$$

for some $A > 0$, where $F(t) = \int_0^t f(\tau)d\tau$ (see [14], [20] and also [8]). This condition is nowadays usually known as Keller-Osserman condition.

Let us return now to problems (1.1) and (1.2). By a solution to either of such problems we mean a function $u \in C^2(\Omega)$ such that $g(u) \in L^1(\Omega)$, it point-wise verifies the corresponding equation in (1.1) or (1.2) and $u(x) \to \infty$ as $d(x) := \text{dist}(x, \partial\Omega) \to 0$. Our aim is to determine necessary and sufficient conditions so that (1.1) and (1.2) have positive solutions for some range of λ's. Since both problems share fundamental features, we are confining ourselves in the sequel to describe the main results on (1.1). A detailed account on (1.2) and related problems will be postponed to Section 6.

Our first result shows that the Keller-Osserman condition (KO) is necessary for the existence of positive solutions to (1.1). Moreover, there is a second restriction, which is a sort of a coupling condition between f and g, which needs to be satisfied. It takes the form

$$\int_A^\infty \frac{g(t)}{\sqrt{F(t)}} dt < \infty \tag{1.5}$$

for some positive A.

More precisely, we have:

Theorem 1. *Assume f and g verify the assumptions (Hf_1) and (Hg), respectively. If problem (1.1) admits a positive solution for some $\lambda > 0$ then:*

(a) $1 + \frac{1}{|\Omega|}\int_\Omega g(u) > 0$;
(b) *f verifies condition (KO);*
(c) *g verifies condition (1.5).*

It is natural to ask whether conditions (Hf_1), (KO), (Hg) and (1.5) are also sufficient for the existence of positive solutions to (1.1). Existence of solutions is intimately connected with the existence of positive solutions to

$$\begin{cases} \Delta u = \alpha f(u) & \text{in } \Omega \\ u = \infty & \text{on } \partial\Omega \end{cases} \tag{1.6}$$

for $\alpha > 0$ (see (6.6) for the auxiliary problem corresponding to (1.2)), and their behavior with respect to α. Thus, we are assuming a situation where (1.6) has a unique positive solution for every $\alpha > 0$. This is the case, for instance, if f verifies

$$(Hf_2) \qquad \begin{array}{c} \text{there exists } p > 1 \text{ such that} \\ f(t)/t^p \text{ is increasing for large } t \end{array}$$

(see [11]). Under this additional condition on f, it is a consequence of our results that positive solutions to (1.1) exist at least in an interval of the form (λ_0, ∞) for some $\lambda_0 > 0$ (see Remark 5). Notice in addition that (Hf_2) already entails condition (KO).

We are also interested in giving conditions under which we can exactly determine the set Λ of values of λ for which solutions do exist. For this aim, we are further requiring that f satisfies

$$(Hf_3) \qquad 0 < \varliminf_{t\to\infty} \frac{f(t)}{\sqrt{2F(t)}} \int_t^\infty \frac{1}{\sqrt{2F}} \le \varlimsup_{t\to\infty} \frac{f(t)}{\sqrt{2F(t)}} \int_t^\infty \frac{1}{\sqrt{2F}} < \infty.$$

On the other hand, we are also showing that (Hf_3) may be replaced by a condition of a different nature. Namely

$$\frac{\int_t^\infty (F(s)-F(t))^{-1/2}\,ds}{\int_t^\infty F(s)^{-1/2}\,ds} \le C, \tag{1.7}$$

for $t \ge A$. For future reference, we are using a shortened version of (1.7) where F_t will stand for $F(t)$.

Both (Hf_3) and (1.7) are verified by the canonical examples $f(t) = t^p$, $p > 1$, and $f(t) = e^t$. They also hold provided, for instance, that $f(t) \sim t^p$ at infinity, or more generally that f is a function of regular variation at infinity of index $p > 1$ (see [6], [25] and Remarks 3–(iii) and (iv)). Nevertheless, it should be observed that neither (1.5) and (Hf_3) (alternatively (1.7)) are required in case g is bounded.

We next state our main existence result where it is found that, depending on the value of $g(0)$, the set Λ is the interval $(0,\infty)$ or a finite interval $(0,\lambda^*)$. In this last case, we have more than one solution for some values of λ (see the bifurcation diagrams in Figure 1).

Theorem 2. *Assume f verifies the assumptions (Hf_1), (Hf_2) and g verifies (Hg). In case g is not bounded suppose it fulfills* (1.5) *while f satisfies either condition (Hf_3) or* (1.7). *Then*

(a) *If $g(0) > -1$, problem* (1.1) *admits a positive solution for every $\lambda > 0$.*
(b) *If $g(0) < -1$, there exists $\lambda^* > 0$ such that problem* (1.1) *admits at least two positive solutions for $\lambda \in (0,\lambda^*)$, at least one positive solution if $\lambda = \lambda^*$ and no positive solutions if $\lambda > \lambda^*$.*

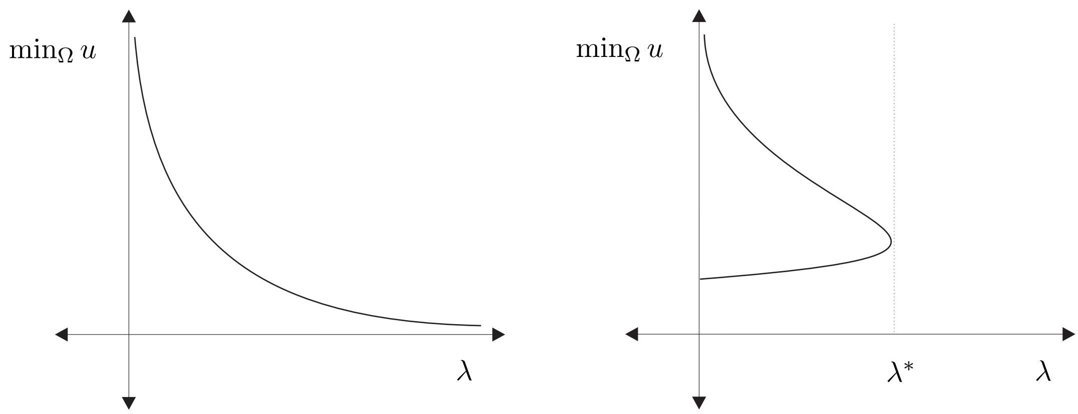

FIGURE 1. Bifurcation diagrams for problem (1.1), plotting $\min_\Omega u$ versus λ. Left: $g(0) > -1$; right: $g(0) < -1$.

Remark 1. It follows from the proof of Theorem 2 that solutions u_λ for large λ satisfy in case (a) $\min_\Omega u_\lambda \to 0$ as $\lambda \to \infty$, while solutions u_λ with small λ verify $\min_\Omega u_\lambda \to \infty$ as $\lambda \to 0$. Similarly, in case (b) there is a large solution u_λ^+ such that $\min_\Omega u_\lambda^+ \to \infty$ as $\lambda \to 0$ while another solution u_λ^- exists verifying $\min_\Omega u_\lambda^- \to c$ as $\lambda \to 0$ for a certain positive constant c.

The conclusions of Theorem 2 can be attained by removing the extra conditions (Hf_3) or (1.7) on f but restricting instead the growth of $g(t)$ for large t. To this purpose in our next statement it will be assumed that

$$g(t) = o(t^{p-1}) \qquad t \to \infty, \tag{1.8}$$

where the exponent p is the one in (Hf_2). Notice that condition (Hf_2) together with (1.5) already entail

$$g(t) = o(t^{\frac{p+1}{2}}) \qquad t \to \infty.$$

Thus (1.8) is a genuine restriction only when $1 < p \le 3$.

Theorem 3. *Assume that f satisfies (Hf_1), (Hf_2) and g satisfies (Hg). In case g is not bounded for large t suppose g fulfills* (1.5) *together with* (1.8)*. Then, all assertions of Theorem 2 hold true.*

We are next concerned with the question of determining the *exact* multiplicity of positive solutions of (1.1) for all values of λ. For this aim, we restrict ourselves to the fundamental example of the nonlinearity $f(t) = t^p$, $p > 1$. That is, we are dealing with the problem

$$\begin{cases} \Delta u = \lambda \dfrac{u^p}{1 + \dfrac{1}{|\Omega|}\displaystyle\int_\Omega g(u)} & x \in \Omega \\ u = \infty & x \in \partial\Omega. \end{cases} \tag{1.9}$$

and we are assuming that g verifies (Hg) and (1.5). As usual in uniqueness results, the function g is asked to be more regular.

Theorem 4. *Assume $g \in C^1[0,\infty)$ verifies (Hg) and $g(0) > -1$. If in addition there exist $\theta < \frac{p-3}{2}$ and $M > 0$ such that*

$$|g'(t)| \le M t^\theta \quad \text{for large } t, \tag{1.10}$$

then there exists $\lambda^{**} > 0$ such that problem (1.9) *has a unique positive solution for $\lambda \ge \lambda^{**}$. Moreover, if*

$$\frac{g(t)}{t^{p-1}} \quad \text{is decreasing for } t > 0 \tag{1.11}$$

then the positive solution to (1.9) *is unique for every $\lambda > 0$.*

Remark 2. Observe that the restriction of θ in condition (1.10) is somehow optimal. Indeed, the function $g(t) = t^{\frac{p-1}{2}}$ does not even meet condition (1.5) with $f(t) = t^p$.

Corollary 5. *Let $g(t) = t^q$ for $q \ge 1$. Then problem* (4) *admits a solution for some λ if and only if $q < \frac{p-1}{2}$. In that case, there exists a unique positive solution for every $\lambda > 0$.*

The proofs rely on the naive observation that solutions to (1.1), (1.2) and other related nonlocal problems, are solutions to (1.6) –or suitable two parametric variants of this problem– when parameters α, λ, are appropriately chosen. This amounts to solving an equation of the form $H(\lambda, \alpha) = \alpha$, where H depends on g and the unique solution to the auxiliary problem. In the case of (1.1), say, a detailed knowledge on how the solution u_α to (1.6) varies with α is required. Of special importance is the behavior when $\alpha \to \infty$ (which has already been considered in [7]) and when $\alpha \to 0$. We analyze the latter, and it is precisely there where either conditions (Hf_3) or (1.7) are needed.

The paper is organized as follows: in Section 2 we gather some preliminary results related to problem (1.6), some of them already known. Section 3 is dedicated to the proof of Theorem 1, while Sections 4 and 5 deal with the proofs of Theorems 2, 3 and 4. Section 6 is devoted to the analysis of (1.2) and further classes of nonlocal boundary blow–up problems.

