BOUNDARY BEHAVIOR FOR LARGE SOLUTIONS TO ELLIPTIC EQUATIONS WITH SINGULAR WEIGHTS *

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ABSTRACT

In this paper we analyze the boundary behavior of large positive solutions to some semilinear elliptic equations which include a singular weight. The most important point is that the growth of the solutions can be determined in terms of the solution to a one-dimensional first order equation. We also consider the questions of existence and uniqueness of positive solutions.

1. INTRODUCTION

This paper deals with the following semilinear elliptic boundary value problem

$$\begin{cases} \Delta u = a(x)f(u) & \text{in} \quad \Omega \\ u = +\infty & \text{on} \quad \partial\Omega \end{cases} \tag{P}$$

where Ω is a bounded $C^{2,\mu}$ domain of $\mathbb{R}^N$, $a(x)$ is a positive weight function and $f(t)$ is a C^1 function defined in $[0, +\infty)$. The boundary condition is to be understood as $u(x) \to +\infty$ when $d(x) := \text{dist}(x, \partial\Omega) \to 0+$.

Problems related to (P) have been extensively studied. We refer to the pioneering paper [6], and to later contributions both for equations (cf. [1], [2], [3], [4], [5], [7], [8], [10], [11], [14], [15], [17], [21], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33] and [34]) and systems (cf. [9], [12], [13], [18], [19], and [20]). These papers deal mainly with the issues of existence and uniqueness (or multiplicity) of positive solutions, and also with the behavior of the solutions near $\partial\Omega$.

As for problem (P), one of its important features in our present situation is that the weight function $a(x)$ is assumed to be singular near the boundary of Ω. Problems with this characteristic have been studied before in [34], [7] and [8], but all of them considered power or exponential nonlinearities, or were of one-dimensional or radial nature. At the best of our knowledge, no results in general domains for nonlinearities different from the power or

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exponential ones have been treated before, except for [15], which was only dedicated to the existence of solutions.

We are concerned here – for the sake of completeness – with the existence and uniqueness of positive solutions, although our main stress will be put in the boundary estimates. We are following a procedure which is entirely new, though it could seem to be useful only for some nonlinearities. The main point is to analyze the boundary behavior in terms of a one-dimensional first order equation related to (P), instead of doing it with the one-dimensional version of (P). However, this last method – commonly employed in the literature – has always been used with nonlinearities where the dependence on x is not very important. We also would like to remark that the one-dimensional solution permits to obtain estimates which are very simple, and that this solution does not depend on the weight $a(x)$.

We come to state the main hypotheses that are to be assumed in the sequel, both on the nonlinearity $f(t)$ and on the weight $a(x)$. We are supposing that f verifies:

(F1) f is a C^1 increasing function such that $f(0) = 0$.

(F2) There exist $p > 1$, $t_0 \geq 0$ such that $f(t)/t^p$ is increasing if $t \geq t_0$.

and the weight function $a(x)$ is a positive μ-locally Hölder continuous function which also satisfies:

(A1) There exist positive constants C_1, C_2 and γ such that

$$C_1 d(x)^{-\gamma} \leq a(x) \leq C_2 d(x)^{-\gamma} \quad \text{in } \Omega.$$

We begin by stating our result on existence of positive solutions. It turns out, as in [8], that positive solutions to (P) can only exist if $0 < \gamma < 2$.

Theorem 1. *Assume f verifies hypotheses* (F1) *and* (F2)*, and $a \in C^\mu(\Omega)$ verifies* (A1)*. Then problem* (P) *admits a positive solution if and only if* $0 < \gamma < 2$.

In order to deal with the more subtle question of estimates of the solutions near $\partial\Omega$, we have to introduce some more hypotheses. Actually, one could not expect that the solutions were "well-behaved" near $\partial\Omega$ if the weight were not. Thus, it seems reasonable to ask that the weight $a(x)$ has a prescribed behavior near $\partial\Omega$. As in [17], [7] and [8], we are assuming

(A2) There exists $C_0 > 0$ such that

$$a(x) \sim C_0 d(x)^{-\gamma} \quad \text{as } d \to 0+,$$

which obviously implies (A1).

