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Nonlinear Analysis 53 (2003) 1063 – 1077

Nonlinear Analysis

www.elsevier.com/locate/na

Multiplicity of solutions to a degenerate diffusion problem☆

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Received 18 December 2001; accepted 3 October 2002

Abstract

In this paper it is shown that the Dirichlet problem $-\Delta_p u = \lambda f(u)$, $u_{\partial B} = 0$ in a ball $B \subset \mathbb{R}^N$, loses the property of uniqueness of positive solutions u under the sole condition $\max_B u \longrightarrow \bar{u}_0$ as $\lambda \to +\infty$, with $\bar{u}_0 > 0$ certain prefixed zero of f, provided $p > k+1$, k being the order of $\bar{u}_0$, what is in contrast with the so-called "nondegenerate case" $p \leqslant k+1$ where such hypothesis implies uniqueness. This also proves that a slightly stronger convergence condition for uniqueness introduced by the authors in a previous work cannot be relaxed.

MSC: primary 35J25; secondary 35B40

Keywords: Degenerate diffusion; Phase plane analysis; Dead cores; Multiplicity of solutions

1. Introduction

The present work continues previous ones of the authors [6,10] on the uniqueness of positive solutions to the problem

$$\begin{cases} -\Delta_p u = \lambda f(u), & x \in B, \\ u_{|\partial B} = 0, \end{cases} \tag{P}$$

when $\lambda > 0$ is large. In (P), Δ_p stands for the p-Laplacian which is distributionally defined in $W_0^{1,p}(B)$, $p > 1$, as $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, where B is the N-dimensional ball $B = \{x \in \mathbb{R}^N : |x| < R\}$.

☆ Supported by MCYT & FEDER, project BFM2001-3894 (Spain).

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0362-546X/03/$ - see front matter
doi:10.1016/S0362-546X(03)00042-7

The uniqueness of positive solutions to (P) and large $\lambda > 0$ for the Laplacian operator Δ ($p=2$) in a bounded domain $\Omega \subset \mathbb{R}^N$ is a topic considerably studied in the literature of nonlinear boundary value problems. Especially, the sublinear case where solutions under consideration satisfy $0 < u(x) \leqslant \bar{u}_0, f(\bar{u}_0) = 0$, and uniqueness is restricted to those solutions fulfilling

$$\sup_{\Omega} u \to \bar{u}_0 \tag{1}$$

as $\lambda \to +\infty$. See for instance [1,2,4,14,5,13] and the corresponding references quoted there. To be more precise, in the case $N \geqslant 2$ the energy condition

$$F(u) < F(\bar{u}_0), \quad 0 \leqslant u < \bar{u}_0, \tag{I}$$

$F(u) = \int_0^u f$, is necessary and sufficient for the plain existence of solutions u satisfying (1) as $\lambda \to \infty$ (see, for instance [13] for the case $p=2$ and [6,10] for general $p > 1$). Such inequality is not required to be strict in the one-dimensional case $N=1$. However, since in that special case all possible kind of solutions satisfying (1) have been studied and classified in [8], we will only be concerned here with $N \geqslant 2$.

On the other hand, it has been recently shown in [6] that (P) admits a unique positive solution satisfying (1) if f is C^1 and $u = \bar{u}_0$ is a zero of f with order $k \geqslant p-1 \geqslant 1$. The complementary situation where $\bar{u}_0$ is a degenerate zero, i.e. $k < p-1$, has been also considered in [10] both in balls B and annuli $A = \{x \in \mathbb{R}^N : 0 < r < |x| < R\}$. In the latter case uniqueness holds also true for solutions satisfying (1), while in the one-dimensional case $N = 1$ the stronger condition (I) provides in addition the uniqueness of such solutions in balls B (i.e., intervals). However, in the case of balls B and $N \geqslant 2$, if $\bar{u}_0$ is a degenerate zero the uniqueness is obtained in [10] under the stronger approximation requirement

$$u(x) \to \bar{u}_0 \quad \text{uniformly on compacts of } B \tag{2}$$

as $\lambda \to +\infty$. It should be also remarked that in the regime $k < p-1$ the convergence condition (1) is attained both in balls B and annuli A under the degenerate form that solutions u develop flat cores $\{u = \bar{u}_0\}$ which progressively fill B (respectively A) when $\lambda \to +\infty$. In the particular case where f is a logistic-type nonlinearity this kind of behaviour was treated in [11,7], on general domains $\Omega \subset \mathbb{R}^N$.

The main objective of this work is to show, by studying a special kind of nonlinearities f of interest in its own right, that the stronger condition (2) for the uniqueness of positive solutions in the degenerate case and the ball B ($N \geqslant 2$), cannot be weakened to the convergence condition (1). Specifically, it will be proved that for fixed $0 < \bar{u}_1 < \bar{u}_0$ a value $0 < \bar{u}_1 < u_\sigma < \bar{u}_0$ exists such that the bistable-like problem,

$$\begin{cases} -\Delta_p u = -\lambda(u - \bar{u}_1)(u - u_\sigma)(u - \bar{u}_0), & x \in B, \\ u_{|\partial B} = 0, \end{cases}$$

admits, for large λ, two solutions u_λ, $\tilde{u}_\lambda$ such that u_λ develops a single flat core $\{u_\lambda = \bar{u}_0\} \to B$ as $\lambda \to +\infty$, while $\tilde{u}_\lambda$ possesses a multiple core $\{\tilde{u}_\lambda = \bar{u}_0\} \cup \{\tilde{u}_\lambda = \bar{u}_1\}$ satisfying $\{\tilde{u}_\lambda = \bar{u}_0\} \to \{0\}$, $\{\tilde{u}_\lambda = \bar{u}_1\} \to B \setminus \{0\}$ as $\lambda \to +\infty$. Observe that the second solution $\tilde{u}_\lambda$ fulfills condition (1) but not (2) proving that the latter cannot be relaxed in order to get uniqueness (u_λ satisfies (1)). An early draft of these results was presented in [9].

The work is organized as follows. Section 2 describes in detail the properties of local existence and uniqueness, global continuation and continuous dependence on initial data and parameters for the natural Cauchy problem associated to radial solutions to the equation $-\Delta_p u = f(u)$. A special emphasis is put in the kind of solutions which equal a zero $\bar{u}_0$ of f in a whole interval. Section 3 contains the main result (Theorem 6) where the existence of multiple solutions to (P) for large λ is shown. As an auxiliary interesting step in the proof of that result, a connection in finite time between zeros of $f(u) = -(u-\bar{u}_1)(u-u_\sigma)(u-\bar{u}_0)$ by an orbit of the radial equation is obtained (Theorem 5). These features are further extended to cover nonlinearities with any prefixed number of zeros.

