UNIQUENESS TO QUASILINEAR PROBLEMS FOR THE p-LAPLACIAN IN RADIALLY SYMMETRIC DOMAINS *

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Abstract

In this work it is shown that the class of perturbed logistic-type problems,

$$\begin{array}{rcll} -\Delta_p u &=& \lambda m u^{p-1} - u^q + g(u) & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega, \end{array} \tag{P}$$

where Δ_p is the p-Laplacian, $p > 2$, $q > p-1$, $m > 0$ and $\Omega \subset \mathbb{R}^N$ is a rotationally invariant bounded domain, admits a *unique* positive solution when the parameter λ becomes large, provided the C^1 perturbation term $g = g(u)$ satisfies the growth conditions $g = o(u^{p-1})$ as $u \to 0+$ and $g = o(u^q)$ as $u \to +\infty$. While uniqueness of positive solutions to (P) in general domains Ω still remains an open question, our uniqueness result permits us to to show that *all* possible positive solutions u to (P), in arbitrary (non necessarily radially symmetric) bounded smooth domains Ω, develop "dead cores" $\mathcal{O}_\lambda = \{x : u = \bar{u}_0(\lambda)\}$, for λ large, $\bar{u}_0(\lambda)$ being the unique zero of the nonlinearity in (P) as $\lambda \to +\infty$. This enables us to give a precise description of the asymptotic profile of all such solutions u as $\lambda \to +\infty$, under the form of an exact estimate of *dist* $(\mathcal{O}_\lambda, \partial\Omega)$ when $\lambda \to +\infty$. These results are also extended to the more general class of radially symmetric sublinear problems $-\Delta_p u = \lambda f(\lambda, u)$ in Ω, $u|_{\partial\Omega} = 0$, where Ω is a ball or an annulus.

Keywords: Degenerate diffusion, weak sub and supersolutions, boundary layers, dead cores.

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1. Introduction

It is well-known that the class of logistic-type problems

$$\begin{array}{rcll} -\Delta_p u &=& \lambda u^{p-1} - u^q & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega, \end{array} \tag{1.1}$$

where $q > p - 1 > 0$, λ is a positive parameter, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and Δ_p stands for the p-Laplace operator, defined in distributional sense on $W_0^{1,p}(\Omega)$ with values

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in $W^{-1,p'}(\Omega)$, $p' = p/(p-1)$, as $\Delta_p u = \text{div }(|\nabla u|^{p-2}\nabla u)$, possess a *unique* positive solution $u = u_\lambda$ only when $\lambda > \lambda_{1,p}$ ([1]), $\lambda_{1,p}$ the first Dirichlet eigenvalue for the p-Laplacian in Ω.

In the present work we are proving, among other facts, that the perturbed boundary value problem,

$$\begin{array}{rcll} -\Delta_p u & = & \lambda u^{p-1} - u^q + g(u) & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega, \end{array} \tag{1.2}$$

where $p > 2$, $q > p-1$ and Ω is a bounded radially symmetric domain D, i. e. D is either a ball $B = \{x \in \mathbb{R}^N : |x| < R\}$ or an annulus $A = \{x \in \mathbb{R}^N : a < |x| < R\}$, a, R positive, admits a *unique* positive solution when $\lambda > 0$ is large, provided that the perturbation term $g = g(u)$ is C^1 and satisfies the growth conditions $g = o(u^{p-1})$ as $u \to 0+$, $g = o(u^q)$ as $u \to +\infty$ (see §4). As explicit examples show in the case $p = 2$, uniqueness for large λ is the best possible result since multiple solutions could appear for particular g's when λ is of the order of $\lambda_{1,p}$.

It is worthy of mention that this kind of uniqueness result has been analyzed in the semilinear case $p = 2$, i. e. $-\Delta u = f(\lambda, u)$ in Ω, $u|_{\partial\Omega} = 0$ with Ω a general domain, in a series of works ([2], [3], [4], [5], [6]) where typically $f(\lambda, u) = \lambda f(u)$. Precisely [5] has been the starting point for the present research. It should be also remarked that the continuation methods used in most of those works, based upon linearization around positive solutions, can not be used when $-\Delta$ is replaced by $-\Delta_p$, $p \neq 2$. In fact, as will be next described, positive solutions to (1.2) achieve their maxima in whole subdomains of D if λ grows, and so the linearized equations around them critically degenerate in our problem. Therefore, (1.2) is by no means a standard perturbation of (1.1) regarding the known techniques for the semilinear case.

A second main achievement of the present work is showing that *all* possible positive solutions u to (1.2) develop dead cores in general bounded domains $\Omega \subset \mathbb{R}^N$, when λ is large enough. To precise this behavior, it should be first observed that the assumptions on the perturbation term $g = g(u)$ (cf. hypothesis (Hg) in §4) imply that the right member in (1.2) exhibits a maximum positive zero $u = \bar{u}_0(\lambda)$ such that $\bar{u}_0(\lambda) \sim \lambda^{1/p}$ as $\lambda \to +\infty$ (notice that $\bar{u}_0(\lambda) = \lambda^{1/p}$ if $g = 0$ in (1.2)). It can be further shown (Theorem 4.1) that all possible positive solutions u to (1.2) are estimated by $\bar{u}_0(\lambda)$, i. e.,

$$0 < u(x) \leq \bar{u}_0(\lambda) \qquad x \in \Omega\,. \tag{A}$$

Since $\bar{u}_0(\lambda)$ turns out to be simple for large λ, the second inequality in (A) must be strict provided $1 < p \leq 2$ (see [7], [8]). Therefore, only when $p > 2$ (as will be supposed here in the context of (1.2)) the region $\mathcal{O}_\lambda = \{x \in \Omega : u(x) = \bar{u}_0(\lambda)\}$ could become nonempty. In that case $\mathcal{O}_\lambda$ is termed as "dead" or "flat" core (see [9], [8], [10]).

The appearance of N-dimensional dead cores for the logistic problem (1.1) when λ is large was first proven in [10], being these results substantially improved in [8] with a precise description of the asymptotic profile of the solution u_λ to (1.1) for large λ. It should be emphasized that both works strongly rely upon the uniqueness of positive solutions to (1.1) in any bounded smooth domain Ω. However, and at the best of our knowledge, such uniqueness still remains an open question for the perturbed problem (1.2) in arbitrary domains Ω.

In the present work we are showing that *every* possible positive solution u to (1.2) necessarily develops a dead core $\mathcal{O}_\lambda = \{x : u = \bar{u}_0(\lambda)\}$ in *any* general smooth domain Ω when

λ is great enough (Theorem 5.1). Thus, we close the qualitative analysis of the perturbed logistic problem (1.2), which was initiated in [12], [8] with the study of boundary layers to that problem in the full regime $p > 1$. Moreover, we are also providing a detailed account of the asymptotic profile of any family $\{u_\lambda\}$ of positive solutions to (1.2), under the form of the following exact estimate of the distance from the dead core $\mathcal{O}_\lambda$ to the boundary $\partial\Omega$,

$$\text{dist}\,(\mathcal{O}_\lambda, \partial\Omega) \sim C\ \lambda^{-1/p} \qquad \text{as } \lambda \to +\infty\,, \tag{1.3}$$

where C is explicitly given in terms of an integral involving p and q (Theorem 5.1). In particular, it is shown by this mean that such asymptotic behaviour exactly coincides with the one in the logistic problem (1.1) obtained in [8].

The strategy to achieve the uniqueness of positive weak solutions to (1.2) consists in proving that *radial* positive solutions to (1.2) are unique when λ is large, showing in addition that *all* possible positive solutions to (1.2) in D must be radial. In the semilinear case ($\Delta_p = \Delta$) and D a ball B, that is a direct consequence of the celebrated Gidas-Ni-Nirenberg result [13] (see [14] for a counterexample in annuli). However, GNN's result is not in general valid for the p-Laplacian. Two known cases where GNN's theorem holds are either when the positive solution reaches its maximum exactly in the center of B (cf. [15]) or when $p = N$ (cf. [16]). Moreover, even in the case $D = B$, [15] does not apply due to the presence of dead cores (see details in §§4 and 5). Accordingly, it is shown in the present work, via a combination of the method of sub and supersolutions and the symmetry of D (cf. Appendix) that all possible positive solutions to (1.2) must be radial when $\lambda > \lambda_{1,p}$ is large, once one already knows that radial positive solutions are unique.