Also, in order to determine the boundary behavior by means of the solution to (1.1), we need that the nonlinear term $f(t)$ has a certain growth at infinity. This is why we are imposing the additional property

(F3) The limit

$$\sigma = \lim_{t \to +\infty} f'(t) \int_t^\infty \frac{d\tau}{f(\tau)}$$

exists and verifies $\sigma > 1$,

which says that, roughly speaking, the nonlinearity is like a power at infinity. We are showing later on that if the limit σ exists and f verifies (F1) and (F2), then we always have $1 \leq \sigma \leq p/(p-1)$, and condition (F3) is not too restrictive. We would like to stress that in some situations with $\sigma = 1$, our estimates are still applicable (see Remark 3).

The key of our estimates is the solution to the one-dimensional problem (observe that it is independent of $a(x)$):

$$\begin{cases} \phi' = -f(\phi), \ t \in (0, +\infty) \\ \phi(0) = +\infty. \end{cases} \tag{1.1}$$

We are now ready to state the result.

Theorem 2. *Assume f verifies hypotheses* (F1), (F2) *and* (F3) *and $a \in C^\mu(\Omega)$ verifies* (A2). *Let u be a positive solution to* (P). *Then*

$$\lim_{d \to 0+} \frac{u}{\phi(Ad^{2-\gamma})} = 1, \tag{1.2}$$

where

$$A = \frac{C_0}{(2-\gamma)((2-\gamma)(\sigma - 1) + 1)}.$$

Finally, we are briefly considering the question of uniqueness of positive solutions to (P). It turns out that for some nonlinearities the monotonicity is not enough to ensure uniqueness (see for instance the problem considered in [7]). Thus we are restricting to nonlinearities which also satisfy:

(F4) The function $f(t)/t$ is nondecreasing.

In this respect, we have the following result.

Theorem 3. *Assume f verifies hypotheses* (F1), (F2), (F3) *and* (F4), *and $a \in C^\mu(\Omega)$ verifies* (A2). *Then problem* (P) *has a unique positive solution, which in addition has the asymptotic behavior given by (1.2).*

The contents of the paper are organized as follows: in Section 2 we state some already known results on particular cases of problem (P). Section 3 deals with the proof of Theorem 1, and Section 4 is devoted to prove Theorems 2 and 3.

2. Preliminaries

In this section we are quoting without proof some results which are to be used in subsequent sections. We begin by considering problem (P) when the weight $a(x)$ is smooth up to the boundary (cf. [4]).

Theorem 4. *Assume $a \in C^\mu(\overline{\Omega})$ and f verifies hypotheses* (F1) *and* (F2). *Then problem* (P) *has a positive solution. If f verifies in addition* (F4), *then the positive solution is unique.*

For the very special case when $f(t) = t^p$ or $f(t) = e^t$, problem (P) when a is singular near $\partial\Omega$, verifying (A1) is well understood. See [8] for the proof of the next result.

Theorem 5. *Assume $a \in C^\mu(\Omega)$ verifies* (A1)*. Then the problem*

$$\begin{cases} \Delta u = a(x)u^p & in \quad \Omega \\ u = +\infty & on \quad \partial\Omega \end{cases}$$

with $p > 1$ admits a unique positive solution $U \in C^{2,\mu}(\Omega)$. If, moreover, a verifies (A2)*, then:*

$$\lim_{d \to 0+} d(x)^\alpha U(x) = \left(\frac{C_0}{\alpha(\alpha+1)} \right)^{\frac{1}{p-1}} \tag{2.1}$$

where $\alpha = (2-\gamma)/(p-1)$. Similarly, if a verifies (A1)*, the problem*

$$\begin{cases} \Delta u = a(x)e^u & in \quad \Omega \\ u = +\infty & on \quad \partial\Omega \end{cases}$$

also admits a unique solution $V \in C^{2,\mu}(\Omega)$, which verifies

$$\lim_{d \to 0+} V(x) + (2-\gamma)\log d = \log\left(\frac{2-\gamma}{C_0} \right) \tag{2.2}$$

if a satisfies (A2)*.*

We finally remark that the following comparison principle (an easy adaptation of Theorem 2.2 in [11]) will be used throughout: if $u, v \in C^2(\Omega)$ verify $\Delta u - a(x)f(u) \geq \Delta v - a(x)f(v)$, with f satisfying (F1), together with $\limsup_{d \to 0+} u/v \leq 1$, then $u \leq v$ in Ω. It will be frequently the case that u remains bounded near $\partial\Omega$, while $v = +\infty$ on $\partial\Omega$.

3. Existence of positive solutions

We are devoting this section to prove Theorem 1. The proof is inspired in the corresponding ones for the potential and exponential cases in [8].