2. Preliminary results. Initial value problems

It will be assumed that $f(u)$ is locally Lipschitz in $\bar{\mathbb{R}}^+ = [0, +\infty)$. By a weak solution to (P) it will be meant a function $u \in W_0^{1,p}(B) \cap L^\infty(B)$ such that

$$\int_B |\nabla u|^{p-2} \nabla u \nabla \varphi \, \mathrm{d}x = \lambda \int_B f(u) \varphi \, \mathrm{d}x,$$

for each $\varphi \in W_0^{1,p}(B)$. The condition $u \in L^\infty(B)$, natural in the context of the present work, provides an extra amount of smoothness on weak solutions. In fact, $u \in C^{1,\beta}(\bar{B})$ for certain $0 < \beta < 1$ (see for instance [15,16]). If, in addition, u is radially symmetric then u will coincide almost everywhere with $u(r) \in C^1[0,R]$, $r = |x|$, which after a careful checking (see [10]) is shown to satisfy that $r^{N-1}|u'|^{p-2}u' \in C^1[0,R]$. Moreover, u will solve the equation

$$-(r^{N-1}|u'|^{p-2}u')' = r^{N-1}\lambda f(u), \quad 0 \leqslant r \leqslant R \tag{E}$$

together with $u'(0) = u(R) = 0$. Observe that the existence of a second derivative is not required at all. Such a derivative in general fails to exist where u' vanishes (see precise estimates in Remarks 2.1(b) and (c)).

An important property of positive radial solutions u to (P) is that $u(r)$ is nonincreasing in $r \in [0,R]$ and thus $u(0) = \max_B u$. To show this assume that $u \in C^1(I)$, $r^{N-1}|u'|^{p-2}u' \in C^1(I)$ solves (E) in an interval $I = [a,b] \subset \bar{\mathbb{R}}^+$ ($u'(0) = 0$ if $0 \in I$). Then the combination:

$$H(u,u') = |u'|^p + p'\lambda F(u), \quad F(u) = \int_0^u f(s) \, \mathrm{d}s$$

is C^1 in r. In fact, $|u'|^p = |w|^{p'}$ with $w = |u'|^{p-2}u'$ and $p' = p/(p-1)$. Since $p' > 1$ and w is C^1 in I then $|u'|^p$ is C^1. Moreover, from (E):

$$\frac{\mathrm{d}}{\mathrm{d}r}(H(u,u')) = -p' \frac{N-1}{r} |u'|^p, \quad r \in I \backslash \{0\}$$

and so H does not increase on solutions to (E). We next show that $u' \leqslant 0$ in I provided $u'(a) \leqslant 0$ together with $u(r) \geqslant u(b)$ in I (positive solutions to (P) satisfy this in $I=[0,R]$). On the contrary, $a \leqslant r_0 < r_1 \leqslant b$ exist such that $u'(r_0)=0$, $u'>0$ in $(r_0,r_1]$. Another $r_2>r_1$ must exist such that $u(r_2)=u(r_0)=u_0$. Thus,

$$H(u(r_0),u'(r_0)) = p'\lambda F(u_0) > H(u(r_2),u'(r_2)) = |u'(r_2)|^p + p'\lambda F(u_0),$$

what is not possible. Thus $u' \leqslant 0$ in I as desired.

To summarize, all these previous remarks are collected in the next statement.

Lemma 1. *Assume that $f(u)$ is locally Lipschitz in $\bar{\mathbb{R}}^+$ and let $u \in W_0^{1,p}(B) \cap L^\infty(B)$ be a weak positive radial solution to (P). Then u almost everywhere coincides with a nonincreasing and C^1 function $u(r)$, $r=|x|$, $0 \leqslant r \leqslant R$, such that furthermore $r^{N-1}|u'|^{p-2}u'$ is also C^1 in $[0,R]$.*

2.1. Local solutions

Let us assume now that u is a positive solution to (P) whose derivative vanishes at some $\tilde{r} \in [0,R)$ (for instance $\tilde{r}=0$, $u(0)=\bar{u}$, or either $\tilde{r}$ lying in the upper boundary of a flat core $\{u=\bar{u}\}$). Then, scaling λ in (E), $u(r)$ defines a solution of

$$\begin{cases} -(r^{N-1}|u'|^{p-2}u')' = r^{N-1}f(u), & r \geqslant d, \\ u(d)=\bar{u}, \quad u'(d)=0, \end{cases} \tag{C}$$

where $d=\lambda^{1/p}\tilde{r} \geqslant 0$ will be observed as a parameter. Problem (C) will play a crucial rôle in the present work. Under the regularity exhibited by radial solutions to (P), problem (C) is equivalent to

$$\begin{cases} u' = |v|^{p'-2}v, \quad u(d)=u_0, \\ v' = -\dfrac{N-1}{r}v - f(u), \quad v(d)=v_0, \end{cases} \tag{S}$$

where $v=|u'|^{p-2}u'$, $p'=p/(p-1)$, and the choice $u_0=\bar{u}$, $v_0=0$ is performed. In the first instance we will be interested in local C^1 solutions on some interval $(d-\delta,d+\delta)$, $\delta>0$ small. If $p>2$, as will be the case later, then $0<p'-1<1$ and (S) falls out the scope of the standard ode's theory [3] if $v_0=0$. In addition, (S) is singular at $r=0$ if $d=0$. This is the reason why the existence and uniqueness affairs for (S) or (C) have not been clearly stated until quite recently (see for instance [12,10]). Especially, critical is the case $f(\bar{u})=0$ in (C) where one is interested in solutions different from the trivial $u=\bar{u}$ and *nonuniqueness* is needed.

The next result summarizes the main features on existence and uniqueness of nontrivial solutions to (C) obtained in [10] and required for the course of the present work. There, a solution u to (C) in an interval $I \subset \bar{\mathbb{R}}^+$, $d \in I$, will mean a C^1 function in I with $r^{N-1}|u'|^{p-2}u'$ also C^1 in I. In the case $\bar{u}$ a zero of f a solution u to (C) is said to be *nontrivial* in the interval $[d,d+\delta]$ (respectively, $[d-\delta,d]$) if $u(r) \neq \bar{u}$ for $r \in (d,d+\delta]$ (respectively, $r \in [d-\delta,d)$).

Theorem 1. *Assume that $f(u)$ is locally Lipschitz in $\bar{\mathbb{R}}^+$, $p>1$.*

(i) *If $f(\bar{u}) \neq 0$ then (C) admits a unique solution in an interval $[d-\delta,d+\delta]$, $\delta>0$.*

(ii) *Suppose that* $\bar{u}$ *is a zero of* $f(u)$ *with order* $k \in \mathbb{R}^+$, $k \geqslant 1$, *in the sense that* $f(\bar{u}) = 0$, f *is differentiable in* $(\bar{u} - \epsilon,\ \bar{u} + \epsilon) \setminus \{\bar{u}\}$, $\epsilon > 0$ *small, and*

$$f'(u) = -k\gamma|u - \bar{u}|^{k-1} + o(|u - \bar{u}|^{k-1}), \quad u \to \bar{u}, \tag{3}$$

then the following properties hold:

(A) Nondegenerate case: *If either* $\gamma < 0$ *in* (3) *or* $p \leqslant k + 1$, *then* $u = \bar{u}$ *is the unique solution to* (C).