When studying the uniqueness of positive radial solutions and dead core generation for (1.2) we are further considering a more general class of problems,

$$\begin{aligned} -\Delta_p u &= \lambda f(\lambda, u) && \text{in } D \\ u &= 0 && \text{on } \partial D\,, \end{aligned} \tag{1.4}$$

where f exhibits a fixed positive zero $u = \bar{u}_0$ with order $k > 0$ (see §2.1), which satisfies a suitable "energy" condition (see §3). We are then showing the uniqueness together with the arising of dead cores $\mathcal{O}_\lambda = \{x \in D : u_\lambda(x) = \bar{u}_0\}$ for large λ, for families $\{u_\lambda\}$ of radial positive solutions to (1.4) which keep progressively close to $\bar{u}_0$ in the sense that

$$\lim_{\lambda\to+\infty} u_\lambda = \bar{u}_0\,,$$

uniformly on compacts $K \subset D$ providing a precise asymptotic estimate (1.3), C explicitly given in terms of f and p. All those results will be obtained in §3 provided that p and the order k of $\bar{u}_0$ keep the natural relation

$$0 < k < p - 1\,, \tag{B}$$

which is a necessary condition for the appearance of dead cores ([7], [8]).

To deal with radial solutions to (1.4) is a matter of ordinary differential equations. In fact, if $u \in W_0^{1,p}(\Omega)$ is a weak radial bounded solution to (1.4) then $u = u(r)$ a. e., $r = |x|$, with u, $r^{N-1}\varphi_p(u') \in C^1(J)$ defining a solution (see §3.1) to,

$$\begin{aligned} &-(r^{N-1}\varphi_p(u'))' = \lambda\ r^{N-1} f(\lambda, u) \\ &\ u'(d) = 0\,,\ \ u(R) = 0\,, \end{aligned} \tag{C}$$

where $\varphi_p(z) = |z|^{p-2}z$, $J = [0, R]$ if D is a ball B, $J = [a, R]$ for the case of the annulus A. Regarding the boundary conditions in (C), $d \in J$ is chosen so that $u(d) = u_0$, where $u_0 = \sup_D u$ (§3.1).

Therefore, a substantial part of the present work is concentrated in developing a complete study of the initial value problem,

$$\begin{aligned} -(r^{N-1}\varphi_p(u'))' &= r^{N-1} f(\lambda, u) \\ u(d) = u_0\,,\ u'(d) &= 0\,, \end{aligned} \tag{1.5}$$

(λ has been scaled) where we are keeping in mind that u_0 plays the rôle of the maximum of a radial solution to (1.4) and where $d\lambda^{-1/p}$ is going to measure the "width" of a possible dead core. This is done in §2 where local classical solutions to (1.5) are analyzed in depth. This means that we will be searching for u, $r^{N-1}\varphi_p(u') \in C^1[d, d+\delta]$, $\delta > 0$ small, solving (1.5) in $I = [d, d+\delta]$.

It should be remarked that (1.5) falls out the scope of the standard existence and uniqueness ode's theory ([17], [18]) when $p > 2$. On the other hand, explicits examples in [19], [20] show that uniqueness may be lost when f vanishes at $u = u_0$.

Therefore, our interest in §2 will be mainly focused in a detailed analysis of the "critical" case $f(\bar{u}_0) = 0$ of (1.5) (λ has been removed for brevity). As a major achievement it will be shown that (1.5) exhibits a unique nontrivial solution $u = u(r) < \bar{u}_0$ in $0 \le r \le \delta$ ($u \equiv \bar{u}_0$ is already a solution to (1.5)!). As for f, it will be assumed that $\bar{u}_0$ defines a zero of order k in the sense that $f(u) \sim \gamma\ (\bar{u}_0 - u)^k$, $\gamma \neq 0$, as $u \to \bar{u}_0-$, for a positive constant γ. It is shown that both $\gamma > 0$ and $0 < k < p-1$ become necessary and sufficient conditions for the existence of nontrivial solutions (Theorems 2.2 and 2.3). Furthermore, if the behaviour of $f(u)$ at both sides of $\bar{u}_0$ is known to be as $f(u) \sim \gamma\ \varphi_{k+1}(\bar{u}_0 - u)$ as $u \to u_0$, our analysis will enable us to find the remaining kind of nontrivial solutions to (1.5), furnishing a complete picture of all of the (infinite) solutions to that problem (Corollary 2.4). It should be remarked that the recent work [21] by Reichel and Walter develops, under a different approach, general existence and uniqueness results for (1.5) (see also the final remarks).

Radial positive solutions $\{u_\lambda\}$ to (1.4) for λ large are studied in §3 under the assumption that $f(\lambda, u)$ and its asymptotic profile $\bar{f}(u) = \lim_{\lambda\to\infty} f(\lambda, u)$ have $u = \bar{u}_0$ as a common zero of order k, satisfying (B).

To show the uniqueness of radial positive solutions to (1.4), the issue of smooth dependence of $u(\cdot, u_0, \lambda, d)$ on initial data and parameters (especially, differentiability with respect to d), with particular emphasis in the singular and limit cases $u_0 = \bar{u}_0$, $d, \lambda \to +\infty$, becomes of paramount importance (see §§2.4, 2.5). In fact, such uniqueness is achieved by proving in §3 that the nontrivial solution $u = u(\cdot, \bar{u}_0, \lambda, d)$ to (1.5) can be globally defined away from $r = 0$ and characterized (Implicit Function Theorem) as the unique perturbation of the corresponding nontrivial solution to the problem,

$$\begin{aligned} -(\varphi_p(u'))' &= \bar{f}(u) \\ u(0) = \bar{u}_0\,,\ u'(0) &= 0. \end{aligned} \tag{1.6}$$

which results from (1.5) by setting $u_0 = \bar{u}_0$ and letting $d, \lambda \to +\infty$ (§2.4).

As a main consequence it is shown in §4 (Theorem 4.2) that the perturbed logistic problem (1.2) exhibits a unique positive solution as λ is large.

2. The Cauchy problem

2.1. The initial value problem, statement of results. As observed in §1 (see also §3.1), the study of weak radial solutions to problem (1.4) leads to the analysis of classical solutions to the Cauchy problem (1.5) (see §1), which we are going to use under the following shifted version,

$$\begin{aligned} -((r+d)^{N-1}\varphi_p(u'))' &= (r+d)^{N-1}f(u) \\ u(0) = u_0\,,\ u'(0) &= 0\,. \end{aligned} \tag{2.1}$$

We will have in mind that u_0 is the maximum value of $u = u(r)$ in a radially symmetric domain D and thus, our attention will be put on solutions to (2.1) which do not exceed the value $u = u_0$.

To begin with the description of our results let us introduce a first statement, concerning the "nonsingular case" where u_0 is not a zero of $f(u)$ (see also [19], [22]).

Theorem 2.1 *Suppose that $f(u_0) \neq 0$, and that there exists $\delta_1 > 0$ such that f is Lipschitz continuous in $|u - u_0| \leq \delta_1$. Then for every $d \geq 0$ there exists $\delta > 0$ such that (2.1) has a unique solution in $[0, \delta]$. Moreover, δ can be chosen independently of d. In addition*

$$u(r) = u_0 - Cr^{p'} + o(r^{p'}), \qquad \textit{as } r \to 0+, \tag{2.2}$$

where $p' = p/(p-1)$, $C = \varphi_{p'}\left(f(u_0)/N\right)/p'$ if $d = 0$ while $C = \varphi_{p'}(f(u_0))/p'$ for $d \neq 0$.

Remarks 2.1.

a) Estimate (2.2) gives a precise account of the behaviour of a positive solution $u = u(r)$ to (1.4) for $r \sim 0$, provided its maximum value $u_0 = \sup_B u = u(0)$ is not a zero of f.

b) The proof of Theorem 2.1 (see §2.2) relies on the contractivity of the operator T,

$$Tu(r) = u_0 - \int_0^r \varphi_{p'}\left(\int_0^s \left(\frac{\rho+d}{s+d}\right)^{N-1} f(u(\rho))\, d\rho\right) ds \tag{2.3}$$

defined in the space $C[0,\delta]$. In fact observe that solutions u to (2.1) are fixed points of T, while, conversely, every fixed point of T in $C[0,\delta]$ is necessarily a C^1 function such that $(r+d)^{N-1}\varphi_p(u') \in C^1[0,\delta]$ and $((r+d)^{N-1}\varphi_p(u'))' = -(r+d)^{N-1}f(u)$, therefore providing a solution to (2.1). Nevertheless, if $f = f(u)$ is only continuous Schauder's fixed point theorem can be used to attain a local existence result for (2.1) no matter the value of $f(u_0)$ is (see for instance [20]).

c) A careful analysis of the proof of Theorem 2.1 shows that the conclusion remains true if $f(u_0) = 0$ provided $1 < p \leq 2$ (alternatively, standard ode's theory is valid in this case), the solution being obviously $u \equiv u_0$. Actually, the key point for this fact is that a "proper" balance between p and the order k of u_0 must hold. More precisely, we have the following result.