2. Preliminaries

In this section we are collecting some properties of positive solutions to the boundary blow-up problem

$$(2.1)\qquad \begin{cases} \Delta u = \alpha f(u) & \text{in } \Omega \\ u = \infty & \text{on } \partial\Omega, \end{cases}$$

where $\alpha > 0$. Under fairly general conditions on the nonlinear term, this problem admits a positive solution. But we are also interested in the uniqueness issue and in the boundary behavior of solutions.

Assume f verifies conditions (Hf_1) and (KO). It then makes sense to define the functions

$$\psi(t) = \frac{1}{\sqrt{2}} \int_t^\infty F^{-1/2} \qquad t > 0,$$

and $\phi(t) = \psi^{-1}(t)$ (the inverse of ψ). It is readily checked that $u(t) = \phi(\sqrt{\alpha}t)$, is the unique positive solution to the one-dimensional problem

$$\begin{cases} u'' = \alpha f(u) & \text{in } x > 0 \\ u(0) = \infty. \end{cases}$$

The next theorem, dealing with existence and uniqueness of problem (2.1) is well-known. We refer to [14], [20], [8] for the existence part, [3], [8] for the asymptotic behavior and [11] for uniqueness.

Theorem 6. *Assume f verifies the hypotheses (Hf_1) and* (KO)*. Then problem* (2.1) *has at least a positive solution u_α for every $\alpha > 0$. Moreover, the boundary behavior of all solutions u is given by*

$$(2.2)\qquad \lim_{d(x)\to 0} \frac{\psi(u(x))}{\sqrt{\alpha}d(x)} = 1.$$

If in addition f verifies (Hf_2) then the solution u_α is unique and it verifies:

$$(2.3)\qquad \lim_{d(x)\to 0} \frac{u_\alpha(x)}{\phi(\sqrt{\alpha}d(x))} = 1.$$

¿From now on we always assume that problem (2.1) has a unique positive solution, and it will be denoted throughout by u_α. Our next step is analyzing the dependence of this solution with varying α. The first result in this line is an direct consequence of comparison and uniqueness.

Lemma 7. *Assume f verifies conditions (Hf_1) and (Hf_2). Then the unique solution u_α of problem* (2.1) *is decreasing and continuous in α, with values in $C^2_{\rm loc}(\Omega)$.*

Proof. If $0 < \alpha < \beta$, then clearly u_α is a supersolution to the problem

$$(2.4)\qquad \begin{cases} \Delta u = \beta f(u) & \text{in } \Omega \\ u = n & \text{on } \partial\Omega \end{cases}$$

for an arbitrary $n \in \mathbb{N}$. It follows by comparison that $u_\alpha \geq u_{\beta,n}$, where $u_{\beta,n}$ is the unique solution to (2.4). Since $u_{\beta,n} \to u_\beta$ as $n \to \infty$, we arrive at $u_\alpha \geq u_\beta$, and the strong maximum principle implies $u_\alpha > u_\beta$ in Ω.

Now let $\alpha_n \to \alpha > 0$. For a small enough positive ε we have $\alpha - \varepsilon \leq \alpha_n \leq \alpha + \varepsilon$ for large n, so that, by monotonicity, $u_{\alpha+\varepsilon} \leq u_{\alpha_n} \leq u_{\alpha-\varepsilon}$ in Ω. This implies that the sequence u_{α_n} is locally uniformly bounded, and thus a standard bootstrap gives that, for a subsequence, $u_{\alpha_n} \to \bar{u}$ in $C^2_{\rm loc}(\Omega)$. Then clearly $\Delta\bar{u} = \alpha f(\bar{u})$ in Ω, and since $\bar{u} \geq u_{\alpha+\varepsilon}$, we also have

$\bar{u} = \infty$ on $\partial\Omega$. Hence $\bar{u}$ is a solution to (2.1), and uniqueness implies $\bar{u} = u_\alpha$. Thus $u_{\alpha_n} \to u_\alpha$ in $C^2_{\mathrm{loc}}(\Omega)$, and since the sequence α_n is arbitrary the continuity is proved. □

Our next ingredient is the precise behavior of u_α when both $\alpha \to \infty$ and $\alpha \to 0$. More precisely, we need that the limits in (2.2) and (2.3) are uniform in α. It turns out that this has been already proved in [7] for large values of α. We remark that an analysis of the proofs there reveals that the differentiability condition that is assumed on f is not necessary. Remember that we are denoting $d(x) = \mathrm{dist}(x, \partial\Omega)$.

Theorem 8. *Under the assumptions of Lemma 7 let u_α be the unique positive solution to* (2.1)*. Then for every $\varepsilon > 0$ there exists $\alpha^* > 0$ such that if $\alpha \geq \alpha^*$*

$$u_\alpha(x) \leq \max\{\phi(\sqrt{\alpha}(1-\varepsilon)d(x)), \varepsilon\} \quad \text{in } \Omega. \tag{2.5}$$

In particular, $u_\alpha \to 0$ uniformly in compact subsets of Ω as $\alpha \to \infty$.

We now deal with the behavior of u_α as $\alpha \to 0$. Remark that there is an essential difference with that of the previous theorem. Namely, since the solution u_α is expected to go to infinity uniformly in Ω, the behavior of f at infinity is relevant for obtaining this behavior. In Section 1, two alternative conditions governing such behavior were presented. In our next result, we impose the first one of them, i.e. condition (Hf_3).

Theorem 9. *Assume f verifies (Hf_1), (Hf_2) and (Hf_3). Let u_α be the unique positive solution to* (2.1)*. Then there exists α_0 and positive constants C_1, C_2 such that for $0 < \alpha \leq \alpha_0$:*

$$\phi(C_1\sqrt{\alpha}d(x)) \leq u_\alpha(x) \leq \phi(C_2\sqrt{\alpha}d(x)) \quad \text{in } \Omega. \tag{2.6}$$

In particular, $u_\alpha \to \infty$ uniformly in Ω as $\alpha \to 0$.

Proof. To show the upper inequality, we construct a supersolution. Let z be the unique solution to $-\Delta z = 1$ in Ω with $z = 0$ on $\partial\Omega$. For a suitable $\varepsilon > 0$ we let

$$\bar{u}(x) = \phi(\varepsilon\sqrt{\alpha}z(x)).$$

A straightforward calculation proves that $\bar{u}$ will be a supersolution provided

$$\varepsilon^2\alpha f(\phi(\varepsilon\sqrt{\alpha}z))|\nabla z|^2 - \varepsilon\sqrt{\alpha}\phi'(\varepsilon\sqrt{\alpha}z) \leq \alpha f(\phi(\varepsilon\sqrt{\alpha}z))$$

in Ω. This is equivalent to

$$|\nabla z|^2 - \frac{\phi'(\varepsilon\sqrt{\alpha}z)}{\varepsilon\sqrt{\alpha}zf(\phi(\varepsilon\sqrt{\alpha}z))}\, z \leq \varepsilon^{-2}$$

in Ω. Now notice that

$$\limsup_{t\to 0} \frac{-\phi'(t)}{tf(\phi(t))} = \limsup_{t\to 0} \frac{\sqrt{2F(\phi(t))}}{tf(\phi(t))} = \limsup_{s\to\infty} \frac{\sqrt{2F(s)}}{\psi(s)f(s)} = \frac{1}{l},$$

with $l > 0$ the value of the inferior limit in (Hf_3). Thus $\bar{u}$ will be a supersolution for small α if for instance $|\nabla z|^2 + 2l^{-1}z \leq \varepsilon^{-2}$, which is certainly possible by taking ε sufficiently small. By means of a comparison as in Lemma 7 we obtain that $u_\alpha \leq \bar{u}$. Finally notice that $z(x) \geq Cd(x)$ in Ω for a positive constant C thanks to Hopf's principle and since ϕ is decreasing we obtain the upper inequality in (2.6).

The lower inequality is shown similarly by means of a subsolution $\underline{u} = \phi(K\sqrt{\alpha}z)$ for large K. If $L < \infty$ stands for the upper limit in (Hf_3), we remark that it is only needed that $|\nabla z|^2 + (2L)^{-1}z > 0$ in $\overline{\Omega}$, which is also a consequence of Hopf's principle. □

Theorem 9 can also be obtained from the alternative condition (1.7) on the behavior of f for large u. Such condition, which is essentially different from (Hf_3), is optimal (Remark 3 below). Let us begin by considering a particular case.

Theorem 10. *Assume f satisfies (Hf_1), (Hf_2) while Ω is a ball B or an annulus A of $\mathbb{R}^N$ in problem (2.1). Then, the following statements are equivalent.*

(a) *f satisfies (1.7), i. e.,*

$$\frac{\int_t^\infty (F - F_t)^{-1/2}}{\int_t^\infty F^{-1/2}} \le C,$$

for all $t \ge A$ and certain $A > 0, C > 0$.

(b) *There exist positive constants C_1, C_2 and α_0 such that the solution u_α to (2.1) satisfies (2.6) for $0 < \alpha \le \alpha_0$.*

Proof. We first suppose that Ω is a ball. No generality is lost by taking $\Omega = B(0,R)$, where $B(0,R) = \{x : |x| < R\}$ (just perform a shift). Let $u_\alpha(r)$ be any radial solution to (2.1) with $u_0 = u_0(\alpha)$ its minimum in $B(0,R)$, i. e. $u_\alpha(0) = u_0$ (the dependence on α of both u_α and u_0 will be dropped in the sequel for simplicity). Following [14] (cf. Proof of Theorem I), $u(r)$ satisfies

$$\sqrt{\frac{\alpha}{N}}(r_2 - r_1) \le \frac{1}{\sqrt{2}} \int_{u(r_1)}^{u(r_2)} (F - F_0)^{-1/2} \le \sqrt{\alpha}(r_2 - r_1) \tag{2.7}$$

for $0 \le r_1 < r_2 < R$, with $F_0 = F(u_0)$. This implies in particular,

$$\frac{1}{\sqrt{2}} \int_{u_0}^{\infty} F^{-1/2} < \sqrt{\alpha} R, \tag{2.8}$$

and so $u_0 \to \infty$ as $\alpha \to 0$. In addition

$$\frac{1}{\sqrt{2}} \int_{u(r)}^{\infty} F^{-1/2} < \frac{1}{\sqrt{2}} \int_{u(r)}^{\infty} (F - F_0)^{-1/2} \le \sqrt{\alpha}(R - r),$$

and hence

$$\phi(\sqrt{\alpha}(R - r)) \le u(r)$$

for $0 \le r < R$. Thus the lower estimate in (2.6) always holds with $C_1 = 1$, irrespective of (1.7), and in the ball the proof of (2.6) reduces to proving the upper estimate.