(B) Degenerate case: *If* $\gamma > 0$ *and* $p > k+1$ *then* (C) *admits unique nontrivial solutions* $u_\pm(r, d+)$ (*respectively*, $u_\pm(r, d-)$), *defined in* $[d, d+\delta]$ (*respectively*, $[d-\delta, d]$) *satisfying* $u_-(r, d+) < \bar{u} < u_+(r, d+)$ *for* $d < r \leqslant d + \delta$ (*respectively*, $u_-(r, d-) < \bar{u} < u_+(r, d-)$ *for* $d - \delta \leqslant r < d$).

Remark 2.1. (a) Condition (3) entails $f(u) = -\gamma|u - \bar{u}|^{k-1}(u - \bar{u}) + o(|u - \bar{u}|^k)$, $u \to \bar{u}$. However, observe that real $k > 0$ are allowed (the results also hold even with $0 < k < 1$, so f is Hölder near $\bar{u}$). Other configurations of zeros in (3) could be also considered, involving different k's and signs of γ at different sides of $\bar{u}$, keeping the same conclusions when one is restricted to a single of such sides.

(b) For later use, initial profiles to solutions in (i) can be shown to be $u(r) = \bar{u} - (1/p')(f(\bar{u})/\kappa)|(f(\bar{u})/\kappa)|^{p'-1}\ |r - d|^{p'} + o(|r - d|^{p'})$ as $r \to \delta$ where $\kappa = N$ if $d = 0$, $\kappa = 1$ if $d > 0$.

(c) By drawing the phase plane u, u' to equation (E) with orbits starting in $u' = 0$ at $r = d$ the four nontrivial solutions $u_\pm(r, d\pm)$ in (B) are reminiscent of the stable and unstable manifolds of a saddle point. However, they all meet $(u, u') = (\bar{u}, 0)$ at finite time $r = d$.

It is thus convenient to fix the following notion for later use in the work.

Definition. The point $(u, v) = (\bar{u}, 0)$ is a degenerate saddle to the system (S), or equivalently, $u = \bar{u}$ is a degenerate saddle zero for the function $f(u)$ if both f and $\bar{u}$ satisfy the conditions of (ii-(B)) in the statement of Theorem 1.

It can be proved that problem (C) also admits infinitely many other solutions aside the nontrivial ones. Namely, those solutions $u(r)$ taking the value $\bar{u}$ in a whole interval $[d_1, d_2]$ containing d and being nontrivial outside, i.e. $u(r) = u_-(r, d_1-)$ or $u(r) = u_+(r, d_1-)$ before $r = d_1$ and $u(r) = u_-(r, d_2+)$ or $u(r) = u_+(r, d_2+)$ after $r = d_2$ (Fig. 1). Returning to the nontrivial solutions $u_\pm(r, d\pm)$ to (C) in a saddle $\bar{u}$, more elaborated estimates can be obtained so that now $u_\pm(r, d\pm) = \bar{u} \pm C|r - d|^\alpha + o(|r - d|^\alpha)$, $r \to d\pm$, with $\alpha = p/(p - k - 1)$, $C = \{\gamma/[\alpha^{p-1}(\alpha k + \kappa)]\}^{1/(p-k-1)}$ with either $\kappa = N$ or 1 if $d = 0$ or $d > 0$ (see [10]). For a deeper insight in the one-dimensional case $N = 1$ we refer to [8] where both the linear ($p = 2$) and nonlinear ($p \neq 2$) diffusion cases are studied in detail.

(d) For future reference observe that in the special case $N = 1$ in (C) the four nontrivial solutions in an order k degenerate saddle point $\bar{u}$ to

$$-(|u'|^{p-2}u')' = f(u),$$

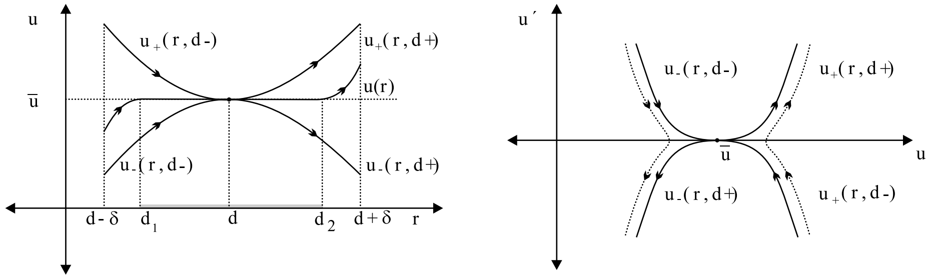

Fig. 1. One of the infinitely many solutions $u(r)$ to (C) at a saddle $\bar{u}$ meets the value $\bar{u}$ in a whole interval $d_1 \leqslant r \leqslant d_2$. A sketch of the phase plane near $(u,u') = (\bar{u},0)$ is also drawn.

are given as $u_{\pm}(r,d+) = U_{\pm}(r-d)$, $r \geqslant d$, $u_{\pm}(r,d-) = U_{\pm}(d-r)$, $r \leqslant d$, where $U = U_+(r)$ is implicit in

$$\int_{\bar{u}}^{U} \frac{\mathrm{d}s}{\{p'(F(\bar{u}) - F(s))\}^{1/p}} = r, \qquad U \geqslant \bar{u},$$

while $U = U_-(r)$ is obtained by solving

$$\int_{U}^{\bar{u}} \frac{\mathrm{d}s}{\{p'(F(\bar{u}) - F(s))\}^{1/p}} = r, \qquad U \leqslant \bar{u}.$$

Notice that the convergence of the integrals is ensured by the condition $p > k+1$.

2.2. Global solutions. Continuous dependence

After scaling λ, positive radial solutions to (P) either initially or after a flat core $\{u = \bar{u}\}$ solve (C) at some $d \geqslant 0$. However, they must progress beyond $r = d$ through the region $\Omega_- := \{(u,u') \in \mathbb{R}^2 : u > 0, u' < 0\}$ in order to meet $u = 0$. The following result, whose proof being standard is omitted, states that any local solution furnished by Theorem 1 can be continued in a unique *maximal* way into Ω_-. It must be remarked that if $f(u)$ is only continuous (S) still has a unique local solution provided $v_0 \neq 0$ [12]. Thus, f is weakened to continuous in the next statement.

Theorem 2. *Suppose* $f \in C(\bar{\mathbb{R}}^+)$ *and let* $u(r)$, $d \leqslant r \leqslant d+\delta$, *be any solution to* (C) entering Ω_- *in the sense that* $u' \leqslant 0$ *while* $u'(r) < 0$ *if* $d \leqslant r_1 < r \leqslant d+\delta$, *for a certain* r_1. *Then* $u(r)$ *admits a continuation* $u_1(r)$ *to an interval* $[d,\omega)$ *such that* $(u_1(r), u_1'(r)) \in \Omega_-$ *for each* $r > r_1$, *being* u_1 *maximal with regard to this property.*

Remark 2.2. In reference to Theorem 2 observe that if $f(\bar{u}) \neq 0$ in (C) then u enters Ω_- only when $f(\bar{u}) > 0$ (Remark 2.1(b)). In the case $f(\bar{u}) < 0$ we will set $\omega := d$. On the other hand, observe that if $\bar{u}$ is a degenerate saddle for f, then infinitely many solutions u enter Ω_- from $\bar{u}$ each of them with its own ω (Remark 2.1(c)). In this case, we will always understand that the maximal solution emanated from $\bar{u}$ is the continuation of the nontrivial solution $u_-(r,d+)$. If the context requires it, the notation $\omega = \omega(\bar{u})$ to link ω to the initial value $\bar{u}$ or even other parameters, will be used.