Theorem 2.2 *Assume that f is only continuous in $|u - \bar{u}_0| \leq \delta_1$ for some $\delta_1 > 0$, while $u = \bar{u}_0$ is a zero of order $k > 0$ of f in the following sense,*

$$\lim_{u\to\bar{u}_0} \frac{f(u)}{\varphi_{k+1}(\bar{u}_0 - u)} = C \tag{2.4}$$

for a certain nonzero constant C. Then the unique solution to (2.1), $u_0 = \bar{u}_0$, is the trivial one $u \equiv \bar{u}_0$ either if $C < 0$ or if $C > 0$ but the inequality $k \geq p - 1$ holds.

Remark 2.2.
Theorem 2.2 asserts that $k < p-1$ is a necessary condition in order to have a nontrivial response of the problem (2.1) with $u_0 = \bar{u}_0$, i. e. the existence of solutions other than $u \equiv \bar{u}_0$, when $\bar{u}_0$ is a zero for f.

We are next describing the main features of the"singular case" $f(\bar{u}_0) = 0$ of the Cauchy problem (2.1). To introduce our hypotheses on f let us first remark that only the case of finite order isolated zeros $u = \bar{u}_0$ for $f(u)$ will be treated here, for simplicity. On the other hand (cf. for instance the proof of Theorem 2.2 below) it can be shown that $f(u) > 0$ for $u < \bar{u}_0$, $u \sim \bar{u}_0$, is a necessary condition in order that $\bar{u}_0$ is a candidate for maximum of a positive solution $u = u(r)$ to (1.4). If it is assumed that $u = \bar{u}_0$ is of order k and we recall Remark 2.2 then the following conditions on $f = f(u)$ seem quite natural. Namely,

(H1) There exists $\bar{u}_0$ such that $f(\bar{u}_0) = 0$, while f is C^1 in $\bar{u}_0 - \delta_1 \le u < \bar{u}_0$ for some $\delta_1 > 0$.

(H2) $\bar{u}_0$ is a zero of order k for f in the sense $\lim_{u \to \bar{u}_0 -} -f'(u)/(k(\bar{u}_0 - u)^{k-1}) = \gamma$, for some positive γ and $0 < k < p-1$.

These conditions will be referred to as (H) in the sequel. Observe that (H) only restricts the behaviour of $f(u)$ in $u < \bar{u}_0$, $u \sim \bar{u}_0$, while the order k of $\bar{u}_0$ is even allowed to be a non integer positive number. Notice that in particular,

$$f(u) \sim \gamma(\bar{u}_0 - u)^k \qquad \text{as} \quad u \to \bar{u}_0 - \; . \tag{2.5}$$

Our main results concerning the singular case of (2.1) are contained in the following theorem.

Theorem 2.3 *Let us assume that f satisfies hypotheses (H). Then the singular initial value problem*

$$\begin{array}{l} -((r+d)^{N-1}\varphi_p(u'))' = (r+d)^{N-1} f(u) \\ u(0) = \bar{u}_0 \, , \; u'(0) = 0 \end{array} \tag{2.6}$$

admits, for each $d \ge 0$ a unique nontrivial solution u in $0 \le r \le \delta$, that is, $u(r) < \bar{u}_0$ for $0 < r \le \delta$, where δ can be chosen not depending on d. Moreover

$$\lim_{r \to 0+} \frac{\bar{u}_0 - u(r)}{r^\alpha} = \begin{cases} C_N, & d = 0 \\ C_1, & d > 0 \end{cases} \tag{2.7}$$

where $\alpha = p/p - k - 1$ and $C_N = (\gamma/((\alpha k + N)\alpha^{p-1}))^{1/(p-k-1)}$.

Remarks 2.3.
a) Under the hypotheses (H) on f a more precise information can be asserted on the singular initial value problem (2.1). Namely, the existence of a unique nontrivial solution u to (2.1) in an interval $[-\delta, 0]$ in the sense that $u(r) < \bar{u}_0$ for $-\delta \le r < 0$, which also satisfies the estimate (2.7).
b) The features drawn in Theorem 2.3 and a) follow from conditions imposed on f only if $u < \bar{u}_0$, $u \sim \bar{u}_0$. A more complete picture of the nontrivial solutions meeting the value $\bar{u}_0$ with zero derivative at $r = d$ can be given, if the behaviour of f near $\bar{u}_0$ covers "both sides" of $u = \bar{u}_0$. In other words, replace (H) by

(H1)' f is C^1 in $0 < |u - \bar{u}_0| < \delta_1$, with $f(\bar{u}_0) = 0$,

(H2)' $\bar{u}_0$ is of order $k > 0$, i. e. , $\lim_{u \to \bar{u}_0} -f'(u)/(k|u - \bar{u}_0|^{k-1}) = \gamma$, for some positive γ.

Under these conditions, a unique nontrivial solution $u_1(r) > \bar{u}_0$ meets $u = \bar{u}_0$ from $r < 0$, while a unique additional nontrivial solution $u_2(r) > \bar{u}_0$ leaves $u = \bar{u}_0$ to $r > 0$. The following statement, a singular version of the saddle-point property ([18]), gives a detailed description of the full set of solutions to (2.1) (see fig. 1). The results are expressed in terms of the original problem (1.5).

Corollary 2.4 *Assume that $f(u)$ satisfies (H)'. Then, for each $d \geq 0$ there exist $\delta = \delta(d)$ and unique solutions $u^-(\cdot, d-)$, $u^+(\cdot, d-)$ defined in $d - \delta \leq r \leq d$ and $u^-(\cdot, d+)$, $u^+(\cdot, d+)$ defined in $d \leq r \leq d + \delta$ to the singular problem (1.5) with $u_0 = \bar{u}_0$, which are nontrivial in the sense that $u^-(\cdot, d-) < \bar{u}_0$ (respectively $u^+(\cdot, d-) > \bar{u}_0$) in $d - \delta \leq r < d$ while $u^-(\cdot, d+) < \bar{u}_0$ (r. $u^+(\cdot, d+) > \bar{u}_0$) for $d < r \leq d + \delta$. In addition, $u^\pm(\cdot, \pm d)$ satisfy the estimates*

$$u^\pm(r, d \mp -) \sim \bar{u}_0 \pm C|d - r|^\alpha + o(|d - r|^\alpha) \quad \text{as } r \to d\mp \tag{2.8}$$

C and α being given in (2.7). Furthermore, for any other possible solution $u = u(r)$ to (1.5) defined in $d - \delta \leq r \leq d + \delta$ some $d - \delta \leq d_1 \leq d \leq d_2 \leq d + \delta$ exist such that either $u = u^-(\cdot, d_1-)$ or $u = u^+(\cdot, d_1-)$ in $d - \delta \leq r \leq d_1$, $u \equiv \bar{u}_0$ in $d_1 < r < d_2$ while either $u = u^-(\cdot, d_2+)$ or $u = u^+(\cdot, d_2+)$ in $d_2 \leq r \leq d + \delta$.

figure 1

Remark 2.4.
If the equation $(r^{N-1}\varphi_p(u'))' = -r^{N-1}f(u)$ is written as the one-dimensional system $u' = v$, $\varphi_p(v)' = -f(u) - \frac{N-1}{r}\varphi_p(v)$ and its phase plane is drawn under the convention that orbits start at $v = 0$ when the "time" $r = 0$, then it is found that $(u, v) = (u^\pm(\cdot, 0-), (u^\pm)'(\cdot, 0-))$ for $-\delta \leq r < 0$ while $(u, v) = (\bar{u}_0, 0)$ if $r \geq 0$, parametrizes a sort of degenerate stable manifold W_s associated to the critical point $(u, v) = (\bar{u}_0, 0)$. Similarly, a kind of degenerate unstable manifold W_u at $(u, v) = (\bar{u}_0, 0)$ can be obtained.

In view of the applications in §3 we need to consider the dependence of the Cauchy problem on initial data and parameters. To this aim we will also assume that f depends on a real parameter λ and verifies the following hypotheses:

(H1)$_\lambda$ $f = f(\lambda, u)$ vanishes at a fixed $u = \bar{u}_0$, and $f \in C(\mathbb{R}^+ \times [\bar{u}_0 - \delta_1, \bar{u}_0])$, $\partial f/\partial u \in C(\mathbb{R}^+ \times [\bar{u}_0 - \delta_1, \bar{u}_0))$ for some $\delta_1 > 0$.