Proof of Theorem 1. Let us first show that there are no positive solutions to (P) when $\gamma \geq 2$. For $x \in \Omega$, let

$$v(y) = A(x)u(x + d(x)y), \ y \in B_{1/2}(0),$$

where $A(x)$ is to be determined and $B_{1/2}(0)$ stands for the ball of center 0 and radius 1/2. Thanks to hypothesis (A1) on the weight $a(x)$, it follows that:

$$\Delta v \geq CA(x)d(x)^{2-\gamma} f(A(x)^{-1}v) \quad \text{in } B_{1/2}(0),$$

where the letter C will denote a positive constant, not necessarily the same everywhere. Now let x close to $\partial\Omega$ so that $u(x) \geq t_0$ (t_0 given by condition (F2) on f). Then

$$\Delta v \geq CA(x)^{1-p}d(x)^{2-\gamma}v^p \quad \text{in } B_{1/2}(0).$$

Setting $A(x) = (Cd(x)^{2-\gamma})^{\frac{1}{p-1}}$, we obtain that $\Delta v \geq v^p$ in $B_{1/2}(0)$, and it follows that $v \leq V$, where V is the unique positive solution to

$$\begin{cases} \Delta V = V^p & \text{in} \quad B_{1/2}(0) \\ V = +\infty & \text{on} \quad \partial B_{1/2}(0) \end{cases}$$

(Theorem 5). Then $u(x) = A(x)^{-1}v(0) \leq A(x)^{-1}V(0) = Cd(x)^{\frac{\gamma-2}{p-1}}$ if x is close to $\partial\Omega$. In particular, since $\gamma \geq 2$, u is bounded near $\partial\Omega$, which is impossible.

We now prove the existence of a positive solution to (P) when $0 < \gamma < 2$. We are showing that for every positive integer n, the problem

$$\begin{cases} \Delta u = a(x)f(u) & \text{in} \quad \Omega \\ u = n & \text{on} \quad \partial\Omega \end{cases} \tag{3.1}$$

admits a unique positive solution u_n. For this sake, we truncate the weight $a(x)$. Let $\varphi \in C^1(\mathbb{R})$ be a cut-off function such that $0 \leq \varphi \leq 1$, $\varphi(t) = 0$ if $0 \leq t \leq 1$ and $\varphi(t) = 1$ if $t \geq 2$, and define $a_k(x) = a(x)\varphi(kd(x))$. It follows that $a(x) \in C^\mu(\overline{\Omega})$. Consider the problem

$$\begin{cases} \Delta v = a_k(x)f(v+n) & \text{in} \quad \Omega \\ v = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.2}$$

Since $\underline{v} = -n$ and $\overline{v} = 0$ are order sub and supersolutions, respectively, we get a solution $v_{n,k}$ to (3.2), which, thanks to the monotonicity of f and the maximum principle, is unique.

We now apply Lemma 4 in [8] (see Lemma 4.9 and Problem 4.6 in [22]), and use that the solutions $v_{n,k}$ are uniformly bounded to obtain that

$$\sup d(x)^{\gamma-2}|v_{n,k}(x)| \leq C, \tag{3.3}$$

where C does not depend on k. Estimate (3.3) gives interior uniform bounds, and it is standard to conclude that (for a subsequence) we have $v_{n,k} \to v_n$ in $C^2_{\text{loc}}(\Omega)$. Thanks to (3.3), it follows that $v_n = 0$ on $\partial\Omega$, and it solves $\Delta v = a(x)f(v+n)$ in Ω. Letting $u_n = v_n + n$, we obtain a solution to (3.1), which, by the maximum principle, is unique. Moreover, the uniqueness of u_n implies that it increases in n.

It remains to obtain local uniform bounds on the solutions u_n. This can be easily done, since for every $\Omega' \subset\subset \Omega$ the problem

$$\begin{cases} \Delta u = a(x)f(u) & \text{in} \quad \Omega' \\ u = +\infty & \text{on} \quad \partial\Omega' \end{cases}$$

admits a positive solution U (cf. Theorem 4), and by comparison $u_n \leq U$. Thus we can obtain that $u_n \to u$ in $C^2_{\text{loc}}(\Omega)$, where u is a positive solution to (P). This proves Theorem 1. □

4. ESTIMATES NEAR THE BOUNDARY. UNIQUENESS

In this section we are proving Theorem 2. As remarked in the introduction, the main point is that the behavior of the solutions can be characterized in terms of a one-dimensional first-order equation.