Thus let us next show that the upper estimate in (2.6) is equivalent to (1.7). Assume first that the upper estimate in (2.6) holds, then setting $r = 0$ we get

$$C_2 \sqrt{\alpha} R \le \psi(u_0)$$

for $\alpha \le \alpha_0$ which is equivalent to $u_0 \ge u_0^*$ for a certain positive u_0^*. Then,

$$\frac{C_2}{\sqrt{2}} \int_{u_0}^{\infty} (F - F_0)^{-1/2} \le \psi(u_0)$$

whence we obtain

$$\frac{\int_{u_0}^\infty (F - F_0)^{-1/2}}{\int_{u_0}^\infty F^{-1/2}} \le \frac{1}{C_2}$$

for $u_0 \ge u_0^*$, i. e., condition (1.7) holds.

Conversely, assume now that (1.7) is satisfied. Then in order to prove (2.6) it suffices, thanks to (2.7) with $r_1 = r$ and $r_2 = R$, to show that the estimate

$$\frac{\int_u^\infty (F - F_0)^{-1/2}}{\int_u^\infty F^{-1/2}} \le C \tag{2.9}$$

holds also true for $u \geq u_0 \geq u_0^*$. But (2.9) follows from (1.7) by observing that, as a function of u, the left hand side in (2.9) is decreasing. Indeed, by differentiating in u it is easily checked that

$$\frac{1}{\sqrt{F(u)}}\int_u^\infty (F-F_0)^{-1/2} < \frac{1}{\sqrt{F(u)-F_0}}\int_u^\infty F^{-1/2},$$

for all $u > u_0$. This completes the equivalence between (a) and (b) for $\Omega = B(0,R)$.

To proceed with the case where Ω is an annulus A, observe that it can be assumed that $A = \{x : R_1 < |x| < R_2\}$ with $0 < R_1 < R_2$. Let us again designate by $u_\alpha(r)$ a radial solution to (2.1) with $u_0 = \min_A u_\alpha$ (subindex α being again removed for simplicity). By performing the change

$$t = g(r) := \frac{1}{N-2}\left(\frac{1}{R_1^{N-2}} - \frac{1}{r^{N-2}}\right),$$

for $N \geq 3$ or

$$t = g(r) := \log\left(\frac{r}{R_1}\right),$$

if $N = 2$, it is found that the transformed u, i. e. $v(t) = u(g^{-1}(t))$ satisfies

$$\alpha K_1^2 f(v) \leq v'' \leq \alpha K_2^2 f(v) \qquad t \in (0,L),$$

where $K_i = R_i^{N-1}$, $i = 1,2$ and $L = g(R_2)$. Since v is strictly convex, it achieves the minimum $v_0 = u_0$ at a unique point $t_0 \in (0,L)$. Thus

$$K_1\sqrt{\alpha}(t_2-t_1) \leq \frac{1}{\sqrt{2}}\int_{v(t_1)}^{v(t_2)} (F-F_0)^{-1/2} \leq K_2\sqrt{\alpha}(t_2-t_1), \tag{2.10}$$

if $t_0 \leq t_1 < t_2 < L$, while

$$K_1\sqrt{\alpha}(t_2-t_1) \leq \frac{1}{\sqrt{2}}\int_{v(t_2)}^{v(t_1)} (F-F_0)^{-1/2} \leq K_2\sqrt{\alpha}(t_2-t_1), \tag{2.11}$$

when $0 < t_1 < t_2 \leq t_0$ (observe the interchange of values in the limits of integration above).

In particular we have

$$K_1\sqrt{\alpha}(L-t_0) \leq \frac{1}{\sqrt{2}}\int_{v_0}^{\infty} (F-F_0)^{-1/2} \leq K_2\sqrt{\alpha}(L-t_0),$$

together with

$$K_1\sqrt{\alpha}t_0 \leq \frac{1}{\sqrt{2}}\int_{v_0}^{\infty} (F-F_0)^{-1/2} \leq K_2\sqrt{\alpha}t_0.$$

Recall that $t_0 = t_0(\alpha)$. Therefore

$$0 < \eta_1 := \varliminf_{\alpha\to 0} t_0 \leq \eta_2 := \varlimsup_{\alpha\to 0} t_0 < L.$$

By setting $d(t) = \operatorname{dist}(t,\{0,L\}))$ (respectively, $d(r) = \operatorname{dist}(r,\{R_1,R_2\}))$ we then obtain

$$K_1\sqrt{\alpha}d(t_0) \leq \frac{1}{\sqrt{2}}\int_{v_0}^{\infty} (F-F_0)^{-1/2} \leq K_2\sqrt{\alpha}d(t_0). \tag{2.12}$$

By observing that $\psi(v_0) \leq K_2\sqrt{\alpha}d(t_0)$ we conclude again that $v_0 \to \infty$ as $\alpha \to 0+$.

Assume now that u_α is a family of radial solutions satisfying (2.6). Then,

$$C_2'\sqrt{\alpha}d(t_0) \leq \psi(v_0) \leq C_1'\sqrt{\alpha}d(t_0),$$

since $D_1 d(r) \le d(g(r)) \le D_2 d(r)$ for certain positive D_1, D_2 and $R_1 \le r \le R_2$. By combining this inequalities with (2.12) we achieve

$$\frac{1}{\sqrt{2}} \int_{v_0}^{\infty} (F - F_0)^{-1/2} \le \frac{K_2}{C_2'},$$

for large v_0. This proves (1.7).

Conversely, let us suppose that (1.7) holds. Then, for any family of radial solutions u_α we find

$$K_1 \sqrt{\alpha}(L-t) \le \frac{1}{\sqrt{2}} \int_{v(t)}^{\infty} (F - F_0)^{-1/2} \le K_2 \sqrt{\alpha}(L-t)$$

for $t \ge t_0$ while

$$K_1 \sqrt{\alpha} t \le \frac{1}{\sqrt{2}} \int_{v(t)}^{\infty} (F - F_0)^{-1/2} \le K_2 \sqrt{\alpha} t$$

holds for $t \le t_0$. Thus,

$$\psi(v(t)) \le \begin{cases} \sqrt{\alpha} K_2 (L-t) & t \ge t_0 \\ \sqrt{\alpha} K_2 t & t \le t_0. \end{cases} \tag{2.13}$$

Assume now that $t \ge t_0$. If $L - t \le t$ then the first option above reads

$$\psi(v(t)) \le K_2 \sqrt{\alpha} d(t).$$

If on the contrary, $t \le L - t$ then

$$\psi(v(t)) \le K_2 \sqrt{\alpha} \frac{(L-t)}{t} t \le K_2 \sqrt{\alpha} \frac{(L-\eta_1)}{\eta_1} d(t)$$

since $d(t) = t$ in this case. By arguing in the same way with the second option of (2.13) we conclude that

$$\psi(v(t)) \le K_2' \sqrt{\alpha} d(t),$$

and so

$$\phi(K_2' \sqrt{\alpha} d(t)) \le v(t),$$

for all α and t, which is the lower estimate in (2.6).

Finally, by setting suitable values in (2.10) and (2.11) and employing (1.7) we arrive at

$$K_1 \sqrt{\alpha} d(t) \le \frac{1}{\sqrt{2}} \int_{v(t)}^{\infty} (F - F_0)^{-1/2} \le \frac{C}{\sqrt{2}} \int_{v(t)}^{\infty} F^{-1/2},$$

for all $0 < t < L$. This yields the upper estimate in (2.6) and the proof is finished. □

Remarks 3. (i) Theorem 10 remains true if (Hf_2) is dropped (but now fulfilling (KO)). In other words, uniqueness of solutions to (2.1) is not required for its validity. It suffices with changing assertion (b) by the following one.

(b') For every family of radial solutions $\{u_\alpha\}$ to (2.1) there exist constants C_1, C_2 and α_0 such that (2.6) holds for $\alpha \le \alpha_0$.

(ii) To get an insight on condition (1.7) just check problem (2.1) in the one–dimensional interval $\Omega = (-L, L)$. A direct convexity argument shows that its solution u_α is symmetric with respect to $x = 0$. Thus, $u_0 := \min u_\alpha = u_\alpha(0)$ and an elementary integration yields

$$\frac{1}{\sqrt{2}} \int_{u_\alpha(x)}^{\infty} (F - F_0)^{-1/2} = \sqrt{\alpha}(L - x) \qquad 0 \le x < L, \tag{2.14}$$

with $F_0 = F(u_0)$.

Consider now the mixed problem

$$\begin{cases} u'' = \alpha f(u) & 0 < x < L \\ u(0) = u_0 \quad u(L) = \infty. \end{cases} \tag{2.15}$$

It is readily checked that $\underline{u}(x) = \phi(\sqrt{\alpha}(L-x))$ is a subsolution ($\phi(\sqrt{\alpha}L) < u_0$) while $\overline{u}(x) = \phi(A\sqrt{\alpha}(L-x))$ is a comparable supersolution if $A \le 1$ and

$$\phi(A\sqrt{\alpha}L) \ge u_0.$$

After this choice, estimate (2.6) reads

$$\underline{u}(x) \le u_\alpha \le \overline{u}(x),$$

provided that $A = A(\alpha) \le 1$ can be found which keeps uniformly bounded away from zero as $\alpha \to 0+$, equivalently as $u_0 \to \infty$. The "optimum" choice for $A(\alpha)$ is just:

$$\phi(A\sqrt{\alpha}L) = u_0,$$

i. e.,

$$A(\alpha) = \frac{\psi(u_0)}{\sqrt{\alpha}L} = \frac{\int_{u_0}^\infty F^{-1/2}}{\int_{u_0}^\infty (F-F_0)^{-1/2}},$$

where relation (2.14) with $x = 0$ has been employed to provide an alternative expression for $\sqrt{\alpha}L$. Condition (1.7) simply reads

$$A_- := \varliminf_{\alpha\to 0} A(\alpha) > 0.$$

Indeed, $C_2^* = A_-$ is just the best choice for constant C_2 in (2.6).