(H2)$_\lambda$ $u = \bar{u}_0$ is uniformly of order k, $0 < k < p - 1$, in the sense that,

$$\frac{\partial f}{\partial u}(\lambda, u) = (-k\gamma(\lambda) + r(\lambda, u))(\bar{u}_0 - u)^{k-1},$$

where $\gamma = \gamma(\lambda)$ is continuous and positive in $\mathbb{R}^+$, $r = r(\lambda, u) \in C(\mathbb{R}^+ \times [\bar{u}_0 - \delta_1, \bar{u}_0])$ with $r(\lambda, \bar{u}_0) = 0$.

When studying the asymptotic behaviour for large λ we will require in addition, (H3)$_\lambda$ $\lim_{\lambda\to+\infty}\gamma(\lambda)=\bar\gamma>0$ and $r(\lambda,u)\to\bar r(u)$, uniformly in $[\bar u_0-\delta_1,\bar u_0]$, as $\lambda\to+\infty$.

After normalization, problem (1.2) is found to verify these hypotheses (see §4) showing the way in which conditions (H)$_\lambda$ are met in applications.

Then we have the following result.

Theorem 2.5 *Let us assume that f satisfies hypotheses (H1)$_\lambda$, (H2)$_\lambda$ while $u(r,u_0,\lambda,d)$ designates the solution (respectively nontrivial solution) to*

$$\begin{aligned} -((r+d)^{N-1}\varphi_p(u'))' &= (r+d)^{N-1}f(\lambda,u) \\ u(0)=u_0\,,\ u'(0) &= 0\,, \end{aligned} \tag{2.1$_\lambda$}$$

for $\bar u_0-\delta_1\le u_0<\bar u_0$ (resp. $u_0=\bar u_0$). Then, for $\varepsilon>0$ small there exists $\delta>0$ such that the mapping $u(\cdot,u_0,\lambda,d)$ is continuous from $(u_0,\lambda,d)\in[\bar u_0-\varepsilon,\bar u_0]\times\mathbb{R}^+\times[0,+\infty)$ with values in $C^1[0,\delta]$. Moreover, $u(\cdot,u_0,\lambda,d)$ converges in $C^1[0,\delta]$, as $d\to+\infty$ to the unique (nontrivial in case $u_0=\bar u_0$) solution to the one-dimensional problem,

$$\begin{aligned} -(\varphi_p(u'))' &= f(\lambda,u) \\ u(0)=u_0\,,\ u'(0) &= 0\,. \end{aligned} \tag{2.9}$$

If in addition f verifies (H3)$_\lambda$ then $u(\cdot,u_0,\lambda,d)$ converges in $C^1[0,\delta]$ as $d,\lambda\to+\infty$ to the unique (nontrivial if $u_0=\bar u_0$) solution to (2.9) with $f(\lambda,u)$ replaced by its uniform limit $\bar f(u)=(-\bar\gamma+\bar r_1(u))(u-\bar u_0)^k$, $\bar r_1$ being defined as $\bar r_1(u)=\frac{1}{(u-\bar u_0)^k}\int_{\bar u_0}^u \bar r(s)(s-\bar u_0)^{k-1}\,ds$.

Our next result extends the smoothness on initial data to differentiability with respect to d. Namely,

Theorem 2.6 *Let $\varepsilon>0$ and $u(r,u_0,\lambda,d)$ be as in the statement of Theorem 2.5. Then, for any $d_0>0$ there exists $\delta>0$ such that the mapping $d\to u(\cdot,u_0,\lambda,d)$ is differentiable with values in $C^1[0,\delta]$ for each $(u_0,\lambda,d)\in[\bar u_0-\varepsilon,\bar u_0]\times\mathbb{R}^+\times[d_0,+\infty)$. Moreover, $v=\frac{\partial u}{\partial d}(\cdot,u_0,\lambda,d)$ satisfies*

$$\begin{aligned} -((r+d)^{N-1}|u'|^{p-2}v')' &= \tfrac{1}{p-1}(r+d)^{N-1}\left(\tfrac{\partial f}{\partial u}(\lambda,u)v-\tfrac{N-1}{(r+d)^2}\varphi_p(u')\right) \\ v(0)=v'(0) &= 0\,, \end{aligned} \tag{2.10}$$

while the mapping $(u_0,\lambda,d)\to\frac{\partial u}{\partial d}(\cdot,u_0,\lambda,d)$ is continuous in $[\bar u_0-\varepsilon,\bar u_0]\times\mathbb{R}^+\times[d_0,+\infty)$ with values in $C^1[0,\delta]$. Furthermore, $\frac{\partial u}{\partial d}(\cdot,u_0,\lambda,d)\to 0$ as $\lambda,\ d\to+\infty$ in $C^1[0,\delta]$.

Remark 2.5.
As a byproduct of Theorem 2.6 it follows that the nontrivial solution $u=u(r,\bar u_0,\lambda,d)$ is smooth in the variables (r,d) which in turn will be essential for the uniqueness results in §§3 and 4.

2.2. The nonsingular case.
Proof of Theorem 2.1. Our proof is based upon contractivity of the operator T given in (2.3), when observed in $C[0,\delta]$ for δ small enough. If $\overline{B}_{\delta_1}(u_0):=\{u\in C[0,\delta]:\ |u-u_0|_\infty\le\delta_1\}$ stands for the closed ball of center u_0 and radius δ_1, it is easy to see that $T(\overline{B}_{\delta_1}(u_0))\subset\overline{B}_{\delta_1}(u_0)$ if δ is small. Indeed,

$$|u_0-Tu|_\infty\le M^{\frac{1}{p-1}}/p'\delta^{p'},$$

where $M := \sup\{|f(u)| : |u - u_0| \leq \delta_1\}$. Also, an application of the mean value theorem leads us to

$$|Tu - Tv|_\infty \leq \frac{L}{p-1} A^{p'-2}\delta^{p'}|u - v|_\infty,$$

where L is the Lipschitz constant of f in $|u - u_0| \leq \delta_1$ and $A = M$ if $1 < p \leq 2$, while $A = f(u_0)/2N$ when $p > 2$ (here δ_1 has to be restricted to have $f(u) \geq f(u_0)/2$ in $u_0 - \delta_1 \leq u \leq u_0 + \delta_1$).

As for estimate (2.2), it follows from the fixed point equation (2.3) by taking into account that $f(u) = f(u_0) + o(1)$ and $\varphi_{p'}(f(u)) = \varphi_{p'}(f(u_0)) + o(1)$ as $u \to u_0$.

Notice that similar arguments provide us with a unique solution in an interval of the form $[-\delta, \delta]$ with $\delta > 0$ small if δ additionally verifies, say, $0 < \delta < \frac{d}{2}$. □

Proof of Theorem 2.2. Let us begin with a general remark. If $u \in C^1$, $(r+d)^{N-1}\varphi_p(u') \in C^1$ is a classical solution to (2.1) in $0 \leq r \leq \delta$ then it can be checked that the expression $H(u, u') := |u'|^p + p'F(u)$, where $F(u) = \int_0^u f(s)ds$, is non-increasing in r.

If we now suppose that $C < 0$ in (2.4) then $H(u, u') = F(\bar{u}_0) + |u'|^p - C/(k+1)|u - \bar{u}_0|^{k+1} + o(|u - \bar{u}_0|^{k+1})$ reaches a local minimum at $(\bar{u}_0, 0)$, so $H(u(r), u'(r))$ keeps constant in $0 \leq r \leq \delta$ what implies $u \equiv \bar{u}_0$, as desired.

As for the case $C > 0$ in (2.4) and $k \geq p - 1$ notice that, since $H(u, u')$ is non increasing we have $-u'(r) \leq (p'(F(\bar{u}_0) - F(u)))^{1/p}$ for $0 \leq r \leq \delta$ and certain nontrivial solution u. This yields after integration,

$$\int_{u(r)}^{\bar{u}_0} \frac{ds}{(F(\bar{u}_0) - F(s))^{1/p}} \leq (p')^{1/p}(r - d)\,,$$

where $d = \sup\{r > 0 : u(r) = \bar{u}_0\}$ which is not possible since the first integral diverges. □

2.3. The singular case. One of the objectives of the present section is to provide a proof of Theorem 2.3. In this regard, such existence and uniqueness result for nontrivial solutions strongly relies upon finding a "selective enough" subspace $\mathcal{E}_\delta \subset C[0, \delta]$ where the nontrivial solutions u to (2.1) must lie. In addition, $\mathcal{E}_\delta$ must be chosen so that the operator (2.3) with $u_0 = \bar{u}_0$ keeps $\mathcal{E}_\delta$ invariant and is contractive there.

Accordingly, a first natural step towards the search of $\mathcal{E}_\delta$ is to find precise estimates of all possible nontrivial solutions u to (2.1) as $r \to 0+$. Such estimates are detailed in the next result.