Thus we consider the solution to the problem

$$\begin{cases} \phi' = -f(\phi), \ t \in (0, +\infty) \\ \phi(0) = +\infty. \end{cases} \tag{4.1}$$

It follows by direct integration that ϕ is the inverse function of

$$\psi(s) = \int_s^{+\infty} \frac{d\tau}{f(\tau)}, \tag{4.2}$$

the integral being convergent thanks to hypothesis (F2) on f. For instance, with $f(t) = t^p$, $p > 1$ (resp. $f(t) = e^t$), we have $\phi(t) = ((p-1)t)^{-\frac{1}{p-1}}$ (resp. $\phi(t) = -\log t$). Before proceeding further, we need a property of the solution ϕ which will be important when proving Theorem 2 (cf. Lemma 2.1 in [4]).

Lemma 6. *Let ϕ be the unique solution to (4.1), and $A > 0$. Then*

$$\limsup_{\varepsilon\to 0+} \limsup_{t\to 0+} \frac{\phi((A-\varepsilon)t)}{\phi(At)} \le 1.$$

Proof. For t, ε sufficiently small, we have that $\phi((A-\varepsilon)t) \ge \phi(At) \ge t_0$. Thus, by hypothesis (F2) on f,

$$\frac{\phi'((A-\varepsilon)t)}{\phi((A-\varepsilon)t)^p} = -\frac{1}{\phi((A-\varepsilon)t)^p} f(\phi((A-\varepsilon)t)) \le -\frac{1}{\phi(At)^p} f(\phi(At)) = \frac{\phi'(At)}{\phi(At)^p}.$$

We now integrate between 0 and t, to obtain that

$$\left(\frac{\phi((A-\varepsilon)t)}{\phi(At)}\right)^{p-1} \le \frac{A}{A-\varepsilon},$$

and the lemma is proved by letting $t \to 0+$ and then $\varepsilon \to 0+$. □

Remark 1. It follows similarly that

$$\liminf_{\varepsilon\to 0+} \liminf_{t\to 0+} \frac{\phi((A+\varepsilon)t)}{\phi(At)} \ge 1.$$

To obtain the estimates in Theorem 2, we perform the change of variable $v = \psi(u)$ in problem (P), to arrive at

$$\begin{cases} -\Delta v + I(v)\dfrac{|\nabla v|^2}{v} = a(x) & \text{in} \quad \Omega \\ v = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where

$$I(t) = -\frac{t\phi''(t)}{\phi'(t)} = tf'(\phi(t)).$$

Hypothesis (F3) in the introduction gives important information on the function $I(t)$. Indeed, by the definition of ψ as the inverse function of ϕ and thanks to (4.2):

$$\lim_{t\to 0+} I(t) = \lim_{t\to 0+} tf'(\phi(t)) = \lim_{s\to+\infty} \psi(s)f'(s) = \lim_{s\to+\infty} f'(s) \int_s^{+\infty} \frac{d\tau}{f(\tau)} = \sigma.$$

Although we are requiring our function f to have $\sigma = 1$ in (F3), we are showing next that this condition is not too restrictive once we assume (F1) and (F2).

Lemma 7. *Assume f verifies* (F1) *and* (F2)*, and that the limit*

$$\sigma = \lim_{t\to+\infty} f'(t) \int_t^{+\infty} \frac{d\tau}{f(\tau)} \tag{4.3}$$

exists. Then

$$1 \le \sigma \le \frac{p}{p-1}. \tag{4.4}$$

Proof. Denote by $J(t)$ the function in the right-hand side of (4.3). Fix $t_1 > 0$ and integrate $J(t)$ from t_1 to some $t > t_1$. If we integrate by parts, we obtain

$$\begin{aligned}\int_{t_1}^t J(s)ds &= \int_{t_1}^t f'(s) \int_s^{+\infty} \frac{d\tau}{f(\tau)} ds \\ &= f(t) \int_t^{+\infty} \frac{d\tau}{f(\tau)} - f(t_1) \int_{t_1}^{+\infty} \frac{d\tau}{f(\tau)} + t - t_1,\end{aligned}$$

so that

$$\frac{1}{t-t_1} \int_{t_1}^t J(s)ds = \frac{f(t)}{t-t_1} \int_t^{+\infty} \frac{d\tau}{f(\tau)} + 1 + o(1). \tag{4.5}$$

Now $f(t) > 0$, and the left-hand side of (4.5) converges to σ, so we obtain that $\sigma \ge 1$.