(iii) A positive measurable function f defined on $[0,\infty)$ is said to be regularly varying with index p at infinity if

$$\lim_{u\to\infty} \frac{f(tu)}{f(u)} = t^p,$$

for all $t > 0$ ([25]). Let us show that such a f satisfies (1.7) provided $p > 1$ ((KO) condition is an immediate consequence of $p > 1$). In fact, for fixed $\theta > 1$,

$$\frac{\int_t^\infty (F-F_t)^{-1/2}}{\int_t^\infty F^{-1/2}} \le \max\left\{ \frac{\int_t^{\theta t} (F-F_t)^{-1/2}}{\int_t^{\theta t} F^{-1/2}}, \frac{\int_{\theta t}^\infty (F-F_t)^{-1/2}}{\int_{\theta t}^\infty F^{-1/2}} \right\}$$

But, $F(\theta u) > \theta F(u)$ for $u > 0$. Thus

$$\frac{\int_{\theta t}^\infty (F-F_t)^{-1/2}}{\int_{\theta t}^\infty F^{-1/2}} \le \left(1 - \frac{F(t)}{F(\theta t)}\right)^{-1/2} \le \left(\frac{\theta}{\theta-1}\right)^{1/2}. \tag{2.16}$$

On the other hand, it follows from Theorem 1.1 in [25] that $F(ts)/F(t) \to s^{p+1}$ uniformly in $[1,\theta]$ as $t \to \infty$. Hence

$$\frac{\int_t^{\theta t} (F-F_t)^{-1/2}}{\int_t^{\theta t} F^{-1/2}} = \frac{\int_1^\theta \left(\frac{F(ts)}{F(t)} - 1\right)^{-1/2} ds}{\int_1^\theta \left(\frac{F(ts)}{F(t)}\right)^{-1/2} ds} \to \frac{\int_1^\theta (s^{p+1}-1)^{-1/2} ds}{\int_1^\theta s^{-(p+1)/2} ds}$$

as $t \to \infty$. In conclusion, f satisfies (1.7).

(iv) Let us show now that f also fulfills (Hf_3) provided it is of regular variation at infinity with index $p > 1$. Actually, this implies that $F^{-1/2}$ is of regular variation with index $-(p+1)/2$. Thus (see [25])

$$\frac{tF^{-1/2}(t)}{\int_t^\infty F^{-1/2}} \sim \frac{p-1}{2}$$

together with

$$\frac{tf(t)}{\int_0^t f} \sim p+1$$

as $t \to \infty$ in both cases. Hence

$$\frac{f}{\sqrt{F}} \int_t^\infty F^{-1/2} \sim 2\frac{p+1}{p-1},$$

as $t \to \infty$ and we are done.

Let us next show that the same conclusions of Theorem 9 can be attained when replacing condition (Hf_3) by (1.7).

Theorem 11. *Suppose f satisfies (Hf_1), (Hf_2) and* (1.7)*. Then, positive constants C_1, C_2 and α_0 exist so that the solution u_α to* (2.1) *satisfies*

$$\phi(C_1\sqrt{\alpha}d(x)) \le u_\alpha(x) \le \phi(C_2\sqrt{\alpha}d(x)) \qquad x \in \Omega,$$

for all $0 < \alpha \le \alpha_0$.

Proof. Due to the smoothness of $\partial\Omega$ there exists a fixed number $\rho > 0$ such that for each $y \in \partial\Omega$ points $z_i \in \Omega$, $z_e \in \mathbb{R}^N \setminus \overline{\Omega}$ can be found with $B(z_i,\rho) \subset \Omega$, $B(z_e,\rho) \subset \mathbb{R}^N \setminus \overline{\Omega}$ and $\partial B(z_i,\rho) \cap \partial\Omega = \partial B(z_e,\rho) \cap \partial\Omega = \{y\}$.

Let $x \in \Omega$ be so that $d(x) \le \delta$, while $y = y(x) \in \partial\Omega$ satisfies $|x-y| = d(x)$. Then,

$$u_\alpha(x) \le u_{\alpha,B}(x - z_i)$$

where z_i is the center of the inner sphere $B(z_i,\rho)$, tangent to $\partial\Omega$ at y, while $u_{\alpha,B}$ is the solution to (2.1) in $\Omega = B(0,\rho)$. Thus, (1.7) implies the existence of C_2 and α_0 such that

$$u_{\alpha,B}(r) \le \phi(C_2\sqrt{\alpha}(\rho - r)) \qquad 0 \le r \le \rho,$$

for $\alpha \le \alpha_0$. Hence, it follows that

$$u_\alpha(x) \le \phi(C_2\sqrt{\alpha}(\rho - |x - z_i|)) = \phi(C_2\sqrt{\alpha}d(x)),$$

provided $d(x) \le \delta$. To complete the upper estimate in (2.6) set

$$d_\Omega = \max_{x\in\Omega} d(x)$$

and define

$$h = \frac{C_2(d_\Omega - \delta)}{d_\Omega}.$$

Since u_α is subharmonic we achieve

$$u_\alpha(x) \le \phi(C_2\sqrt{\alpha}\delta) = \phi((C_2 - h)\sqrt{\alpha}d_\Omega) \le \phi((C_2 - h)\sqrt{\alpha}d(x)),$$

for all $x \in \Omega$ such that $d(x) \ge \delta$. Therefore

$$u_\alpha(x) \le \phi((C_2 - h)\sqrt{\alpha}d(x)),$$

holds true for all $x \in \Omega$ and $\alpha \le \alpha_0$. The upper estimate in (2.6) is thus accomplished.

To obtain the lower estimate observe that a fixed $R > 0$ can be found so that $\Omega \subset z_e(y) + A$ for all $y \in \partial\Omega$ where A stands for the "stationary" annulus $A = \{x : \rho < |x| < R\}$.

If $x \in \Omega$ and $y \in \partial\Omega$ satisfies $|x-y| = d(x)$ then

$$u_\alpha(x) \geq u_{\alpha,A}(x - z_e(y)),$$

where $u_{\alpha,A}$ defines the solution to (2.1) in A. Then condition (1.7) implies that

$$u_\alpha(x) \geq \phi(C_1\sqrt{\alpha}(|x - z_e(y)| - \rho)) = \phi(C_1\sqrt{\alpha}d(x)).$$

Since x is arbitrary and A does not depend on x the proof is finished. □

Remark 4. If f satisfies (KO) then (Hf_2) can be once again dropped in the statement of Theorem 11 by replacing the reference to a "unique" solution u_α by an arbitrary family $\{u_\alpha\}$ of solutions to (2.1) with $\alpha \to 0$. In that case, the upper estimate is shown for the maximal solution $\overline{u}_\alpha$ and to this purpose, one works in turn with the maximal solution to the auxiliary problem in the ball B which is certainly radial. A dual approach works for the lower estimate.

3. Necessary conditions

We now come to the proof of the necessity of conditions (KO) and (1.5) for the existence of positive solutions.

Proof of Theorem 1. Assume a positive solution to (1.1) exists, and let

$$\alpha = \frac{\lambda}{1 + \frac{1}{|\Omega|}\int_\Omega g(u)}.$$

Then u is a solution to the problem (2.1) and consequently $\alpha > 0$ and f verifies condition (KO) (see for instance [8]). This shows (a) and (b).

To prove (c), recall that $g(u) \in L^1(\Omega)$. Now choose $\varepsilon > 0$ small. Since u is a solution to (2.1), it follows from Theorem 6 that there exists $\delta > 0$ with

$$(1-\varepsilon)\sqrt{\alpha}d(x) \leq \psi(u(x)) \leq (1+\varepsilon)\sqrt{\alpha}d(x).$$

provided $d(x) < \delta$. Since ϕ is decreasing, this implies $\phi((1+\varepsilon)\sqrt{\alpha}d(x)) \leq u(x) \leq \phi((1-\varepsilon)\sqrt{\alpha}d(x))$ if $d(x) < \delta$. Choosing a smaller δ if necessary we may assume $\phi((1+\varepsilon)\sqrt{\alpha}d(x)) \geq t_0$ for $d(x) < \delta$, where t_0 is given in condition (Hg). We only need to consider the case where $g(t)$ is increasing for $t \geq t_0$ (otherwise g remains bounded). Since $g(u) \in L^1(\Omega)$, we have

$$\int_{\Omega_\delta} g(\phi((1+\varepsilon)\sqrt{\alpha}d(x))) < \infty,$$

where $\Omega_\delta = \{x \in \Omega : \ d(x) < \delta\}$. We perform a change of variables in the integral, letting $x = \bar{x} - t\nu(y)$, where $\bar{x} \in \partial\Omega$, $0 < t < \delta$ and $\nu(y)$ is the outward unit normal at y. This change is smooth, since we are assuming that $\partial\Omega$ is smooth. If we denote the Jacobian of the change by $J(t,y)$, then we have

$$\int_0^\delta \int_{\partial\Omega} g(\phi((1+\varepsilon)\sqrt{\alpha}t))J(t,y)d\sigma(y)dt < \infty. \tag{3.1}$$

Thanks to the smoothness and compactness of $\partial\Omega$, we can find positive constants κ_1, κ_2 such that $\kappa_1 \leq J(t,y) \leq \kappa_2$ and then (3.1) implies

$$\int_0^\delta g(\phi((1+\varepsilon)\sqrt{\alpha}t))dt < \infty.$$

By performing the change $\phi((1+\varepsilon)\sqrt{\alpha}t) = s$, we obtain (recalling the definition of ϕ as the inverse of ψ):

$$\int_{\phi((1+\varepsilon)\sqrt{\alpha}\delta)}^{\infty} \frac{g(s)}{\sqrt{F(s)}} dt < \infty,$$

and thus the necessity of condition (1.5) for existence is shown. This concludes the proof of part (c) of the theorem. □

4. Existence of solutions

This section is devoted to prove existence and multiplicity of positive solutions to (1.1). As seen in the previous section, if $u \in C^2(\Omega)$ is positive and verifies $u = \infty$ on $\partial\Omega$, denoting

$$\alpha = \frac{\lambda}{1 + \frac{1}{|\Omega|}\int_\Omega g(u)},$$

then u will be a solution to (1.1) if and only if $u = u_\alpha$. That is, solving problem (1.1) amounts to solving the equation

$$H(\alpha) = \lambda$$

for $\lambda, \alpha > 0$, where

$$H(\alpha) = \alpha \left(1 + \frac{1}{|\Omega|}\int_\Omega g(u_\alpha)\right). \tag{4.1}$$

Thus our main effort will be put in analyzing the precise behavior of $H(\alpha)$ for $\alpha > 0$. We first show that H is continuous.