Proposition 2.7 *Assume that f satisfies hypotheses (H) and let u be a nontrivial solution to (2.1) with $u_0 = \bar{u}_0$ and $u(r) < \bar{u}_0$ for $0 < r \leq \delta$. Then u verifies estimate (2.7).*

Proof. The strategy to achieve estimate (2.7) consists in a suitable use of the "normalized" problem obtained from (2.1) by dropping the higher order terms in $u - \bar{u}_0$. Namely,

$$\begin{aligned} -((r+d)^{N-1}\varphi_p(u'))' &= (r+d)^{N-1}\mu(\bar{u}_0 - u)^k \\ u(0) = \bar{u}_0\,,\ u'(0) &= 0, \end{aligned} \tag{2.11}$$

for $d \geq 0$ and $\mu > 0$. With respect to the one-dimensional case $N = 1$, it should be observed that problem (2.11), has a unique nontrivial solution explicitely given by

$$u_\mu(r) = \bar{u}_0 - \left(\frac{\mu}{(\alpha k + 1)\alpha^{p-1}}\right)^{\frac{1}{p-k-1}} r^\alpha. \tag{2.12}$$

For later use it is convenient to set the notation

$$T_\mu u(r) = \bar{u}_0 - \int_0^r \varphi_{p'}\left(\int_0^s \mu(\bar{u}_0 - u(\rho))^k \, d\rho\right) ds$$

where $u \in C[0,\delta]$.

Now notice that condition (2.5) on f implies that for every $\varepsilon > 0$ a certain $\delta' > 0$ can be chosen so that $(\gamma-\varepsilon)(\bar{u}_0 - u)^k \le f(u) \le (\gamma+\varepsilon)(\bar{u}_0 - u)^k$ for $0 \le \bar{u}_0 - u \le \delta'$. Let now u be any nontrivial solution to (2.1). If δ is small enough then $0 \le \bar{u}_0 - u(r) \le \delta'$ provided $0 \le r \le \delta$. Thus $f(u(\rho)) \le (\gamma+\varepsilon)(\bar{u}_0 - u(\rho))^k$ for every $0 \le \rho \le \delta$. Since u is a fixed point of the operator T it follows

$$u(r) \ge T_{\gamma+\varepsilon} u(r) = \bar{u}_0 - \int_0^r \varphi_{p'}\left(\int_0^s (\gamma+\varepsilon)(\bar{u}_0 - u(\rho))^k \, d\rho\right) ds,$$

that is, u is a supersolution of $T_{\gamma+\varepsilon}$ (cf. [23] for a general account on sub and supersolutions). By reducing δ if necessary, u is always comparable to a subsolution of $T_{\gamma+\varepsilon}$. Therefore we have $u \ge u_\mu$, u_μ given by (2.12) with $\mu = \gamma+\varepsilon$. Thus we have proved the estimate

$$\limsup_{r\to 0+} \frac{\bar{u}_0 - u(r)}{r^\alpha} \le \left(\frac{\gamma+\varepsilon}{(\alpha k+1)\alpha^{p-1}}\right)^{\frac{1}{p-k-1}}.$$

Letting ε go to zero, we obtain

$$\limsup_{r\to 0+} \frac{\bar{u}_0 - u(r)}{r^\alpha} \le C_1. \tag{2.13}$$

Let us proceed now to estimate the complementary inferior limit. We are considering in turn the cases $d > 0$ and $d = 0$. Let us begin with $d > 0$ and observe that $(\rho+d)/(s+d) \ge d/(d+\delta)$ for $0 \le s \le r \le \delta$. Thus

$$u(r) \le \bar{u}_0 - \int_0^r \varphi_{p'}\left(\int_0^s \left(\frac{d}{d+\delta}\right)^{N-1} (\gamma-\varepsilon)(\bar{u}_0 - u(\rho))^k \, d\rho\right) ds,$$

and u is a subsolution of T_μ with $\mu = \left(\frac{d}{d+\delta}\right)^{N-1}(\gamma-\varepsilon)$. On the other hand, it is easy to see that $\bar{u} = \bar{u}_0 - \eta(\bar{u}_0 - u)$ is a supersolution of T_μ for $\eta > 0$ small (notice that $\bar{u} \ge u$). Therefore, the existence of a unique fixed point u_μ of T_μ implies $u_\mu \ge u$. Now, the explicit expression (2.12) for u_μ implies

$$\liminf_{r\to 0+} \frac{\bar{u}_0 - u(r)}{r^\alpha} \ge \left(\frac{(\gamma-\varepsilon)}{(\alpha k+1)\alpha^{p-1}}\right)^{\frac{1}{p-k-1}} \left(\frac{d}{d+\delta}\right)^{\frac{N-1}{p-k-1}}.$$

Hence, the arbitrary choice of ε and δ and (2.13) lead to

$$\lim_{r\to 0+} \frac{\bar{u}_0 - u(r)}{r^\alpha} = C_1, \tag{2.14}$$

provided $d > 0$, which is a part of the estimate in (2.7).

To prove the inferior estimate in the case $d = 0$ we need to obtain first a solution to the problem (2.11) with a convenient growth near $r = 0$. In fact, by choosing sub and supersolutions of the form $u_i = \bar{u}_0 - \beta_i r^\alpha$ we have that (u_1, u_2) will be a pair of ordered sub and supersolutions if β_1, β_2 are chosen to satisfy

$$\beta_2 \leq \left(\frac{\mu}{(\alpha k + N)\alpha^{p-1}}\right)^{\frac{1}{p-k-1}} \leq \left(\frac{\mu}{(\alpha k + 1)\alpha^{p-1}}\right)^{\frac{1}{p-k-1}} \leq \beta_1.$$

Thus, we have a nontrivial solution $u = u_{d,\mu}$ to (2.11).

Now let $d = 0$ and u any nontrivial solution to (2.1) and keep the choice of ε and δ as above. For any $0 < d \leq \delta/2$ define $w(r) = \bar{u}_0$ for $0 \leq r \leq d$, $w(r) = u_{d,\gamma-\varepsilon}(r - d)$ if $d < r \leq d + \delta/2$. Using the fixed point equation it is readily seen that $u(r) < w(r)$ for $0 < r \leq d + \delta/2$. In particular, $\bar{u}_0 - u(r) > \bar{u}_0 - u_{d,\gamma-\varepsilon}(r - d)$ if $d \leq r \leq d + \delta/2$ for every $0 < d \leq \delta/2$. Therefore, estimate (2.14) leads to

$$\frac{\bar{u}_0 - u(r)}{r^\alpha} \geq \left(\frac{r-d}{r}\right)^\alpha \left(\frac{\gamma - \varepsilon}{(\alpha k + N)\alpha^{p-1}}\right)^{\frac{1}{p-k-1}} \tag{2.15}$$

for $d \leq r \leq d + \delta/2$. If we perform the choice $d = r/2$ in (2.15), take inferior limits as $r \to 0+$, followed by a further limit as $\varepsilon \to 0+$, we finally find that

$$\liminf_{r\to 0+} \frac{\bar{u}_0 - u(r)}{r^\alpha} \geq \left(\frac{1}{2}\right)^\alpha C_N.$$

Hence, we can conclude, via (2.13), that every nontrivial solution u to (2.1) with $d = 0$ satisfies,

$$0 < a := \liminf_{r\to 0+} \frac{\bar{u}_0 - u(r)}{r^\alpha} \leq b := \limsup_{r\to 0+} \frac{\bar{u}_0 - u(r)}{r^\alpha} < +\infty\ ,$$

that is, $\bar{u}_0 - (b + \varepsilon)r^\alpha \leq u \leq \bar{u}_0 - (a - \varepsilon)r^\alpha$ for $0 < r < \delta$, $\varepsilon, \delta > 0$ small. An iterated application of T to this inequality leads directly to the desired result, since $T^n u = u$ for every $n \in \mathbb{N}$. This finishes the proof of Proposition 2.7. □

Once estimates (2.7) near $r = 0$ for all nontrivial solutions to the singular initial value problem (2.1) have been established, we are ready to show the existence and uniqueness of such solutions.