In the next result we provide a global version of the local statement on continuous dependence given in [10] (see also [12]). However, we have deliberately restricted the scope of the statements for the sake of clarity. In particular, we have omitted any reference to differentiability on parameters since it is not going to be used here (see details in [10]).

Theorem 3. *Let* $f \in C(\bar{\mathbb{R}}^+)$ *be locally Lipschitz,* $\bar{u}_0 > 0$.

(i) *Suppose* $f(\bar{u}_0) > 0$ *while* $U \subset \mathbb{R}^+$ *is any open interval with* $\bar{u}_0 \in U$, $f > 0$ *in* U, *being* $u(r) = u(r, \bar{u}, d)$, $d \leqslant r < \omega$ *the maximal continuation of the local solution to* (C) *in* Ω_-. *Then the set*

$$\Theta := \{(r, \bar{u}, d) \in \mathbb{R}^3 : \bar{u} \in U, d \geqslant 0, d \leqslant r < \omega\}$$

is open relative to $\mathbb{R} \times U \times \bar{\mathbb{R}}^+ \cap \{(r, \bar{u}, d) : r \geqslant d\}$ *while* $u(r, \bar{u}, d)$ *is continuous in* Θ.

(ii) *If* $f(\bar{u}_0) = 0$ *and* $\bar{u}_0$ *is a degenerate saddle,* $U = (a, \bar{u}_0]$ *is an interval where* $f(u) > 0$ *for each* $u \neq \bar{u}_0$, *and for* $\bar{u} \in U$, $u(r, \bar{u}, d)$ *defines the maximal continuation of the solution to* (C) *in* Ω_- ($u_-(r, \bar{u}, d+)$ *in the case* $\bar{u} = \bar{u}_0$) *then the set,*

$$\Theta := \{(r, \bar{u}, d) \in \mathbb{R}^3 : \bar{u} \in U, d \geqslant 0, d \leqslant r < \omega\}$$

is also open relatively to $\mathbb{R} \times U \times \bar{\mathbb{R}}^+ \cap \{(r, \bar{u}, d) : r \geqslant d\}$ *being* $u(r, \bar{u}, d)$ *continuous in* Θ.

(iii) (Behaviour as $d \to +\infty$) *Either under the conditions in* (i) *or* (ii) *let* $u(r) = u_\infty(r, \bar{u})$, $0 \leqslant r < \omega_\infty$ *be the maximal solution in* Ω_- *to*

$$\begin{cases} -(|u'|^{p-2}u')' = f(u), & r \geqslant 0, \\ u(0) = \bar{u}, \quad u'(0) = 0. \end{cases}$$

Then for every fixed $0 < b < \omega_\infty$,

$$u(r + d, \bar{u}, d) \to \bar{u}_\infty(r, \bar{u}), \quad r \in [0, b],$$

uniformly together with its derivative in the interval $[0, b]$ *as* $d \to +\infty$.

Remark 2.3. (a) The proof of Theorem 3 follows in a standard way (see [3]) from the local results in [10] and the property of local uniqueness to (S) when $v_0 < 0$. On the other hand the statement can be strengthened by including additional parameters $\sigma \in \mathbb{R}^k$ in f. In the case of (ii) the only requirement is that k in (3) does not depend on σ while $\gamma = \gamma(\sigma) \geqslant \gamma_0 > 0$ in the parameters regime under consideration. Finally, differentiability on d holds provided f is C^1 (see [10]).

(b) The continuation region Ω_- was chosen in Theorems 3 and 4 thinking on positive solutions to (P). However, the conclusions remain valid if a small amount of negativeness is permitted to global solutions to (C). Namely, if Ω_- is replaced by $\Omega_{-\alpha} = \{(u, u') : u > -\alpha, u' < 0\}$, $\alpha > 0$. In this last case, $f(u)$ should be previously extended to the interval $-\alpha \leqslant u \leqslant 0$.

To conclude our analysis of global solutions to (C) we next state a result where the behaviour of solutions $u(r)$ with r close to ω is precisely elucidated.

Theorem 4. *Consider $f \in C(\bar{\mathbb{R}}^+)$ while (C) admits a local solution $u(r)$ entering Ω_-. If $u(r)$, $d \leqslant r < \omega$, also stands for its maximal continuation in Ω_- then the following alternative holds*:

(i) *Either $\omega < +\infty$ which implies both the existence of*

$$\bar{u}_\omega := \lim_{r\to\omega-} u = \inf_{0\leqslant r<\omega} u, \quad \bar{u}'_\omega := \lim_{r\to\omega-} u',$$

where either $\bar{u}_\omega = 0$ or $\bar{u}'_\omega = 0$.

(ii) *Or $\omega = +\infty$ in which case*

$$\lim_{r\to+\infty} (u, u') = (\bar{u}_\omega, 0)$$

with $f(\bar{u}_\omega) = 0$. Moreover, if $f(u)$ is as in Theorem 1 then, among several possibilities, $\bar{u}_\omega$ cannot be a degenerate saddle in this case.

Proof. Firstly observe that for some $d \leqslant r_1 < \omega$, $u' < 0$ in $r_1 < r < \omega$, with $u(d) = \bar{u}$. Thus $\bar{u}_\omega < u(r) \leqslant \bar{u}$ in that range. On the other hand, as pointed out in the beginning of the section, the group $H(r) = H(u(r),\ u'(r)) = |u'(r)|^p + p'F(u(r))$ is nonincreasing and so u' keeps bounded. Thus, provided that $\omega < +\infty$, the boundedness of both $u(r), u'(r)$ in $d \leqslant r < \omega$ together with Wintner's lemma (see [3]) implies the existence of $\lim_{r\to\omega-} (u(r), u'(r)) := (\bar{u}_\omega, \bar{u}'_\omega)$ meanwhile the own definition of ω gives $(\bar{u}_\omega, \bar{u}'_\omega) \in \partial\Omega_-$. This yields (i).

Assume $\omega = +\infty$ and define $0 \leqslant \bar{u}_\omega < \bar{u}$ as $\bar{u}_\omega = \inf_{0\leqslant r<\omega} u$. Since H is not increasing there exists $H_\omega = \lim_{r\to+\infty} H(r)$ and hence $\bar{u}'_\omega := \lim_{r\to\infty} u' = -(H_\omega - p'F(\bar{u}_\omega))^{1/p}$ also exists. This implies $\bar{u}'_\omega = 0$. To show that $f(\bar{u}_\omega) = 0$ observe that the equation in (C):

$$-(|u'|^{p-2}u')' = \frac{N-1}{r} |u'|^{p-2}u' + f(u),$$

provides the existence of the limit $\lim_{r\to\infty} (|u'|^{p-2}u')'$ what combined with the existence of $\lim_{r\to\infty} |u'|^{p-2}u'$ (which is exactly zero) gives $\lim_{r\to\infty} (|u'|^{p-2}u')' = 0$. This leads to the conclusion.