Lemma 12. *Assume f verifies (Hf_1), (Hf_2) and g verifies (Hg) and (1.5). If u_α denotes the unique positive solution to (2.1) for $\alpha > 0$, then the function $H(\alpha)$ given by (4.1) is well-defined and continuous for $\alpha > 0$.*

Proof. The proof that $g(u_\alpha) \in L^1(\Omega)$ is similar to part (c) in Theorem 1: first notice that if $g(t)$ is decreasing for $t \geq t_0$, then it is bounded in $[\min_\Omega u_\alpha, \infty)$, and there is nothing to prove. Thus we may assume $g(t)$ is increasing for $t \geq t_0$. Let $\varepsilon > 0$ and choose $\delta > 0$ so that $\phi((1+\varepsilon)\sqrt{\alpha}d(x)) \leq u(x) \leq \phi((1-\varepsilon)\sqrt{\alpha}d(x))$ if $d(x) < \delta$. We may also assume $u_\alpha \geq t_0$ in $d(x) < \delta$.

Observe that we only need to show $g(u_\alpha) \in L^1(\Omega_\delta)$ with $\Omega_\delta = \{d(x) < \delta\}$. Since $g(u_\alpha)$ is positive in Ω_δ and

$$\begin{aligned}\int_{\Omega_\delta} g(u_\alpha) &\leq \int_{\Omega_\delta} g(\phi((1-\varepsilon)\sqrt{\alpha}d(x))) = \int_0^\delta \int_{\partial\Omega} g(\phi((1-\varepsilon)\sqrt{\alpha}t))J(t,y)d\sigma(y)dt \\ &\leq \kappa_2|\partial\Omega| \int_0^\delta g(\phi((1-\varepsilon)\sqrt{\alpha}t))dt \leq \frac{\kappa_2|\partial\Omega|}{\sqrt{\alpha}(1-\varepsilon)} \int_{\phi((1-\varepsilon)\sqrt{\alpha}\delta)}^{\infty} \frac{g(s)}{\sqrt{2F(s)}} ds,\end{aligned}$$

this is clear thanks to condition (1.5).

To conclude the proof we only need to show that $\alpha \to \int_\Omega g(u_\alpha)$ is continuous in $\alpha > 0$. Let $\alpha_n \to \alpha$ be an arbitrary sequence. For small ε and large n we have $u_{\alpha+\varepsilon} \leq u_{\alpha_n} \leq u_{\alpha-\varepsilon}$ in Ω.

The case where $g(t)$ is decreasing for $t \geq t_0$ is straightforward, since this implies that g is bounded in $[\min_\Omega u_{\alpha+\varepsilon}, \infty)$ and thus dominated convergence implies

$$\int_\Omega g(u_{\alpha_n}) \to \int_\Omega g(u_\alpha).$$

In the other case, since we can select a small $\delta > 0$ such that $u_{\alpha_n} \geq t_0$ in Ω_δ for large n, we obtain $g(u_{\alpha_n}) \leq g(u_{\alpha-\varepsilon}) \in L^1(\Omega_\delta)$ in Ω_δ, and dominated convergence gives

$$\int_{\Omega_\delta} g(u_{\alpha_n}) \to \int_{\Omega_\delta} g(u_\alpha).$$

Finally, since also $u_{\alpha_n} \leq \sup_{\Omega\setminus\Omega_\delta} u_{\alpha-\varepsilon}$ in $\Omega \setminus \Omega_\delta$, we also have

$$\int_{\Omega\setminus\Omega_\delta} g(u_{\alpha_n}) \to \int_{\Omega\setminus\Omega_\delta} g(u_\alpha).$$

This concludes the proof. □

The next step is analyzing the values of H at both zero and infinity. This turns out to be the crucial ingredient in our proof of existence (and multiplicity).

Theorem 13. *Assume f verifies (Hf_1), (Hf_2) and g verifies (Hg) and (1.5). Then*

$$\lim_{\alpha\to\infty} \frac{H(\alpha)}{\alpha} = 1 + g(0). \tag{4.2}$$

If f verifies either (Hf_3) or (1.7) then we also have

$$\lim_{\alpha\to 0+} H(\alpha) = 0 \tag{4.3}$$

and $H(\alpha) > 0$ for small $\alpha > 0$.

Proof. Let us prove first (4.2). Thanks to Theorem 8 we have $u_\alpha \to 0$ point-wise in Ω as $\alpha \to \infty$, so the case where g is decreasing for large values of t (and hence bounded) is immediate thanks to dominated convergence.

Thus we may assume $g(t)$ is positive and increasing for $t \geq t_0$. Introduce the sets

$$\Omega_\alpha^- = \{x \in \Omega :\ u_\alpha(x) < t_0\} \qquad \Omega_\alpha^+ = \{x \in \Omega :\ u_\alpha(x) \geq t_0\},$$

and choose $\alpha_1 > 0$ large enough. For $\alpha \geq \alpha_1$ we get the estimate:

$$|g \circ u_\alpha| \leq \max_{0\leq t\leq t_0} |g(t)| + \chi_{\Omega_\alpha^+} g \circ u_\alpha \leq \max_{0\leq t\leq t_0} |g(t)| + \chi_{\Omega_{\alpha_1}^+} g \circ u_{\alpha_1},$$

where $g \circ u_\alpha(x) = g(u_\alpha(x))$ and the monotonicity of u_α with respect to α has been employed to achieve the estimate. Since $g \circ u_\alpha \to 0$ point-wise in Ω as $\alpha \to \infty$, dominated convergence yields again the desired result.

Finally we prove that either under the additional conditions (Hf_3) or (1.7) we have $H(0) = 0$. Notice that neither of them are necessary when g is bounded for then $\alpha \int_\Omega g(u_\alpha)$ is positive for small α and tends to zero as $\alpha \to 0$.

Thus we may assume g is increasing for $t \geq t_0$. Thanks to either Theorem 9 or Theorem 11 we have that $u_\alpha \to \infty$ uniformly in Ω as $\alpha \to 0$ and $u_\alpha \leq \phi(C_2\sqrt{\alpha}d(x))$ for some positive C_2 if α is sufficiently small. Thus we have $u_\alpha \geq t_0$ for small α. This implies that

$$\int_\Omega g \circ u_\alpha \to \infty,$$

as $\lambda \to 0$. Consequently $H(\alpha) > \alpha$ for small α, and so $\liminf_{\alpha\to 0+} H(\alpha) \geq 0$.

On the other hand, by arguing as in the proof of Lemma 12 we get for small α and δ:

$$\int_{\Omega_\delta} g(u_\alpha) \leq \frac{\kappa_2|\partial\Omega|}{C_2\sqrt{\alpha}} \int_{\phi(C_2\sqrt{\alpha}\delta)}^{\infty} \frac{g(s)}{\sqrt{2F(s)}}\, ds = o(\alpha^{-1/2}), \tag{4.4}$$

as $\alpha \to 0$. Moreover:

$$\int_{\Omega\setminus\Omega_\delta} g(u_\alpha) \leq |\Omega| g(\phi(C_2\sqrt{\alpha}\delta)). \tag{4.5}$$

Now notice that condition (1.5) imposes a growth restriction on g. Namely,

$$g(t)\psi(t) \le \int_t^\infty \frac{g(s)}{\sqrt{F(s)}}ds$$

when $t \ge t_0$, and thus

$$g(t) = o\left(\frac{1}{\psi(t)}\right)$$

as $t \to \infty$. Hence, for small α:

$$g(\phi(C_2\sqrt{\alpha}\delta)) = \frac{o(1)}{\psi(\phi(C_2\sqrt{\alpha}\delta))} = \frac{o(1)}{C_2\sqrt{\alpha}\delta} = o(\alpha^{-1/2}), \tag{4.6}$$

as $\alpha \to 0$. Summing up, from (4.4), (4.5) and (4.6) we obtain that the integral of $g(u_\alpha)$ in Ω is $o(\alpha^{-1/2})$ as $\alpha \to 0$. Hence $H(\alpha) = o(\alpha^{1/2})$ as $\alpha \to 0$. This concludes the proof. □

Now we are ready to perform the proof of our main existence theorems.

Proof of Theorem 2. It has been already pointed out that solutions to (1.1) correspond to solutions of the equation $H(\alpha) = \lambda$. Now $H(\alpha)$ is continuous and verifies $H(0) = 0$. According to 4.2 in Theorem 13 we have $H(\alpha) \to +\infty$ when $\alpha \to \infty$ in case (a), while $H(\alpha) \to -\infty$ in case (b). Thus in case (a) the equation $H(\alpha) = \lambda$ is solvable for every $\lambda > 0$.

In case (b), denote $\lambda^* = \sup_{\mathbb{R}+} H$ and take a value $\alpha^* > 0$ such that $H(\alpha^*) = \lambda^*$. Notice that $H(\alpha) > 0$ for small positive α, so that $\lambda^* > 0$. It is clear that for $\lambda > \lambda^*$, the equation $H(\alpha) = \lambda^*$ has no solution, while for $\lambda \in (0, \lambda^*)$, there are at least two values $\alpha_1 \in (0, \alpha^*)$, $\alpha_2 \in (\alpha^*, \infty)$ such that $H(\alpha_i) = \lambda$ for $i = 1, 2$. This concludes the proof. □

Remark 5. Since $H(\alpha) \to \infty$ as $\alpha \to \infty$ provided $g(0) > -1$, the existence of positive solutions to (1.1) for large values of λ is guaranteed irrespective of conditions (Hf_3) or (1.7).

Proof of Theorem 3. By reasoning as in the proof of Theorem 2 it suffices with showing that

$$\lim_{\alpha \to 0} H(\alpha) = 0,$$

with $H(\alpha) = \alpha(1 + \frac{1}{|\Omega|}\int_\Omega g \circ u_\alpha)$ where, of course, we confine ourselves to the case g not bounded. By employing the same terminology as in the proof of Theorem 11 we have

$$u_\alpha(x) \le u_{\alpha,B}(x - z_i(y)) \qquad x \in B(z_i(y), \rho)$$

where $y \in \partial\Omega$ satisfies $|x - y| = d(x)$, $B(z_i(y), \rho)$ is the inner ball to Ω which is tangent to $\partial\Omega$ at y and ρ is the uniform radius of inner tangent balls to $\partial\Omega$. In addition $u_{\alpha,B}$ stands for the solution to (2.1) in $\Omega = B(0, \rho)$.