Proof of Theorem 2.3. Observe that, provided f satisfies (H) and δ is small, Proposition 2.7 asserts that every nontrivial solution to (2.1) belongs to the space $\mathcal{E}_\delta := \{u \in C[0,\delta] : \sup_{0<r\leq\delta} |\bar{u}_0 - u(r)|/r^\alpha < +\infty\}$. It can be checked that $\mathcal{E}_\delta$ is a complete metric space when endowed with the metric $d_{\mathcal{E}_\delta}(u,v) = \sup_{0<r\leq\delta} |u(r) - v(r)|/r^\alpha$. In the light of Proposition 2.7 it is natural to consider the following closed part of $\mathcal{E}_\delta$: $B_\eta := \{u \in \mathcal{E}_\delta :\ C_1 - \eta \leq (\bar{u}_0 - u(r))/r^\alpha \leq C_1 + \eta,\ 0 < r \leq \delta\}$ where $\eta > 0$ (notice that B_η is nothing else but the closed ball with center $\bar{u}_0 - C_1 r^\alpha$ and radius η). Our task is to prove that for η, δ small enough the operator

$$Tu(r) = \bar{u}_0 - \int_0^r \varphi_{p'}\left(\int_0^s \left(\frac{\rho + d}{s + d}\right)^{N-1} f(u(\rho))\, d\rho\right) ds$$

is contractive in B_η. For the sake of simplicity it will be assumed henceforth that $d > 0$ and $k \geq 1$, the other cases being treated in a completely similar way.

First observe that conditions (H) on f imply, for every fixed $\varepsilon > 0$, each $u \in B_\eta$ and δ small that

$$\begin{aligned} (\gamma - \varepsilon)(C_1 - \eta)^k r^{\alpha k} \leq f(u(r)) \leq (\gamma + \varepsilon)(C_1 + \eta)^k r^{\alpha k} \\ (\gamma k - \varepsilon)(C_1 - \eta)^{k-1} r^{\alpha(k-1)} \leq -f'(u(r)) \leq (\gamma k + \varepsilon)(C_1 + \eta)^{k-1} r^{\alpha(k-1)} \end{aligned} \tag{2.16}$$

if $0 \leq r \leq \delta$. It is now a straightforward matter to prove that $T(B_\eta) \subset B_\eta$, by choosing δ small.

Indeed, notice that, as a consequence of (2.16) we have, for $u \in B_\eta$,

$$\bar{u}_0 - Tu(r) \leq \frac{1}{\alpha} \left(\frac{(\gamma + \varepsilon)(C_1 + \eta)^k}{\alpha k + 1} \right)^{\frac{1}{p-1}} r^\alpha .$$

Thus $\bar{u}_0 - Tu(r)/r^\alpha \leq C_1 + \eta$ provided that ε is chosen small enough. In a similar way, a lower estimate can be obtained in the form:

$$\frac{\bar{u}_0 - Tu(r)}{r^\alpha} \geq \frac{1}{\alpha} \left(\frac{(\gamma - \varepsilon)(C_1 - \eta)^k}{\alpha k + 1} \right)^{\frac{1}{p-1}} \left(\frac{d}{d + \delta} \right)^{\frac{N-1}{p-1}} ,$$

so that $Tu \in B_\eta$ provided ε and δ are taken small. It is clear that the choice of ε and δ does not depend on d if $d \geq d_0 > 0$.

Next we are proving that the operator T is contractive (with respect to the distance $d_{\mathcal{E}_\delta}$ introduced above) when restricted to B_η, by chosing η and δ small enough.

Let u, v be arbitrary functions in B_η. Then, by using the mean-value theorem,

$$\begin{aligned} &|Tu(r) - Tv(r)| \leq \\ &\int_0^r \left| \varphi_{p'} \left(\int_0^s \left(\frac{\rho + d}{s + d} \right)^{N-1} f(u(\rho))\, d\rho \right) - \varphi_{p'} \left(\int_0^s \left(\frac{\rho + d}{s + d} \right)^{N-1} f(v(\rho))\, d\rho \right) \right| ds \leq \\ &\frac{1}{p-1} \int_0^r |\xi(s)|^{p'-2} \int_0^s \left(\frac{\rho + d}{s + d} \right)^{N-1} |f(u(\rho)) - f(v(\rho))|\, d\rho\, ds, \end{aligned} \tag{2.17}$$

where $\xi(s)$ is a function comprised between the two integrals in the second line of (2.17). Let us estimate the term $|\xi(s)|^{p'-2}$. We have to distinguish between the cases $1 < p \leq 2$ and $p > 2$. For simplicity we will assume $p > 2$ in what follows. We have $p' - 2 < 0$ so that we need to estimate $\xi(s)$ from below:

$$\int_0^s \left(\frac{\rho + d}{s + d} \right)^{N-1} f(u(\rho))\, d\rho \geq \frac{(\gamma - \varepsilon)(C_1 - \eta)^k}{\alpha k + 1} \left(\frac{d}{d + \delta} \right)^{N-1} s^{\alpha k + 1}$$

the same estimate being valid for the integral involving $f(v)$. Thus

$$|\xi(s)|^{p'-2} \leq \left(\frac{(\gamma - \varepsilon)(C_1 - \eta)^k}{\alpha k + 1} \right)^{p'-2} \left(\frac{d}{d + \delta} \right)^{(N-1)(p'-2)} s^{(\alpha k + 1)(p'-2)}.$$

On the other hand, $f(u(\rho)) - f(v(\rho)) = f'(\chi(\rho))(u(\rho) - v(\rho))$, $\chi(\rho)$ being an intermediate function between $u(\rho)$ and $v(\rho)$. Then $|f(u(\rho)) - f(v(\rho))| \le (\gamma k + \varepsilon)(C_1 + \eta)^{k-1}\rho^{\alpha(k-1)}|u(\rho) - v(\rho)|$, and observing that $|u(\rho) - v(\rho)| \le \rho^\alpha d_{\mathcal{E}_\delta}(u, v)$, the last integral in (2.17) results estimated as follows

$$\int_0^s \left(\frac{\rho + d}{s + d}\right)^{N-1} |f(u(\rho)) - f(v(\rho))| \, d\rho \le \frac{(\gamma k + \varepsilon)(C_1 + \eta)^{k-1}}{\alpha k + 1} \, d_{\mathcal{E}_\delta}(u, v) s^{\alpha k + 1}.$$

By substituting all the estimates in (2.17) and setting $C(d) = (d/(d + \delta))^{(N-1)(p'-2)}$, we finally arrive at

$$d_{\mathcal{E}_\delta}(Tu, Tv) \le \frac{C(d)}{\alpha(p - 1)} \left(\frac{(\gamma - \varepsilon)(C_1 - \eta)^k}{\alpha k + 1}\right)^{p'-2} \frac{(\gamma k + \varepsilon)(C_1 + \eta)^{k-1}}{\alpha k + 1} d_{\mathcal{E}_\delta}(u, v),$$

and for $\varepsilon = \eta = \delta = 0$, the Lipschitz constant of T reduces to $k/(p-1) < 1$. Thus, choosing ε, η and δ small, T is also contractive in B_η. However notice that δ is independent of d only when bounded away from zero.

Finally, the local existence and uniqueness of nontrivial solutions to (2.1) is a consequence of Banach's contraction theorem. The slightly sharper assertion on the existence of a not depending on d common definition interval $[0, \delta]$, for the nontrivial solution to (2.1), $u_0 = \bar{u}_0$, will be shown later, in the course of the proof of Theorem 2.5. □

Proof of Corollary 2.4. If $v = v(r)$ designates the nontrivial solution to (2.6) obtained in Theorem 2.3 then it is rather clear thar $u^-(\cdot, d+)$ is given by

$$u^-(r, d+) = v(r - d) \qquad d \le r \le d + \delta\,.$$

To obtain $u^+(\cdot, d+)$ it must be shown the existence of a unique nontrivial solution $v_1(r) > \bar{u}_0$ in $0 < r \le \delta$ to (2.6) for some $\delta > 0$. In fact, since (H)' implies that $f(u)$ behaves in a symmetric way with respect to $\bar{u}_0$, $u \sim \bar{u}_0$, the change of the group $\bar{u}_0 - u$ by the symmetric one $u - \bar{u}_0$ in the proof of Theorem 2.3 leads to the existence of such a solution $v_1(r)$ in the same interval $[0, \delta]$. Thus,

$$u^+(r, d+) = v_1(r - d) \qquad d \le r \le d + \delta\,,$$

which satisfies (2.8). In the same way, the existence of $u^-(\cdot, d-)$ follows from the existence of a unique solution $v_2(r) < \bar{u}_0$ to (2.6) in $[-\delta, 0]$. Since $u^\pm(r, d-) = u^\pm(-r, d+)$ if $d = 0$ it will be assumed in the sequel that $d > 0$. On the other hand, the reflection $w_2(r) := v_2(-r)$ reduces the problem to find a solution $w_2(r) < \bar{u}_0$ in $[0, \delta]$ to (2.6) with $-d$ replacing d. As can be checked, this minor change does not substantially affect the arguments in the proof of Theorem 2.3, hence providing the existence of $u^-(\cdot, d-)$. The existence of $u^+(\cdot, d-)$ is obtained by arguing as in the first part of the proof.