Finally, suppose that $u(r) \to \bar{u}_\omega$ as $r \to +\infty$. Then, for a small $\delta > 0$, $u(r+m) \to \bar{u}_\omega$, $u'(r+m) \to 0$ both as a functions of $r \in [0, \delta]$ as $m \to \infty$. However, if $\bar{u}_\omega$ is a degenerate saddle then Theorem 3 implies that $u(m+r) \to U_+(r)$ (Remark 2.1(d)) uniformly in $[0, \delta]$ together with its derivative as $m \to +\infty$. This is not possible since $U_+(\rho) > \bar{u}_\omega$ with $U'_+(\rho) > 0$ in $0 < \rho \leqslant \delta$. □

Remark 2.4. (a) It should be remarked that a possible option in the case $\omega < +\infty$ in (i) is that $u(r)$ meets a degenerate saddle $\bar{u}_1$ in finite time $r = \omega$. In that case and following the notation of Theorem 1, $u(r) = u_+(r, \bar{u}_1, d_1-)$ with $d_1 = \omega$.

(b) Aside degenerate saddles, other zeros of f are excluded of the limit behaviour of u in (ii). Namely, those zeros $\bar{u}$ at which $F(u)$ reaches a local minimum. In fact, it can be proved in that case that solutions are forced to oscillate infinitely many times around $\bar{u}$ and thus cannot monotonically decay to $\bar{u}$.

3. Flat core solutions

We are dealing now with positive radial solutions u to (P) which admit flat cores, i.e. nonempty interior regions where $u = \bar{u}$. This will necessarily imply the conditions $f(\bar{u})=0$, $F(u) < F(\bar{u})$ for $0 \leqslant u < \bar{u}$ and additionally the convergence of the improper integral:

$$\int_{\bar{u}-\varepsilon}^{\bar{u}} \frac{\mathrm{d}s}{\{p'(F(\bar{u}) - F(s))\}^{1/p}} < +\infty, \qquad \varepsilon > 0, \tag{4}$$

(see for instance [7]). Observe that a single solution u could possibly exhibit several flat cores $\{u = \bar{u}\}$ corresponding to different $\bar{u}$'s.

3.1. Single flat core solutions

We are beginning by discussing the existence of solutions u with a single flat core $\{u=\bar{u}_0\}$ which converges to B as $\lambda \to +\infty$. Thus, we are assuming that $f(u)$ satisfies the conditions of Theorem 1 and that $\bar{u}_0$ is a degenerate saddle, what implies (4) with $\bar{u}=\bar{u}_0$. In view of the preceding remarks, we are also admitting the validity of condition (I):

$$F(u) < F(\bar{u}_0), \qquad 0 \leqslant u < \bar{u}_0.$$

In this case, the expression

$$\int_{u}^{\bar{u}_0} \frac{\mathrm{d}s}{\{p'(F(\bar{u}_0) - F(s))\}^{1/p}} = r,$$

defines the solution $u = u_\infty(r)$ of $-(|u'|^{p-2}u')' = f(u)$, $u(0) = \bar{u}_0$, $u'(0) = 0$ satisfying $u' < 0$ in $0 < r < \omega_\infty$ with $u(\omega_\infty) = 0$, being

$$\omega_\infty = \int_0^{\bar{u}_0} \frac{\mathrm{d}s}{\{p'(F(\bar{u}_0) - F(s))\}^{1/p}}.$$

Note that $u'_\infty(\omega_\infty) = -(p'F(\bar{u}_0))^{1/p} < 0$.

On the other hand, Theorem 3 (see Remark 2.3(b)) implies the existence of large $d_1 > 0$ such that for each $d \geqslant d_1$, problem (C) with $\bar{u} = \bar{u}_0$ admits a unique solution $u(r,d) = u_-(r,\bar{u}_0,d+)$ defined in $d \leqslant r < \omega(d)$ further continued in some $\Omega_{-\alpha}$ up to $r = d + \omega_\infty + \eta$, $\eta > 0$, such that,

$$u(r+d,d) \to \bar{u}_\infty(r), \qquad 0 \leqslant r \leqslant \omega_\infty,$$

as $d \to +\infty$ in $C^1[0, \omega_\infty + \eta]$ (we need to extend first f to some interval of the form $[-\alpha, 0]$, $\alpha > 0$). Since $u_\infty(r)$ has a simple zero at $r = \omega_\infty$ and $u(r,d)_{|r=\omega(d)} = 0$, then it follows that $\omega(d) \sim d + \omega_\infty$ as $d \to +\infty$. In particular, the equation

$$\omega(d) = R\lambda^{1/p} \tag{5}$$

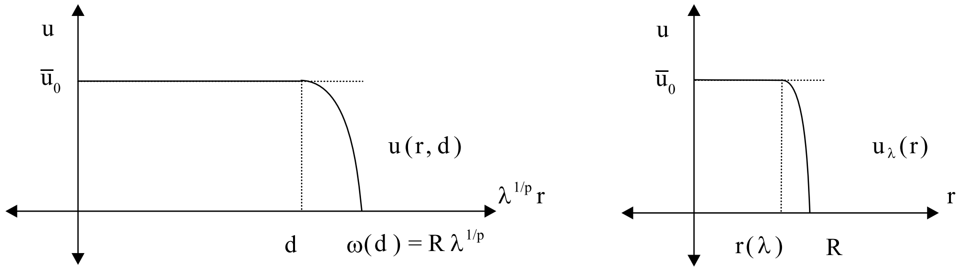

Fig. 2. A single flat core solution.

can be solved for some $d = d(\lambda)$ as $\lambda \to +\infty$. Therefore by setting

$$u_\lambda(r) = \begin{cases} \bar{u}_0, & 0 \leqslant r \leqslant r(\lambda), \\ u(\lambda^{1/p} r, d), & r(\lambda) < r \leqslant R \end{cases}$$

with $r(\lambda) := \lambda^{-1/p} d(\lambda)$, a solution $u_\lambda(r)$ is obtained with flat core $\{r \leqslant r(\lambda)\}$. Note that from (5),

$$\frac{\omega(d(\lambda)) - d(\lambda)}{\lambda^{1/p}} + r(\lambda) = R$$

and so $r(\lambda) \sim R - \omega_\infty \lambda^{-1/p}$ as $\lambda \to +\infty$ what implies that $\{u_\lambda = \bar{u}_0\}$ progressively covers B as $\lambda \to +\infty$ (Fig. 2). In conclusion, the following statement has been already proved.