On the other hand, let $f(t) = t^p g(t)$, where g is increasing for $t \ge t_0$. Then, according to (4.5), we obtain for $t \ge t_0$:

$$\begin{aligned}\frac{1}{t-t_1} \int_{t_1}^t J(s)\ ds &= \frac{t^p g(t)}{t-t_1} \int_t^{+\infty} \frac{d\tau}{\tau^p g(\tau)} + 1 + o(1) \\ &\le \frac{t^p}{t-t_1} \int_t^{+\infty} \frac{d\tau}{\tau^p} + 1 + o(1) = \frac{1}{p-1} \frac{t}{t-t_1} + 1 + o(1).\end{aligned}$$

Letting $t \to +\infty$ we obtain the second inequality in (4.4). This proves the lemma. □

Remark 2. It is worth mentioning that the two limit cases $\sigma = 1$ and $\sigma = p/(p-1)$ are obtained when $f(t) = Ae^{kt}$ and $f(t) = Bt^p$, respectively (the first of these does not verify $f(0) = 0$, but this is of no importance for most of the results). In these cases the function J is indeed constant. On the other hand, it can be proved that these are the only functions which make J constant.

For the sake of completeness, we are providing next with a family of g's for which $\sigma > 1$.

Lemma 8. *Assume $f(t) = t^p g(t)$ for some $p > 1$, where $g(t)$ is increasing, C^2 for large t and $\lim_{t\to+\infty} g(t) = +\infty$. Let*

$$\alpha = \lim_{t\to+\infty} \frac{tg''(t)}{g'(t)}\ ,$$

which we are assuming to exist. Then if $tg'(t)$ is bounded, we have $\sigma = p/(p-1) > 1$. If, on the contrary, $\lim_{t\to+\infty} tg'(t) = +\infty$, then $\sigma = 1 + 1/(\alpha + p)$, so that when $\alpha > -p$ we have $\sigma > 1$.

Proof. By l'Hôpital rule, we have that

$$\sigma = \lim_{t\to+\infty} \frac{f'(t)^2}{f(t)f''(t)},$$

provided this last limit exists. Using the expression for f, we arrive at

$$\begin{aligned}\sigma &= \lim_{t\to+\infty} \frac{p^2 + 2p\dfrac{tg'(t)}{g(t)} + \left(\dfrac{tg'(t)}{g(t)}\right)^2}{p(p-1) + 2p\dfrac{tg'(t)}{g(t)} + \dfrac{t^2g''(t)}{g(t)}} \\ &= \lim_{t\to+\infty} \frac{p^2+2p\beta+\beta^2}{p(p-1)+2p\beta+\alpha\beta}\end{aligned} \tag{4.6}$$

where $\beta = \lim_{t\to+\infty} tg'(t)/g(t)$, provided this limit exists. But this is precisely the case in our hypotheses. Indeed, if $tg'(t)$ is bounded we directly obtain $\beta = 0$, and so $\sigma = p/(p-1)$ by (4.6). In case $tg'(t) \to +\infty$, we have again by l'Hôpital

$$\lim_{t\to+\infty} \frac{tg'(t)}{g(t)} = \lim_{t\to+\infty} \frac{g'(t)+tg''(t)}{g'(t)} = 1+\alpha,$$

and thus (4.6) gives that $\sigma = 1 + 1/(\alpha + p)$. This concludes the proof. □

Let us now finally proceed to the proof of Theorem 2.

Proof of Theorem 2. For $\varepsilon > 0$ fixed, there exists $\delta > 0$ such that $I(v) \le \sigma + \varepsilon$ in $\Omega_\delta = \{x \in \Omega : \ d(x) < \delta\}$. Then

$$-\Delta v + (\sigma+\varepsilon)\frac{|\nabla v|^2}{v} \ge a(x) \quad \text{in } \Omega_\delta.$$

Setting $q_\varepsilon = \sigma + \varepsilon - 1$ and $w_\varepsilon = v^{-q_\varepsilon}$, it is not hard to see that w_ε verifies

$$\Delta w_\varepsilon \ge q_\varepsilon a(x) w_\varepsilon^{1+1/q_\varepsilon} \quad \text{in } \Omega_\delta$$