Let us show now the last assertion of the statement. So, let $u \not\equiv \bar{u}_0$ be any solution to (1.5) defined in $|r - d| \le \delta$ and set $d_2 := \sup\,\{r \ge d : u \equiv \bar{u}_0 \text{ in } [d, r]\}$. No generality is lost by assuming the existence of a sequence $r_n > d_2$, $r_n \to d_2$ such that $u(r_n) < \bar{u}_0$, while $|u(r) - \bar{u}_0| \le K$ in $d_2 \le r \le d + \delta$, where $K > 0$ is such that $f > 0$ in $\bar{u}_0 - K \le u < \bar{u}_0$.

We claim that $u'(r_n) < 0$ for each n. In fact, if $u'(r_{n_0}) \ge 0$ for some n_0 then $u'(r) > 0$ when $r < r_{n_0}$, $r \sim r_{n_0}$ (use Theorem 2.1 in case of equality). Set now $r^* := \inf\,\{r < r_{n_0} :$

$u' > 0$ in $(r, r_{n_0})\}$. It is then clear that $r^* > d_2$ while necessarily $f(u(r^*)) > 0$ together with $u'(r^*) = 0$. Then, Theorem 2.1 entails that $u(r) < u(r^*)$ for $r > r^*$, r close enough to r^*, what is certainly impossible. This shows the claim and so $u'(r_n) < 0$ for each n.

Pick now any such r_n, say r_{n_1}, and set $d_2^* := \inf\{r < r_{n_1} : u' < 0 \text{ in } (r, r_{n_1})\}$. Necessarily $d_2^* \geq d_2$. We claim that $d_2 = d_2^*$ and therefore u must be the nontrivial solution to (2.6) corresponding to $d = d_2$ which is the objective of the assertion. To prove the claim suppose $d_2^* > d_2$. By using the previous arguments, that is only compatible with $u(d_2^*) = \bar{u}_0$. On the other hand, another $d_2 < r_{n_2} < d_2^*$ must exist so that $u'(r_{n_2}) < 0$. Then, Rolle's theorem will provide us with some $r_{n_2} < r_1^* < d_2^*$ such that $u'(r_1^*) = 0$ together with $u(r_1^*) < \bar{u}_0$. As already seen that is not possible and the claim is proved.

Finally, the existence of d_1 and the corresponding behaviour of u before d_1 is proven in a symmetric way. This concludes the proof of Corollary 2.4. □

2.4. Continuity with respect to initial data and parameters.

Proof of Theorem 2.5. Our first objective is to prove that every solution to (2.1) with initial datum $\bar{u}_0 - \varepsilon \leq u_0 \leq \bar{u}_0$ is defined in a common interval $[0, \delta]$, with δ not depending on d (this is completing the proof of that assertion in Theorem 2.3). Since the parameter λ is not relevant for the proof it will be dropped.

So, choose $\varepsilon > 0$ small for f to be decreasing in $(\bar{u}_0 - 2\varepsilon, \bar{u}_0]$, and denote $\underline{u} = u(\cdot, \bar{u}_0 - \varepsilon, \infty)$ and $\overline{u} = u(\cdot, \bar{u}_0, 0)$, both solutions being defined in an interval of the form $[0, \delta]$, where $\delta = \delta(\varepsilon)$. After a further restriction of δ it can be assumed that $\bar{u}_0 - 2\varepsilon \leq \underline{u} \leq \bar{u}_0$ in $[0, \delta]$. It is clear that $\underline{u}(r) < \overline{u}(r)$ for $0 \leq r \leq \delta$. Take $u_0 \in [\bar{u}_0 - \varepsilon, \bar{u}_0]$. Clearly, $\underline{u}$, $\overline{u}$ form a pair of sub and supersolutions for the fixed point equation

$$u(r) = u_0 - \int_0^r \varphi_{p'} \left(\int_0^s \left(\frac{\rho + d}{s + d} \right)^{N-1} f(u(\rho))\, d\rho \right) ds \tag{2.18}$$

and thus there is a solution v to (2.18) in $[0, \delta]$ which verifies $\underline{u}(r) \leq v(r) \leq \overline{u}(r)$. By local uniqueness, $v(r) = u(r, u_0, d)$ in an interval of the form $[0, \delta_1]$ for some $\delta_1 > 0$. However, this equality holds throughout $[0, \delta]$. Indeed, let $\delta_0 = \sup\{r_1 > 0 : \ v(r) = u(r, u_0, d),\ 0 \leq r \leq r_1\}$ and suppose $\delta_0 < \delta$. Then $u(r, u_0, d)$ is a solution to

$$\begin{aligned} -((r+d)^{N-1}\varphi_p(u'))' &= (r+d)^{N-1} f(u) \\ u(\delta_0) = v(\delta_0)\,,\ & u'(\delta_0) = v'(\delta_0). \end{aligned} \tag{2.19}$$

If $v'(\delta_0) \neq 0$ then as a consequence of the standard ode's theory we have that $u(r, u_0, d) = v(r)$ for $r > \delta_0$, close to δ_0, which contradicts the maximality of δ_0. In case of $v'(\delta_0) = 0$, observe that $v(\delta_0) \leq \overline{u}(\delta_0) < \bar{u}_0$ and thus, by Theorem 2.1, (2.19) admits again a unique solution to the right of δ_0, contradicting the choice of δ_0. Thus $\delta_0 = \delta$ and the assertion is proved.

We have found that $\underline{u}(r) \leq u(r, u_0, d) \leq \overline{u}(r)$ for every $0 \leq r \leq \delta$, $\bar{u}_0 - \varepsilon \leq u_0 \leq \bar{u}_0$ and $d \geq 0$. This, in combination with representation of u in (2.18) leads to uniform bounds of $u(\cdot, u_0, d)$ and its derivative $u'(\cdot, u_0, d)$. Therefore (Ascoli-Arzelá), the solution set $\{u(\cdot, u_0, d) : \ \bar{u}_0 - \varepsilon \leq u_0 \leq \bar{u}_0,\ d \geq 0\}$ is precompact in $C[0, \delta]$. By poceeding as in [17, Theorem 3, p. 17] this entails the continuous dependence of $u(\cdot, u_0, d)$ with respect to (u_0, d).

Finally, it should be remarked that $u(\cdot, u_0, d) \to \hat{u}$ in $C^1[0, \delta]$ as $d \to \infty$, where $\hat{u}$ solves,

$$\hat{u}(r) = u_0 - \int_0^r \varphi_{p'} \left(\int_0^s f(\hat{u}(\rho))\, d\rho \right) ds\,.$$

Observe that $\hat{u}$ defines the unique solution to the one-dimensional problem (2.9). $\square$

2.5. Differentiability with respect to d. To prove Theorem 2.6 we are only considering the more complicated case of the nontrivial solution $u(r, \bar{u}_0, d)$ (λ will be removed from f for the sake of brevity). It is also convenient to introduce some preliminary properties. First of all, if equation (2.1) is differentiated with respect to d and the function $\frac{\partial u}{\partial d}(r, u_0, d)$ is denoted by v, we obtain

$$\begin{array}{c} -((r+d)^{N-1}|u'|^{p-2}v')' = \frac{1}{p-1}(r+d)^{N-1}\left(f'(u)v - \frac{N-1}{(r+d)^2}\varphi_p(u')\right) \\ v(0) = v'(0) = 0. \end{array} \tag{2.20}$$

For simplicity let us designate by $h(r) = \frac{1}{p-1}\frac{N-1}{(r+d)^2}\varphi_p(u'(r))$ and $g(r) = -\frac{1}{p-1}f'(u(r))$. Remark that $h(r) \sim \frac{N-1}{d^2(p-1)}(C_1\alpha)^{p-1}r^{\alpha k+1}$, while $g(r) \sim \frac{1}{p-1}\gamma k C_1^{k-1} r^{\alpha(k-1)}$ as $r \to 0+$. Moreover, as a consequence of the proof of Theorem 2.3, it can be further asserted that for an arbitrary $\varepsilon > 0$ some $\delta > 0$ small enough exists such that

$$\frac{|g(r)|}{r^{\alpha(k-1)}} \le \frac{\gamma k C_1^{k-1}}{p-1} + \varepsilon, \qquad \frac{r^{(\alpha-1)(p-2)}}{|u'(r)|^{p-2}} \le \frac{1}{(C_1\alpha)^{p-2}} + \varepsilon \tag{2.21}$$

for $0 < r \le \delta$ not depending on $d \ge d_0$, for some fixed $d_0 > 0$.