Lemma 2. *Let $\bar{u}_0$ be a degenerate saddle for $f(u)$, a locally Lipschitz function in $\bar{\mathbb{R}}^+$, while $F(u) = \int_0^u f$ satisfies condition (I). Then for λ large (P) admits a positive radial solution $u_\lambda(r)$ which exhibits a flat core $\{r : u_\lambda(r) = \bar{u}_0\} = [0, r(\lambda)]$ such that $r(\lambda) \to R$ as $\lambda \to +\infty$. More precisely, $r(\lambda) \sim R - \omega_\infty \lambda^{1/p}$ as $\lambda \to +\infty$.*

Remark 3.1. If $f(u)$ is additionally C^1 in $u < \bar{u}_0$ then it can be further proved that $\omega(d)$ is differentiable in d with $\omega'(d) \sim 1$ as $d \to +\infty$ and the solution $d = d(\lambda)$ to (5) is unique as $\lambda \to +\infty$ [10]. As a main consequence of this fact it can be shown that (P) exhibits a unique positive radial solution u satisfying (2) as $\lambda \to +\infty$.

3.2. Connecting saddles. Multiple cores

We are next introducing the more elementary nonlinearity $f(u)$ allowing in (P) the existence of positive solutions u with multiple flat cores under the form $\{u = \bar{u}_0\} \cup \{u = \bar{u}_1\}$. Our first step in this direction is to show that equation (E) exhibits with such nonlinearity a finite time connection between the zeros $u = \bar{u}_0$ and $\bar{u}_1$.

Theorem 5. *Fix $p > 2$ and positive values $\bar{u}_0 > \bar{u}_1$ together with $u_\sigma = \bar{u}_0 + \sigma(\bar{u}_1 - \bar{u}_0)$ where $0 \leqslant \sigma \leqslant 1$. Then there exists $d_1 \geqslant 0$ such that for each $d \geqslant d_1$ a $\sigma = \sigma(d) \in (0, 1]$ exists so that the maximal solution $u = u_-(r, \bar{u}_0, d+)$, $d \leqslant r < \omega$,*

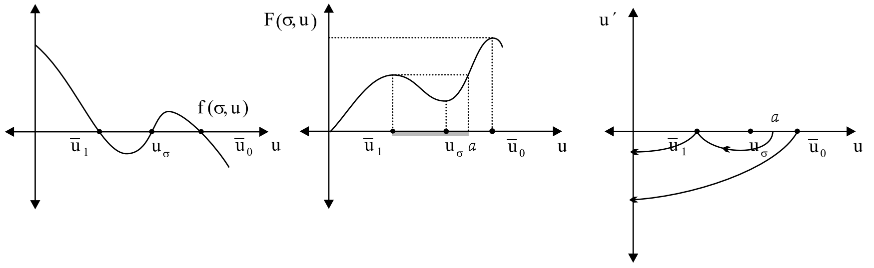

Fig. 3. Profiles of $f = f(\sigma,u)$, $F = F(\sigma,u)$ and a possible sketch of the phase plane for $\sigma^* < \sigma \leqslant 1$.

in Ω_- *to*

$$\begin{cases} -(r^{N-1}|u'|^{p-2}u')' = -r^{N-1}(u-\bar{u}_1)(u-u_\sigma)(u-\bar{u}_0)_{|\sigma=\sigma(d)}, & r \geqslant d, \\ u(d) = \bar{u}_0, \quad u'(d) = 0, \end{cases} \tag{6}$$

satisfies,

$$u_-(r,\bar{u}_0,d+)_{|r=\omega} = \bar{u}_1, \quad u_-'(r,\bar{u}_0,d+)_{|r=\omega} = 0.$$

Remark 3.2. Note that $\bar{u}_0$, $\bar{u}_1$ are simple zeros ($k=1$) with negative derivatives of the bistable nonlinearity when $0 < \sigma < 1$. Accordingly, they are degenerate saddles only when $p > 2$. However, $\bar{u}_1$ is double ($k = 2$) when $\sigma = 1$, being $p > 3$ in this case the condition for degenerate saddle. As will seen below, the conclusion of Theorem 5 holds for any $d \geqslant 0$ in this last case, without restriction in the size of d.

Proof. It is based in a continuation argument in σ together with a phase plane analysis of equation (E). Thus, it is important to describe the profiles of $F(\sigma,u) = \int_0^u f(\sigma,t)\,dt$ with varying σ. Firstly, observe that F increases in $[0,\bar{u}_1]\cup[u_\sigma,\bar{u}_0]$ being decreasing in $[\bar{u}_1,u_\sigma]$. On the other hand, $F(\sigma,\bar{u}_1)$ and $F(\sigma,\bar{u}_0)$ are linear in σ, $F(0,\bar{u}_1) > F(0,\bar{u}_0)$, $F(1,\bar{u}_1) < F(1,\bar{u}_0)$ and hence a unique σ^* exists such that $F(\sigma^*,\bar{u}_1) = F(\sigma^*,\bar{u}_0)$. This implies that $F(\sigma,u)$ possesses a well $\{u \in (\bar{u}_1,\bar{u}_0) : F(\sigma,u) < F(\sigma,\bar{u}_1)\} := (\bar{u}_1, a(\sigma))$ where $a(\sigma) \to \bar{u}_0$ as $\sigma \to \sigma^*+$, $a(\sigma) \to \bar{u}_1$ as $\sigma \to 1-$ (Fig. 3). Hence, as a consequence of the nonincreasing character of $H(u,u') = |u'|^p + p'F(\sigma,u)$ it follows that the desired connection can only happen in the range $\sigma^* < \sigma \leqslant 1$. Finally, note that condition (I) at the zero $\bar{u}_0$ is satisfied in such range for σ.

To proceed with the perturbation argument we have to distinguish between the cases $p > 3$ and $2 < p \leqslant 3$, because—as mentioned in the previous remark—$\bar{u}_1$ becomes a double zero at $\sigma = 1$, and we could fall in the degenerate regime or in the nondegenerate one. We first assume $p > 3$. Fix $d \geqslant 0$ arbitrary and set $\sigma = 1$. Then $\bar{u}_0$ is a degenerate saddle while $\bar{u}_1$ satisfies the assumptions of (ii(B)) in Theorem 1 when observed in the range $u \leqslant \bar{u}_1$. Now, since $\bar{u}_1$ is of saddle type then $\omega(\bar{u}) < \infty$ for every $0 < \bar{u} \leqslant \bar{u}_0$. In fact, option (ii) in Theorem 4 is not possible at any $\bar{u}$. In particular $\omega(\bar{u}_0) < +\infty$. If $u(r) = u_-(r,\bar{u}_0,\sigma,d+)_{|\sigma=1}$ reaches $\bar{u}_1$ at $r = \omega$ we achieve the conclusion of the theorem with $\sigma = 1$. Otherwise, due to $f(1,u) > 0$ for $u \neq \bar{u}_1$ we necessarily obtain that $u_-(r,\bar u_0,\sigma,d+)_{|\sigma=1}=0$ with negative derivative at $r=\omega(\bar u_0,\sigma)_{|\sigma=1}$. Introduce in that case,

$$S=\{\sigma\in(\sigma^*,1]:u_-(r,\bar u_0,\sigma,d+)=0,\ u_-'(r,\bar u_0,\sigma,d+)<0 \text{ at } r=\omega(\bar u_0,\sigma)\}.$$