Put $u_0 = \min_{B(0,\rho)} u_{\alpha,B}$ and set $u_{B,\alpha}(x) = v(r)$, $r = |x|$ (α being dropped from both v and u_0 for simplicity). Notice that $v(0) = u_0$. Choose $\theta > 1$, then a unique $0 < r_\theta < \rho$ exists such that $v(r_\theta) = \theta u_0$. Thus we obtain that,

$$u_\alpha(x) \le v(\rho - d(x)),$$

for $d(x) < \rho - r_\theta$ while

$$u_\alpha(x) \le v(\theta u_0),$$

if $d(x) \ge \rho - r_\theta$ due to the subharmonicity of u_α.

Now,

$$\int_{\{d(x)<\rho-r_\theta\}} g\circ u_\alpha \le \kappa_2|\partial\Omega| \int_0^{\rho-r_\theta} g(v(\rho-t))\,dt.$$

By a suitable use of (2.7) and the ideas leading to (2.16) we get

$$\sqrt{\frac{\alpha}{N}}(\rho-r) \le \frac{1}{\sqrt{2}}\int_{v(r)}^{\infty}(F-F_0)^{-1/2} \le \left(1-\frac{1}{\theta}\right)^{-1/2}\psi(v(r)),$$

for $r_\theta \le r < \rho$, where $F_0 = F(u_0)$. That is why

$$v(\rho-t) \le \phi(C_\theta\sqrt{\alpha}t)$$

for all $0 < t \le \rho - r_\theta$ and a certain constant C_θ. Thus,

$$\int_{\{d(x)<\rho-r_\theta\}} g\circ u_\alpha \le \frac{C}{\sqrt{\alpha}}\int_{\phi(C_\theta\sqrt{\alpha}(\rho-r_\theta))}^{\infty}\frac{g}{\sqrt{F}},$$

for a certain positive constant C. This implies

$$\int_{\{d(x)<\rho-r_\theta\}} g\circ u_\alpha = o(\frac{1}{\sqrt{\alpha}}) \qquad \alpha\to 0. \tag{4.7}$$

We now claim that, in view of (Hf_2)

$$u_0 = O(\alpha^{-\frac{1}{p-1}}) \qquad \alpha\to 0.$$

Therefore, such estimate together with (1.8) leads to

$$\alpha\int_{\{d(x)>\rho-r_\theta\}} g\circ u_\alpha \le |\Omega|\alpha g(\theta u_0) = o(1) \qquad \alpha\to 0.$$

By combining this with (4.7) we conclude that $H(\alpha)\to 0$.

To show the claim simply observe that from (2.7) we get

$$\sqrt{\frac{\alpha}{N}}\rho \le \frac{1}{\sqrt{2}}\int_{u_0}^{\infty}(F-F_0)^{-1/2},$$

On the other hand, (Hf_2) entails the existence of a positive c such that $f(t)\ge ct^p$ for large t. Hence,

$$\int_{u_0}^{\infty}(F-F_0)^{-1/2} \le K u_0^{-\frac{p-1}{2}},$$

K being a positive constant whose explicit value is here irrelevant. Thus,

$$u_0 \le K\alpha^{-\frac{1}{p-1}} \tag{4.8}$$

for a new constant K and large u_0 (small α). □

5. Proof of Theorem 4

In this section we analyze the particular instance of a power nonlinearity in (1.1). That is, we are taking $f(t) = t^p$, $p>1$. The expression of the solution u_α to (2.1) in this case is explicit in terms of α. Indeed,

$$u_\alpha = \alpha^{-\frac{1}{p-1}}U$$

where $U = u_1$. Since in the present case $\phi(s) = As^{-\gamma}$, where $\gamma = 2/(p-1)$ and $A = (\gamma(\gamma+1))^{\frac{1}{p-1}}$, according to the boundary behavior (2.3) we find that there exist positive constants C_1, C_2 such that $C_1 d(x)^{-\gamma} \le U(x) \le C_2 d(x)^{-\gamma}$ in Ω.

The function H takes the form

$$H(\alpha) = \alpha \left(1 + \frac{1}{|\Omega|} \int_\Omega g(\alpha^{-\frac{1}{p-1}} U) \right).$$

The important point in the proof of Theorem 4 is that condition (1.10) ensures that H is differentiable for $\alpha > 0$, and the limit of $H'(\alpha)$ as $\alpha \to \infty$ can be determined.

Proof of Theorem 4. Notice that condition (1.10) implies that $|g(s)| \le As^{\theta+1} + B$ for $s \ge 0$, and thus (1.5) holds (notice that $\theta + 1 < \frac{p+1}{2} - 1$). Theorem 2 (a) gives then that problem (1.9) has a solution for every $\lambda > 0$.

The differentiability of the integral term in H follows because its expected derivative is well-defined. Indeed, it follows by (1.10) that the function of two variables

$$W(\alpha, x) = g(\alpha^{-\frac{1}{p-1}} U(x))$$

verifies $|W(\alpha, x)| \le Cd(x)^{-\gamma(\theta+1)} + B$ for some positive constant C independent of $\alpha \ge \alpha_0 > 0$. Moreover, the derivative of W with respect to α is continuous and it verifies

$$\left| \frac{\partial W}{\partial \alpha}(\alpha, x) \right| = \frac{\alpha^{-\frac{p}{p-1}}}{p-1} \left| g'(\alpha^{-\frac{1}{p-1}} U) U \right| \le Cd(x)^{-\gamma(\theta+1)}$$

where C does not depend on α if $\alpha \ge \alpha_0 > 0$. Thus we conclude that the integral term $\int_\Omega W(\alpha, x) dx$ is differentiable in α for $\alpha > 0$. Hence the derivative of H becomes

$$H'(\alpha) = 1 + \frac{1}{|\Omega|} \int_\Omega g(\alpha^{-\frac{1}{p-1}} U) - \frac{\alpha^{-\frac{1}{p-1}}}{(p-1)|\Omega|} \int_\Omega g'(\alpha^{-\frac{1}{p-1}} U) U.$$

It follows by the proof of Theorem 13 that the first two terms converge to $1 + g(0)$. We claim that the last term goes to zero as $\alpha \to \infty$. Indeed, notice that $|g'(t)t| \le Ct^{\theta+1}$ if $t > 1$, say, while $|g'(t)t| \le M$ for $0 \le t \le 1$. Then by dominated convergence, since $tg'(t) \to 0$ as $t \to 0$ we obtain

$$\alpha^{-\frac{1}{p-1}} \int_\Omega g'(\alpha^{-\frac{1}{p-1}} U) U \to 0.$$

We have shown thus that

$$\lim_{\alpha \to \infty} H'(\alpha) = 1 + g(0) > 0,$$

so that $H'(\alpha) > 0$ for large α, and hence the equation $H(\alpha) = \lambda$ is uniquely solvable for all large values of λ. Thus (1.9) has a unique positive solution for every large λ.

Finally, notice that condition (1.11) implies that $g(t) - \frac{1}{p-1} g'(t) t \ge 0$ for $t > 0$ and thus $H' > 0$ in the complete range $(0, \infty)$. This gives the uniqueness of the solution to $H(\alpha) = \lambda$ for every $\lambda > 0$. The proof is concluded. □

6. Some further nonlocal boundary blow-up problems

In this final section we give some ideas on how the same methods we have employed along the paper apply to some more general boundary blow-up problems with a nonlocal reaction.

6.1. **A logistic problem.** We are dealing now with a slightly more general version of (1.2), i. e. the problem

$$\begin{cases} \Delta u = \lambda u(\{1 + \frac{1}{|\Omega|}\int_\Omega g(u)\}h(u) - 1) & x \in \Omega \\ u = \infty & x \in \partial\Omega, \end{cases} \tag{6.1}$$

with $\Omega \subset \mathbb{R}^N$ a bounded smooth domain and where for biological significance, $g(u)$ is chosen as a positive and increasing continuous function in $u \geq 0$ with $g(0) = 0$, $g(\infty) = \infty$. As for the function h, we assume that it is increasing and continuous in $u \geq 0$ with $h(0) = 0$, locally Lipschitz in $u > 0$ while certain $\theta > 0$ exists such that

$$\frac{h(u)}{u^\theta}, \tag{6.2}$$

is increasing for large u. Of course, our reference model is the logistic problem (1.2) where $h(u) = u^{p-1}$ with $p > 1$. Notice also that thanks to (6.2), the nonlinearity in (6.1) satisfies the imperative (KO) condition.

The main properties of (6.1) are next stated.

Theorem 14. *Under the preceding conditions on functions g and h, problem* (6.1) *admits for every $\lambda > 0$ a unique classical solution $u_\lambda \in C^2(\Omega)$. Moreover, u_λ continuously varies with λ while*

$$\lim_{\lambda \to 0} u_\lambda = \infty, \tag{6.3}$$

uniformly in Ω. In addition,

$$\lim_{\lambda \to \infty} u_\lambda = h^{-1}\left(\frac{1}{\alpha_\infty}\right), \tag{6.4}$$

uniformly on compacts of Ω, $\alpha = \alpha_\infty$ being the unique solution to

$$\alpha = 1 + g\left(h^{-1}\left(\frac{1}{\alpha}\right)\right). \tag{6.5}$$

The proof of Theorem 14 relies upon the following result.

Theorem 15. *Assume h is chosen as in Theorem 14. Then, the problem*

$$\begin{cases} \Delta u = \lambda u(\alpha h(u) - 1) & x \in \Omega \\ u = \infty & x \in \partial\Omega, \end{cases} \tag{6.6}$$

with λ, α positive parameters, possesses the following properties.

(a) *For each pair $\lambda > 0, \alpha > 0$ it admits a unique solution $u = u_{\lambda,\alpha} \in C^2(\Omega)$ such that*

$$u_{\lambda,\alpha}(x) > h^{-1}\left(\frac{1}{\alpha}\right) \qquad x \in \Omega. \tag{6.7}$$

(b) *$u_{\lambda,\alpha}$ is separately decreasing on λ and α, and it is continuous in (λ, α).*

(c) *For fixed $\alpha > 0$, $u_{\lambda,\alpha} \to \infty$ uniformly in Ω as $\lambda \to 0$ while $u_{\lambda,\alpha} \to h^{-1}(1/\alpha)$ uniformly on compacts of Ω as $\lambda \to \infty$.*

(d) *For fixed $\lambda > 0$, $u_{\lambda,\alpha} \to \infty$ uniformly in Ω as $\alpha \to 0$ while $u_{\lambda,\alpha} \to 0$ uniformly on compacts of Ω as $\alpha \to \infty$.*

Outline of the proof. The existence and uniqueness assertions follow directly from Theorem 6 meanwhile (6.7) is implied by the maximum principle. The continuity and monotonicity of $u_{\lambda,\alpha}$ are a consequence of uniqueness and their proofs follow the lines of the one of Lemma 7.