(observe that $q_\varepsilon > 0$). Let U_ε be the unique solution (furnished by Theorem 5) to $\Delta U = a(x)U^{1+1/q_\varepsilon}$ in Ω, $U_\varepsilon = +\infty$ on $\partial\Omega$. For any $K > 0$ we have $\Delta(q_\varepsilon^{-1/q_\varepsilon} U_\varepsilon + K) \le q_\varepsilon a(x)(q_\varepsilon^{-q_\varepsilon} U_\varepsilon + K)^{1+1/q_\varepsilon}$. Thus, taking K so that $q_\varepsilon^{-q_\varepsilon} U_\varepsilon + K \ge w_\varepsilon$ on $d(x) = \delta$, we obtain by means of the comparison principle that $w_\varepsilon \le q_\varepsilon^{-q_\varepsilon} U_\varepsilon + K$ in Ω_δ, that is,

$$v \ge (q_\varepsilon^{-q_\varepsilon} U_\varepsilon + K)^{-1/q_\varepsilon} \quad \text{in } \Omega_\delta.$$

Thanks to (2.1) in Theorem 5, we obtain that

$$\liminf_{d\to0} d^{\gamma-2} v \ge \frac{C_0}{(2-\gamma)(\alpha_\varepsilon+1)},$$

where $\alpha_\varepsilon = (2-\gamma)q_\varepsilon$. Letting $\varepsilon \to 0+$ and recalling the definition of v, we arrive at

$$\liminf_{d\to0} d^{\gamma-2} \psi(u) \ge \frac{C_0}{(2-\gamma)((2-\gamma)(\sigma-1)+1)} = A.$$

Taking again a small $\varepsilon > 0$, we obtain $\psi(u) \geq (A-\varepsilon)d(x)^{2-\gamma}$ for d small enough. This implies

$$\frac{u}{\phi(Ad^{2-\gamma})} \leq \frac{\phi((A-\varepsilon)d^{2-\gamma})}{\phi(Ad^{2-\gamma})}.$$

Letting $d \to 0$, then $\varepsilon \to 0$ and using Lemma 6 we arrive at

$$\limsup_{d\to 0+} \frac{u}{\phi(Ad^{2-\gamma})} \leq 1. \tag{4.7}$$

The complementary limit is achieved in the same way. It is only worthy of mention that $\sigma - \varepsilon > 1$ for small ε and everything goes as before. □

Remark 3. Observe that estimate (4.7) can also be obtained even if $\sigma = 1$. In this case, however, the inferior estimate can not be dealt with in the same way. But it still works if we assume that $I(t) \geq 1$ for small t. In that case, let φ be the unique solution to $-\Delta\varphi = 1$ in Ω with $\varphi_{|\partial\Omega} = 0$, and set $w = -\log v + K\varphi$, for some positive K. Then

$$\Delta w \leq a(x)e^{w} \quad \text{in } \Omega_\delta.$$

If V denotes the unique positive solution to $\Delta V = a(x)e^V$ in Ω with $V = +\infty$ on $\partial\Omega$ (cf. Theorem 5) and K is chosen so that $w \geq V$ on $d = \delta$, we obtain $w \geq V$ in Ω_δ, and thanks to (2.2) this leads to

$$\limsup_{d\to 0+} d^{\gamma-2} v \leq A.$$

The proof finishes as before.

The estimate provided by Theorem 2 leads to uniqueness of positive solutions, when combined with some additional monotonicity property of the nonlinearity.

Proof of Theorem 3. We adapt the proof in [17]. If u, v are positive solutions to (P), then it follows from Theorem 2 that

$$\lim_{d\to 0+} \frac{u}{v} = 1. \tag{4.8}$$

Thanks to (4.8), for every $\varepsilon > 0$ there exists $\delta > 0$ such that $(1-\varepsilon)v \leq u \leq (1+\varepsilon)v$ in $\Omega_\delta = \{x \in \Omega : \ d(x) < \delta\}$. Notice that, since $f(t)/t$ is increasing, the functions $\underline{v} = (1-\varepsilon)v$ and $\overline{v} = (1+\varepsilon)v$ are sub and supersolution, respectively, to

$$\begin{cases} \Delta w = a(x)f(w) & \text{in} \quad \Omega \setminus \Omega_\delta \\ w = u & \text{on} \quad \partial\Omega \setminus \Omega_\delta. \end{cases} \tag{4.9}$$

Problem (4.9) has a unique solution w which coincides with u. Thus it follows that $(1-\varepsilon)v \leq u \leq (1+\varepsilon)v$ in $\Omega \setminus \Omega_\delta$ and hence in Ω. Letting $\varepsilon \to 0+$ gives $u = v$, and this proves the uniqueness. □

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