Observe that by continuous dependence $[1-\epsilon,1]\subset S$ for $\epsilon>0$ small. Set $\tilde\sigma=\inf S$. Then three options are possible:

(a) $u_-(r,\bar u_0,\tilde\sigma)=0$, $u_-'(r,\bar u_0,\tilde\sigma)<0$ at $r=\omega(\bar u_0,\tilde\sigma)$,
(b) $u_-(r,\bar u_0,\tilde\sigma)>\bar u_1$, $u_-'(r,\bar u_0,\tilde\sigma)=0$ at $r=\omega(\bar u_0,\tilde\sigma)$,
(c) $u_-(r,\bar u_0,\tilde\sigma)=\bar u_1$, $u_-'(r,\bar u_0,\tilde\sigma)=0$ at $r=\omega(\bar u_0,\tilde\sigma)$.

However, options (a) and (b) contradict by continuous dependence the choice of $\tilde\sigma$. Thus only (c) is possible and the desired connection is achieved.

To conclude the proof suppose now that $2<p\leqslant 3$. In this case the possible behaviour $u_-(r,\bar u_0,\sigma,d+)_{|\sigma=1}\to\bar u_1$ as $r\to+\infty$ is avoided by taking d large enough. In fact, observe that the nontrivial solution $U_-(r)$ to

$$\begin{cases} -(|u'|^{p-2}u')'=-(u-\bar u_1)(u-u_\sigma)(u-\bar u_0)_{|\sigma=1}, & r\geqslant d,\\ u(0)=\bar u_0, \quad u'(0)=0, \end{cases}$$

reaches $u=0$ with $u'<0$ at finite time $r=\omega_\infty$ (see Remark 2.1(d)). Theorem 3(iii) can then be used to conclude that for large d, $u_-(r,\bar u_0,\sigma,d+)_{|\sigma=1}$ behaves similarly to $U_-(r)$ and also vanishes with negative derivative at finite $r=\omega(\bar u_0,\sigma)_{|\sigma=1}$. Once this point is ensured, the rest of the previous argument remains the same and the proof is finished. □

The following result shows that problem (P) with a simple nonlinearity exhibits at least two solutions satisfying, when λ is large, the approximation condition (1).

Theorem 6. *Let* $p>2$, $0<\bar u_1<\bar u_0$ *fixed while* $u_\sigma=\bar u_1+(1-\sigma)(\bar u_0-\bar u_1)$ *with* $0\leqslant\sigma\leqslant 1$. *Then there exists* $0<\sigma<1$ *such that the problem*

$$\begin{cases} -\Delta_p u=-\lambda(u-\bar u_1)(u-u_\sigma)(u-\bar u_0), & x\in B,\\ u_{|\partial B}=0, \end{cases}$$

admits for large λ *two radial positive solutions* $u_\lambda(r)$, $\tilde u_\lambda(r)$ *such that*

$$\max_B u_\lambda=\max_B \tilde u_\lambda=\bar u_0.$$

Moreover,

(i) $\{r\in[0,R]:u_\lambda(r)=\bar u_0\}=[0,r(\lambda)]$ *where* $r(\lambda)\to R$ *as* $\lambda\to+\infty$.
(ii) $\{r\in[0,R]:\tilde u_\lambda(r)=\bar u_0\}=[0,\tilde r(\lambda)]$, $\{r\in[0,R]:\tilde u_\lambda(r)=\bar u_1\}=[r_0(\lambda),r_1(\lambda)]$, *where now* $0<\tilde r(\lambda)<r_0(\lambda)\to 0$, $r_1(\lambda)\to R$ *as* $\lambda\to+\infty$.

Proof. Firstly observe, following the notation of Theorem 5, that $\bar u_0$ is a degenerate saddle satisfying condition (I):

$$F(\sigma,u)<F(\sigma,\bar u_0), \quad 0\leqslant u<\bar u_0,$$

provided that $\sigma^* < \sigma \leqslant 1$. Thus, for σ in that range the existence of the solutions $u_\lambda(r)$ follows directly from Lemma 2 in Section 3.1.

In order to construct $\tilde{u}_\lambda(r)$ fix $d \geqslant 0$, either arbitrary if $p > 3$ or conveniently large in the case $2 < p \leqslant 3$. Set $\sigma = \sigma(d)$, the value obtained in Theorem 5 and write:

$$u_0(r) := u_-(r, \bar{u}_0, \sigma, d+)_{|\sigma=\sigma(d)}, \quad d \leqslant r \leqslant \omega_0 := \omega(\bar{u}_0, \sigma)_{|\sigma=\sigma(d)},$$

the nontrivial solution which has been shown to satisfy $u_0(\omega_0) = \bar{u}_1$, $u_0'(\omega_0) = 0$. As a second step choose $d_1 > \omega_0$ as a parameter. For every value of σ, $\bar{u}_1$ is a degenerate saddle fulfilling the integral condition (I) with $\bar{u}_1$ replacing $\bar{u}_0$, i.e.,

$$F(\sigma, u) < F(\sigma, \bar{u}_1), \quad 0 \leqslant u < \bar{u}_1.$$

Then the construction of Section 3.1 can be performed to attain that the nontrivial solution $u_1(r) = u_-(r, \bar{u}_1, \sigma, d_1+)_{\sigma=\sigma(d)}$ to the equation in (6) with initial conditions,

$$u(d_1) = \bar{u}_1, \quad u'(d_1) = 0,$$

satisfies $u_1(r) = 0$, $u_1'(r) < 0$ at $r = \omega(\bar{u}_1, d_1)$. Setting $\omega_1(d_1) := \omega(\bar{u}_1, d_1)$ we can solve again the equation $\omega_1(d_1) = R\lambda^{1/p}$ to obtain a unique solution $d_1 = d_1(\lambda)$ such that $\omega_1(d_1) \sim d_1 + \omega_{\infty,1}$ as $\lambda \to \infty$, where $\omega_{\infty,1} = \int_0^{\bar{u}_1} \{p'(F(\sigma, \bar{u}_1) - F(\sigma, s))\}^{-1/p}\, ds$, $\sigma = \sigma(d)$. Thus,

$$\tilde{u}_\lambda(r) = \begin{cases} \bar{u}_0, & 0 \leqslant r \leqslant \tilde{r}(\lambda), \\ u_0(\lambda^{1/p} r), & \tilde{r}(\lambda) < r < r_0(\lambda), \\ \bar{u}_1, & r_0(\lambda) \leqslant r \leqslant r_1(\lambda), \\ u_1(\lambda^{1/p} r), & r_1(\lambda) < r \leqslant R, \end{cases}$$

where $\tilde{r}(\lambda) = \lambda^{-1/p} d$, $r_0(\lambda) = \lambda^{-1/p}\omega_0$, $r_1(\lambda) = \lambda^{-1/p} d_1(\lambda)$, defines the desired solution. □

Remark 3.3. The existence of the connection $u_0(r)$, $d \leqslant r \leqslant \omega_0$, between the saddles $\bar{u}_0$ and $\bar{u}_1$ in Theorem 5 *only* depends on the behaviour of $f(u) = -(u - \bar{u}_1)(u - u_\sigma)(u - \bar{u}_0)_{|\sigma=\sigma(d)}$ in the interval $\bar{u}_1 \leqslant u \leqslant \bar{u}_0$.