As for the asymptotic profile of $u_{\lambda,\alpha}$ as either $\lambda \to 0$ or $\alpha \to 0$ observe that $u_{\lambda,\alpha}$ satisfies the lower estimate

$$u_{\lambda,\alpha} > U_\gamma,$$

with $\gamma = \lambda\alpha$, where $u = U_\gamma$ solves the problem

$$\begin{cases} \Delta u = \gamma u h(u) & x \in \Omega \\ u = \infty. \end{cases}$$

By exploiting estimate (2.8), which is solely based on (KO) condition, it is possible to show that $U_\gamma \to \infty$ uniformly in Ω as $\gamma \to 0$. Thus, the same happens to $u_{\lambda,\alpha}$ when either $\lambda \to 0$ or $\alpha \to 0$.

Finally, let us show the limit behavior of $u_{\lambda,\alpha}$ when, say $\alpha \to \infty$. By estimating as in the proofs of Theorems 3 and 11 it suffices with showing the behavior in the case where Ω is the ball $B(0,\mathbb{R})$. By writing in this scenario $u_\alpha(r)$ instead $u_{\lambda,\alpha}$, we find for small $\varepsilon > 0$ (and assuming $N \geq 3$ for simplicity)

$$\frac{u_\alpha(R-\varepsilon)}{\lambda\alpha} \geq \frac{1}{N-2}\int_0^{R-\varepsilon} t\big(1 - (\frac{t}{R-\varepsilon})^{N-2}\big) u_\alpha(h(u_\alpha) - 1/\alpha).$$

Since $u = \lim_{\alpha\to\infty} u_\alpha$ exists then Fatou's lemma implies

$$\int_0^{R-\varepsilon} t\big(1 - (\frac{t}{R-\varepsilon})^{N-2}\big) u h(u) = 0.$$

Thus $u = 0$. Due to the fact that $u_\alpha(r)$ is increasing in r, convergence is uniform in $r \leq R - \varepsilon$. In case of a general smooth bounded domain Ω, the uniform convergence on compacts follows from the subharmonicity of $u_{\lambda,\alpha}$. □

Proof of Theorem 14. As in Section 4 set

$$H(\lambda,\alpha) = 1 + \frac{1}{|\Omega|}\int_\Omega g(u_{\lambda,\alpha}),$$

where $u_{\lambda,\alpha}$ is the solution to (6.6). Then, u solves (6.1) for given $\lambda > 0$ provided $u = u_{\lambda,\alpha}$ where α is a solution to

$$\alpha = H(\lambda,\alpha). \tag{6.8}$$

For fixed λ, $H(\lambda,\cdot)$ is continuous and decreasing with $H(\lambda,\alpha) \to \infty$ as $\alpha \to 0$ while $H(\lambda,\alpha) \to 1$ as $\alpha \to \infty$ (recall $g(0) = 0$). Hence, (6.8) has for each λ a unique solution $\alpha(\lambda)$. As a function of λ, α is continuous and decreasing.

On the other hand $H(\lambda,\cdot) \to \infty$ uniformly in $\alpha \geq 0$ as $\lambda \to 0$ while

$$H(\lambda,\alpha) \to 1 + g(h^{-1}(\frac{1}{\alpha})),$$

uniformly in compacts of $(0,\infty)$ as $\lambda \to \infty$. Thus $\alpha(\lambda) \to \infty$ as $\lambda \to 0$ while $\alpha(\lambda) \to \alpha_\infty$ as $\lambda \to \infty$ where $\alpha = \alpha_\infty$ solves (6.5).

We now set $u_\lambda = u_{\lambda,\alpha(\lambda)}$ and are proving the asymptotic behavior announced in the statement. For the behavior as $\lambda \to 0$ notice that (notation of the proof of Theorem 15 is kept)

$$u_\lambda > U_\gamma$$

with $\gamma = \lambda\alpha(\lambda)$. Thus we are done since

$$\lambda\alpha(\lambda) \to 0 \qquad \lambda \to 0.$$

In fact, assume on the contrary that a sequence $\lambda_n \to 0$ exists such that

$$\lambda_n \alpha(\lambda_n) \geq A > 0.$$

Setting $u_n = u_{\lambda_n}$ and observing that $\alpha(\lambda_n) > B > 0$ for large n we easily conclude that

$$u_n(x) \leq v(x) \qquad x \in \Omega,$$

where v is the unique solution to

$$\begin{cases} \Delta v = Av(h(v) - \frac{1}{B}) & x \in \Omega \\ v = \infty. \end{cases}$$

But then

$$\int_\Omega g(u_n) = O(1),$$

and so $H(\lambda_n, \alpha(\lambda_n)) = O(1)$. This contradicts the fact that $\alpha(\lambda_n) \to \infty$.

With respect to the behavior as $\alpha \to \infty$ observe that

$$u_\lambda = u_{\lambda,\alpha(\lambda)} < u_{\lambda,\alpha_\infty}.$$

Hence,

$$h^{-1}\left(\frac{1}{\alpha(\lambda)}\right) < u_\lambda < u_{\lambda,\alpha_\infty}.$$

This proves the assertion in the statement. □

Remarks 6. (i) In the case of the genuine logistic problem (1.2)

$$u_\lambda = u_{\lambda,\alpha(\lambda)} = \alpha(\lambda)^{-\frac{1}{p-1}} v_\lambda$$

where v_λ is the solution to (6.6) corresponding to $\alpha = 1$ and $h(u) = u^{p-1}$, while α_∞ is the unique solution to

$$\alpha = 1 + g(\alpha^{-\frac{1}{p-1}}),$$

However, no further relevant information is gained in this particular case.

(ii) Consider the finite-valued boundary value problems (1.3) subject to either Dirichlet, Neumann or Robin boundary conditions parameterized by $\mu > 0$ (Section 1) and $\lambda > 0$ fixed. Then, it can be shown by the methods of this section that each one of such problems exhibits for each $\mu > 0$ a unique positive solution $u_\mu^{\mathcal{D}}$, $u_\mu^{\mathcal{N}}$ and $u_\mu^{\mathcal{R}}$, respectively. In all cases, such solutions converge to the solution u_λ to (1.2) as $\mu \to \infty$. To give a brief insight, consider the Dirichlet problem,

$$\begin{cases} \Delta u = \lambda u(\{1 + \frac{1}{|\Omega|}\int_\Omega g(u)\}u^{p-1} - 1) & x \in \Omega \\ u = \mu & x \in \Omega. \end{cases}$$

Its solution takes the form

$$u_\mu^{\mathcal{D}} = \alpha^{-\frac{1}{p-1}} v_\alpha,$$

where $v = v_\alpha$ is the solution to

$$\begin{cases} \Delta v = \lambda v(v^{p-1} - 1) & x \in \Omega \\ v = \alpha^{\frac{1}{p-1}}\mu & x \in \Omega, \end{cases}$$

and α solves the equation

$$\alpha = 1 + \frac{1}{|\Omega|}\int_\Omega g(\alpha^{-\frac{1}{p-1}} v_\alpha).$$

Discussion on the existence, uniqueness and limit profile of $u_\mu^{\mathcal{D}}$ as $\mu \to \infty$ patterns the ones already performed in this section. The analysis corresponding to the remaining boundary conditions mimics, modulo minor features, the above argument and further details are omitted (see [13] for a study of the same affairs in a class of nonlinear diffusion operators complemented with local reaction terms).

6.2. **Exponential nonlinearity. Generalizations.** We next describe how other types of nonlinearities could be considered in (1.1). For instance $f(t) = e^t$, a classical choice in boundary blow-up problems, does not fit the profile in (Hf_1). Nevertheless, the arguments in Section 4 still work after suitable modification. Assume for instance that $g(t)$ is continuous in $\mathbb{R}$, $g(-\infty) = \lim_{t\to-\infty} g(t) > -\infty$ while g is positive and increasing for $t \geq t_0$ (other options being here discarded for brevity). Condition (1.5) reads now

$$\int_A^\infty e^{-s/2} g(s)\ ds < \infty. \tag{6.9}$$

Solution u_α to the auxiliary problem (2.1) can be expressed as $u_\alpha = V - \log\alpha$ where $u_1 = V$ is the solution corresponding to $\alpha = 1$. The asymptotic estimate (cf. [16])

$$V(x) = \log d(x)^{-2} + \log 2 + o(1) \qquad d(x) \to 0+$$

permits showing

$$\lim_{\alpha\to\infty} \frac{1}{|\Omega|}\int_\Omega g(V(x) - \log\alpha)\ dx = 1 + g(-\infty),$$

together with

$$\frac{1}{|\Omega|}\int_\Omega g(V(x) - \log\alpha)\ dx = o(\alpha^{-1/2}) \qquad \alpha \to 0.$$

Therefore, conclusions of Theorem 2 are valid for problem (1.1) with $f(t) = e^t$ and g satisfying the precedent conditions, if one replaces $g(0)$ with $g(-\infty)$ in the statement.

A uniqueness result in the vein of Theorem 4 can be obtained by a more selective choice of g. Suppose, for instance that $g \in C^1(\mathbb{R})$ is increasing, $g(-\infty) > -\infty$ while $\theta > 2$ exists so that

$$g - \theta g' \geq 0 \qquad t \in \mathbb{R}.$$

Then it follows that both g, g' satisfy (6.9) and the function

$$H(\alpha) = \alpha\left(1 + \frac{1}{|\Omega|}\int_\Omega g(V(x) - \log\alpha)\ dx\right)$$

has a negative derivative. Therefore, problem (1.1) has a unique solution u_λ for all $\lambda > 0$ provided $1 + g(-\infty) > 0$.

By employing the methods in this work other variants of problem (1.1) can be dealt with. For instance, the expression $1/(1 + \frac{1}{|\Omega|}\int_\Omega g(u))$ could be replaced by some general continuous function of the average $\frac{1}{|\Omega|}\int_\Omega g(u)$, say

$$G\left(\frac{1}{|\Omega|}\int_\Omega g(u)\right),$$

provided f and g satisfy (Hf_1), (Hf_2) and (Hg) (in addition (1.5) if g is not bounded for large t). Under these conditions, the sign of $G(g(0))$ plays the rôle of the sign of $1 + g(0)$ in Theorem 2. By the same token, provided that f fulfills either (Hf_3) or (1.7), then condition

$$\frac{1}{G(t)} = o(t^2) \qquad t \to \infty,$$

ensures the existence of solutions to such general version of (1.1) for small λ.

6.3. **Further nonlocal problems.** A different class of diffusion problems driven by a reaction term of a nonlocal nature arises when the average of u is replaced by the localized value $u(x_0)$ of u at a single given point $x_0 \in \Omega$ (see [5] where such nonlocal terms are studied in the context of blow–up for parabolic equations). As a simple example, just consider

$$\begin{cases} \Delta u = \dfrac{\lambda}{1+u(x_0)} f(u) & x \in \Omega \\ u = \infty & x \in \partial\Omega, \end{cases} \tag{6.10}$$

$\Omega \subset \mathbb{R}^N$ a bounded smooth domain where f satisfies (Hf_1) and (Hf_2). Then the solution $u = u_\alpha$ to (2.1) defines a solution to (6.10) if and only if

$$\lambda = H(\alpha), \qquad H(\alpha) := \alpha(1 + u_\alpha(x_0)).$$

Since, via Theorem 8, $u_\alpha \to 0$ uniformly on compacts of Ω as $\alpha \to \infty$, it follows $H(\alpha) \sim \alpha$ as $\alpha \to \infty$. Thus, the continuity of H (Lemma 7) implies that (6.10) admits a positive solution u_λ for large λ. Moreover

$$\min_\Omega u_\lambda \to 0,$$

when $\lambda \to \infty$, for every family $\{u_\lambda\}$ of positive solutions to (6.10) corresponding to large values of λ.

On the other hand, and also due to the continuity of H, a positive solution to (6.10) exists for each $\lambda > 0$ provided that $H(\alpha) \to 0$ as $\alpha \to 0$. This amounts to having

$$\alpha u_\alpha(x_0) \to 0 \qquad \text{as} \qquad \alpha \to 0. \tag{6.11}$$

In order to estimate $u_\alpha(x_0)$ observe that

$$w_\alpha(x_1) \le u_\alpha(x_0) \le u_{\alpha,B}(0),$$

where $u = u_{\alpha,B}(x)$ is the solution to (2.1) with $\Omega = B(0,R)$, and $R = d(x_0)$, while $w_\alpha(x_1)$ is the corresponding solution in a strip $a < x_1 < a + 2L$ enclosing Ω in between. By setting $u_0^- = \min w_\alpha = w_\alpha(a+L)$, $u_0^+ = \min u_{\lambda,B} = u_{\alpha,B}(0)$ we get the estimate

$$u_0^- \le u_\alpha(x_0) \le u_0^+,$$

for all α.

Define

$$T(t) = \frac{1}{\sqrt{2}} \int_t^\infty (F - F_t)^{-1/2},$$

$F_t = F(t)$. Then, it follows from (Hf_1) that T decreases in $(0,\infty)$. In fact,

$$T(t) = \frac{1}{\sqrt{2}} \int_0^\infty \left(\int_0^s f(\cdot + t) \right)^{-1/2} ds$$

and since f is increasing then the integrand strictly decreases with t. Moreover, $T(0) = \infty$ while $T(\infty) = 0$. In addition (see relations (2.7))

$$\sqrt{\frac{\alpha}{N}} R \le T(u_0^+) \le T(u_\alpha(x_0)) \le T(u_0^-) \le \sqrt{\alpha} L,$$

for all α. Thus

$$T^{-1}(L\sqrt{\alpha}) \le u_\alpha(x_0) \le T^{-1}(R\sqrt{\alpha}/\sqrt{N}).$$

This means that the complete behavior of $u_\alpha(x_0)$ as $\alpha \to 0$ is dictated by the asymptotic behavior of $T(t)$ as $t \to \infty$. However, to properly estimate the size of T for large t a more precise information on the growth of f at infinity should be required.

Nevertheless, under the sole condition (Hf_2) on f it follows that (see (4.8))

$$\alpha u_\alpha(x_0) \le K\alpha^{\frac{p-2}{p-1}},$$

for small α. Thus (6.11) holds provided the value of p in (Hf_2) satisfies $p > 2$. This is just a simple condition ensuring the existence of a solution to (6.10) for all $\lambda > 0$. Observe in addition that for small λ, positive solutions verify $\min_\Omega u_\lambda \to \infty$ as $\lambda \to 0$.

Finally, some light on the possible responses when $1 < p \le 2$ can be cast by testing the nonlinearity $f(t) = t^p$. Being in this case $u_\alpha(x_0) = \alpha^{-1/(p-1)}U(x_0)$ (Section 5) an explicit value $\lambda^* > 0$ can be found such that, for $1 < p < 2$, no solutions exist for $0 < \lambda < \lambda^*$, meanwhile (6.10) has exactly two solutions for $\lambda > \lambda^*$ and a unique one at $\lambda = \lambda^*$. For $p = 2$ a unique solution exists only when $\lambda \ge \lambda^* > 0$. Roughly speaking, the same response is found for a general f which slowly varies with index $1 < p < 2$ at infinity.

Acknowledgements. Supported by Spanish Ministerio de Ciencia e Innovación and FEDER under grant MTM2008-05824.

References

[1] S. Alarcón, J. García-Melián, A. Quaas, *Keller-Osserman type conditions for some elliptic problems with gradient terms.* to appear in J. Diff. Eqns.

[2] C. Bandle, M. Essén, *On the solutions of quasilinear elliptic problems with boundary blow-up. Partial differential equations of elliptic type* (Cortona, 1992), 93–111, Sympos. Math., XXXV, Cambridge Univ. Press, Cambridge, 1994.

[3] C. Bandle, M. Marcus, *'Large' solutions of semilinear elliptic equations: Existence, uniqueness and asymptotic behaviour*, J. Anal. Math. **58** (1992), 9–24.

[4] L. Bieberbach, *$\Delta u = e^u$ und die automorphen Funktionen*, Math. Ann. **77** (1916), 173–212.

[5] J. Chadam, A. Peirce, H. M. Yin, *The blowup property of solutions to some diffusion equations with localized nonlinear reactions*, J. Math. Anal. Appl. **169** (1992), 313–328.

[6] F. C. Cîrstea, V. Rădulescu, *Asymptotics for the blow-up boundary solution of the logistic equation with absorption.* C. R. Math. Acad. Sci. Paris **336** (2003), no. 3, 231–236.

[7] Y. Du, Z. Guo, *Uniqueness and layer analysis for boundary blow-up solutions*, J. Math. Pures Appl. (9) **83** (2004), no. 6, 739–763.

[8] S. Dumont, L. Dupaigne, O. Goubet, V. Rădulescu, *Back to the Keller-Osserman condition for boundary blow-up solutions.* Adv. Nonlinear Stud. **7** (2007), no. 2, 271–298.

[9] P. Freitas, G. Sweers, *Positivity results for a nonlocal elliptic equation*, Proc. Roy. Soc. Edinbugh 128 A (1998), 697–715.

[10] J. Furter, M. Grinfeld, *Local vs. non–local interactions in population dynamics.* J. Math. Biology **27** (1989), 65–80.

[11] J. García-Melián, *Uniqueness of positive solutions for a boundary blow-up problem.* J. Math. Anal. Appl. **360** (2009), no. 2, 530–536.

[12] J. García-Melián, *Multiplicity of positive solutions to boundary blow up elliptic problems with sign-changing weights.* J. Funct. Anal. **261** (2011), 1775–1798.

[13] J. García-Melián, J. D. Rossi, J. Sabina de Lis, *Layer profiles of solutions to elliptic problems under parameter-dependent boundary conditions*, Zeitschrift fr Analysis und ihre Anwendungen (Journal for Analysis and its Applications) 29 (2010), 1–17.

[14] J. B. Keller, *On solutions of $\Delta u = f(u)$.* Comm. Pure Appl. Math. **10** (1957), 503–510.

[15] A. A. Lacey, *Thermal runaway in a non-local problem modeling Ohmic heating: Part I*, Euro. J. of Appl. Maths. **6** (1995), 127-144.

[16] A. C. Lazer, P. J. McKenna, *Asymptotic behavior of solutions of boundary blowup problems*, Differential Integral Equations **7** (1994), 1001–1019.

[17] C. Loewner, L. Nirenberg, *Partial differential equations invariant under conformal of projective transformations*, Contributions to Analysis (a collection of papers dedicated to Lipman Bers), Academic Press, New York, 1974, p. 245–272.
[18] J. D. Murray, *Mathematical biology. I. An introduction.* Springer-Verlag, New York, 2002.
[19] ———, *Mathematical biology. II. Spatial models and biomedical applications.* Springer-Verlag, New York, 2003.
[20] R. Osserman, *On the inequality* $\Delta u \geq f(u)$. Pacific J. Math. **7** (1957) 1641–1647.
[21] C. V. Pao, *Nonlinear Parabolic and Elliptic Equations.* Plenum Press, New York, 1992.
[22] P. Quittner, P. Souplet, *Superlinear parabolic problems: blow-up, global existence and steady states*, Birkhäuser Advanced Texts / Basler Lehrbücher, 2007.
[23] H. Rademacher, *Einige besondere Probleme partieller Differentialgleichungen*, in: P. Frank, R. von Mises (Eds.) Die Differential- und Integralgleichungen der Mechanik und Physik I, 2nd. ed., Rosenberg, New York (1943), 838–845.
[24] V. Rădulescu, *Singular phenomena in nonlinear elliptic problems: from boundary blow-up solutions to equations with singular nonlinearities*, in "Handbook of Differential Equations: Stationary Partial Differential Equations", Vol. **4** (Michel Chipot, Editor) (2007), 483–591.
[25] Seneta, *Regularly Varying Functions.* Lect. Notes in Maths 508, Springer, New York, 1976.

J. García-Melián and J. C. Sabina de Lis
Departamento de Análisis Matemático, Universidad de La Laguna.
C/. Astrofísico Francisco Sánchez s/n, 38271 – La Laguna, SPAIN
and
Instituto Universitario de Estudios Avanzados (IUdEA) en Física Atómica,
Molecular y Fotónica, Facultad de Física, Universidad de La Laguna
C/. Astrofísico Francisco Sánchez s/n, 38203 – La Laguna , SPAIN
E-mail address: jjgarmel@ull.es, josabina@ull.es