This remark permits to extend the construction argument in Theorem 6 under the following lines. Consider $n+1$ fixed values $0 < \bar{u}_n < \cdots < \bar{u}_1 < \bar{u}_0$ and n additional intermediate values $u_{\sigma_i} = \bar{u}_i + (1 - \sigma_i)(\bar{u}_{i-1} - \bar{u}_i)$, $0 \leqslant \sigma_i \leqslant 1$, $1 \leqslant i \leqslant n$. Define the locally Lipschitz function $f(\sigma_1, \ldots, \sigma_n, u)$ in $\mathbb{R}^+$ in such a way that $f = -(u - \bar{u}_i)(u - u_{\sigma_i})(u - \bar{u}_{i-1})$ in the interval $[\bar{u}_i, \bar{u}_{i-1}]$, $1 \leqslant i \leqslant n-1$ and as $f = -(u - \bar{u}_n)(u - u_{\sigma_n})(u - \bar{u}_{n-1})$ in the whole interval $[0, \bar{u}_{n-1}]$. By proceeding progressively in each interval $[\bar{u}_i, \bar{u}_{i-1}]$ as in the proof of Theorem 5 it is possible to determine values $d_0 < d_1 < \cdots < d_{n-1}$ and associated numbers $\sigma_\ell = \sigma_\ell(d_{\ell-1}) \in (0, 1]$, $1 \leqslant \ell \leqslant n$, such that for each $0 \leqslant i \leqslant n-1$ the maximal nontrivial solution $u_i(r) := u_-(r, \bar{u}_i, d_i+)$, $d_i \leqslant r \leqslant \omega_i := \omega(\bar{u}_i, d_i)$, to the initial value problem

$$\begin{cases} -(r^{N-1}|u'|^{p-2}u')' = r^{N-1} f(\sigma_1, \ldots, \sigma_n, u)_{|\sigma_i=\sigma_i(d_{i-1})}, & r \geqslant d_i, \\ u(d_i) = \bar{u}_i, \quad u'(d_i) = 0, \end{cases}$$

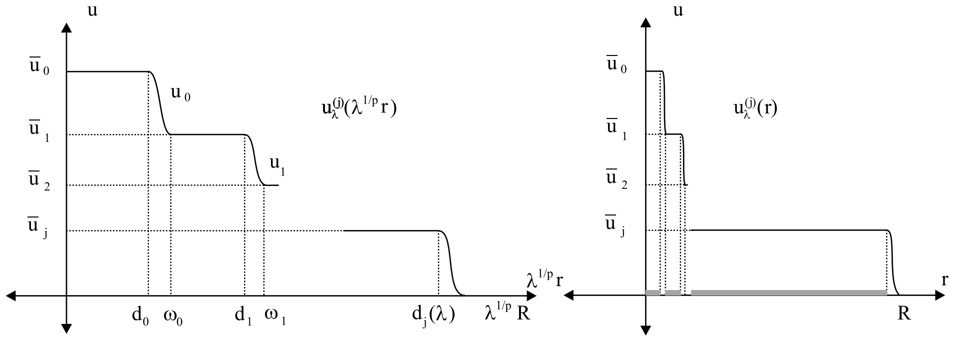

Fig. 4. The profile of $u_\lambda^{(j)}$.

satisfies the connection conditions $u_i(r)=\bar{u}_{i+1}$, $u_i'(r)=0$ at $r=\omega_i$. The values d_i can be arbitrarily chosen under the conditions $d_0 \geqslant 0$ and $d_{i+1} \geqslant \omega_i$, $0 \leqslant i \leqslant n-1$, if $p>3$ or must satisfy $d_{i+1} \geqslant \omega_i$, $0 \leqslant i \leqslant n-1$, and additionally be conveniently large when $2<p \leqslant 3$.

For the choice $\sigma_1=\sigma_1(d_0),\ldots,\sigma_n=\sigma_n(d_{n-1})$ the problem

$$\begin{cases} -\Delta_p u=\lambda f(\sigma_1,\ldots,\sigma_n,u), & x\in B,\\ u_{|\partial B}=0, \end{cases}$$

exhibits different families of positive radial solutions u with multiple flat cores and satisfying $\max_B u=\bar{u}_0$. Specifically, for each $0 \leqslant j \leqslant n$ and large λ, a positive solution $u_\lambda^{(j)}(r)$ can be found such that,

$$\{r: u_\lambda^{(j)}(r)=\bar{u}_0\}=[0,r_0(\lambda)],\quad \{r: u_\lambda^{(j)}(r)=\bar{u}_1\}=[r_1^-(\lambda),r_1^+(\lambda)],\ldots,$$

$$\{r: u_\lambda^{(j)}(r)=\bar{u}_j\}=[r_j^-(\lambda),r_j^+(\lambda)].$$

In addition $0<r_0(\lambda)<r_1^-(\lambda)<r_1^+(\lambda)<\cdots<r_j^-(\lambda)$ converge to 0 while $r_j^+(\lambda)\to R$ as $\lambda\to+\infty$. To construct $u_\lambda^{(j)}(r)$ it suffices with setting

$$u_\lambda^{(j)}(r)=\begin{cases} \bar{u}_0, & 0\leqslant r\leqslant r_0(\lambda),\\ u_-(\lambda^{1/p}r,\bar{u}_0,d_0+), & r_0(\lambda)<r<r_1^-(\lambda),\\ \vdots & \vdots\\ \bar{u}_i, & r_i^-(\lambda)\leqslant r\leqslant r_i^+(\lambda),\\ u_-(\lambda^{1/p}r,\bar{u}_i,d_i+), & r_i^+(\lambda)<r<r_{i+1}^-(\lambda),\\ \vdots & \vdots\\ u_-(\lambda^{1/p}r,\bar{u}_j,d_j(\lambda)+), & r_j^+(\lambda)<r\leqslant R, \end{cases}$$

where $r_0(\lambda)=\lambda^{-1/p}d_0,\ldots,r_i^-(\lambda)=\lambda^{-1/p}\omega_{i-1},r_i^+(\lambda)=\lambda^{-1/p}d_i,\ldots,r_j^-(\lambda)=\lambda^{-1/p}\omega_{j-1}$, while $d_j=d_j(\lambda)>\omega_{j-1}$ is defined through the solution, for large λ, of the equation

$\omega_j(\bar{u}_j, d_j) = R\lambda^{1/p}$ whose existence has been established under the general framework of Section 3.1 (Fig. 4).

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