On the other hand, one must find, once again, a convenient subspace of $C[0, \delta]$ where the incremental quotients

$$\partial_h u := \frac{u(\cdot, \bar{u}_0, d+h) - u(\cdot, \bar{u}_0, d)}{h} \tag{2.22}$$

lie, and where they are expected to converge to $\partial u/\partial d$ as $h \to 0+$. It is easily seen that $\partial_h u \in \mathcal{F}_\delta = \{u \in C[0, \delta] :\ \sup_{0<r\le\delta} |u(r)|/r^\alpha < +\infty\}$, with α as in Theorem 2.3. We provide $\mathcal{F}_\delta$ with a norm given by $\|u\| = \sup_{0<r\le\delta} |u(r)|/r^\alpha$. $\mathcal{F}_\delta$ is in fact a Banach space as can be checked.

Our immediate step is to show that a unique solution to (2.20) can indeed be found in $\mathcal{F}_\delta$.

Lemma 2.8 *For every $d_0 > 0$, there exists $\delta > 0$ such that problem (2.20) has a unique solution v_d in $\mathcal{F}_\delta$ for $d \ge d_0$.*

Proof. Our intention is to prove that the operator

$$\mathcal{T}v(r) = \int_0^r \frac{1}{|u'(s)|^{p-2}} \int_0^s \left(\frac{\rho+d}{s+d}\right)^{N-1} (h(\rho) + g(\rho)v(\rho))\, d\rho\, ds$$

is contractive in $\mathcal{F}_\delta$. It is an easy matter to check that $\mathcal{T}$ is well defined. Indeed, for $v \in \mathcal{F}_\delta$, we have thanks to (2.21) the inequality

$$|\mathcal{T}v(r)| \le \frac{1}{\alpha}\frac{(p-1)D + (\gamma k C_1^{k-1} + \varepsilon(p-1))\|v\|}{(p-1)(\alpha k+1)}\left(\frac{1}{(C_1\alpha)^{p-2}} + \varepsilon\right) r^\alpha,$$

where $D := \sup_{0<r\le\delta} h(r)/r^{\alpha k}$. Thus $\|\mathcal{T}v\| < +\infty$. Proving the contractivity is now easier than in §§ 2.2 and 2.3. For if $v, w \in \mathcal{F}_\delta$ then

$$|\mathcal{T}v(r) - \mathcal{T}w(r)| \le \frac{1}{\alpha}\frac{\gamma k C_1^{k-1} + \varepsilon(p-1)}{(p-1)(\alpha k+1)}\left(\frac{1}{(C_1\alpha)^{p-2}} + \varepsilon\right)\|v-w\| r^\alpha$$

and it follows that $\mathcal{T}$ is Lipschitz continuous, with a constant that for $\varepsilon = 0$ reduces to $k/p - 1 < 1$. Thus, taking ε (and consequently δ) small, $\mathcal{T}$ is contractive on $\mathcal{F}_\delta$, independently of $d \geq d_0$, and the proof is finished with the aid of Banach's contraction theorem. $\square$

Let us proceed now to obtain estimates of the incremental quotients $\partial_h u$ given by (2.22). For simplicity denote $u_d = u(\cdot, \bar{u}_0, d)$. It turns out that the optimal estimates involve the norm $\|\cdot\|$.

Lemma 2.9 *For $d \geq d_0$, $d_0 > 0$ fixed and h_0 small there exists $M > 0$ not depending on d such that $\|\partial_h u\| \leq M$ for every $0 < |h| \leq h_0$.*

Proof. By subtraction of the fixed point equations verified by u_{d+h} and u_d and an iterated application of the mean value theorem we arrive at the equality

$$\begin{aligned} \partial_h u(r) &= \frac{-1}{p-1} \int_0^r |\xi_h|^{p'-2} \int_0^s \left[\left(\frac{\rho + d}{s + d} \right)^{N-1} f'(\eta_h) \partial_h u \right. \\ &+ \left. (N-1) \frac{s - \rho}{(s + \zeta_h)^2} \left(\frac{\rho + \zeta_h}{s + \zeta_h} \right)^{N-2} f(u_{d+h}) \right] d\rho\, ds. \qquad (2.23) \end{aligned}$$

where $\xi_h = \xi_h(s)$ is an intermediate function between the two integrals with respect to ρ in the corresponding fixed point equations; $\eta_h(\rho)$ is comprised between $u_d(\rho)$ and $u_{d+h}(\rho)$ and ζ_h between d and $d + h$.

Let us estimate now these integrals. Recall that for $\varepsilon > 0$ there exists $\delta > 0$ such that $C_1 - \varepsilon \leq \bar{u}_0 - u_d(r)/r^\alpha \leq C_1 + \varepsilon$, if $0 < r \leq \delta$, independently of d in $[d_0, +\infty)$. Actually, this fact is implicit in the proof of Theorem 2.3. Thus using inequality (2.16), $|f'(\eta_h(\rho)| \leq (\gamma k + \varepsilon)(C_1 + \varepsilon)^{k-1} \rho^{\alpha(k-1)}$, for $0 \leq \rho \leq \delta$. In addition, the same argument leads to $|f(u_{d+h}(\rho))| \leq (\gamma + \varepsilon)(C_1 + \varepsilon)^k \rho^{\alpha k}$ for $0 \leq \rho \leq \delta$. Hence,

$$|\partial_h u(r)| \leq \frac{1}{p-1} \int_0^r |\xi|^{p'-2} \left(\frac{(\gamma k + \varepsilon)(C_1 + \varepsilon)^{k-1}}{\alpha k + 1} \|\partial_h u\| + \frac{(N-1)(\gamma + \varepsilon)(C_1 + \varepsilon)^k}{d^2(\alpha k + 1)} s \right) s^{\alpha k + 1} ds.$$

It should be recalled that, as noted before, $\partial_h u \in \mathcal{F}_\delta$ for every h. It is time now to estimate $\xi_h(s)$. Since $u_{d+h} \to u_d$ in $C^1[0, \delta]$ as $h \to 0$ (cf. Theorem 2.5), then for $\varepsilon > 0$ fixed, we have for h small enough

$$(1 - \varepsilon) \int_0^s \left(\frac{\rho + d}{s + d} \right)^{N-1} f(u_d(\rho))\, d\rho \leq \int_0^s \left(\frac{\rho + d + h}{s + d + h} \right)^{N-1} f(u_{d+h}(\rho))\, d\rho \leq$$

$$(1 + \varepsilon) \int_0^s \left(\frac{\rho + d}{s + d} \right)^{N-1} f(u_d(\rho))\, d\rho.$$

On the other hand notice that this last integral equals $-\varphi_p(u_d')$, and hence $|\xi(s)|^{p'-2} \leq (1 - \varepsilon)^{p'-2} / |u_d'(s)|^{p-2}$ for $p > 2$ (for $1 < p \leq 2$ a similar argument can be given). By performing a further restriction on δ as to achieve (2.21) we arrive at

$$|\partial_h u(r)| \leq \frac{1}{p-1} \left(\frac{(\gamma k + \varepsilon)(C_1 + \varepsilon)^{k-1}}{\alpha k + 1} \|\partial_h u\| + \frac{(N-1)(\gamma + \varepsilon)(C_1 + \varepsilon)^k}{d^2(\alpha k + 1)} r \right).$$

$$\cdot \frac{(1-\varepsilon)^{p'-2}}{\alpha}\left(\frac{1}{(C_1\alpha)^{p-2}}+\varepsilon\right) r^{\alpha},$$

which in turn implies for some $B > 0$ that $\|\partial_h u\| \leq A(\varepsilon)\|\partial_h u\| + B$. Setting $\varepsilon = 0$ we have $A(0) = k/p - 1 < 1$, so that fixing ε small one has $\|\partial_h u\| \leq A\|\partial_h u\| + B$, for some $0 < A < 1$. This entails the existence of $M > 0$ such that $\|\partial_h u\| \leq M$, independently of h. □

Proof of Theorem 2.6. Since $|\partial_h u|_\infty \leq \delta^\alpha \|\partial_h u\| \leq \delta^\alpha M$, we also obtain a uniform estimate for $|\partial_h u|_\infty$ what, in view of (2.23), leads to an uniform estimate for $|(\partial_h u)'|_\infty$. Thus we have shown that every sequence $\{\partial_{h_n} u\}$ with $h_n \to 0$ admits a subsequence, again denoted by $\{\partial_{h_n} u\}$, which converges to some $\hat{v} \in C[0,\delta]$. To finish the proof it suffices with seeing that $\hat{v} = v_d$, the unique solution of (2.20) in $\mathcal{F}_\delta$.

To show the assertion first observe that $\partial_{h_n} u \to \hat{v}$ in $C[0,\delta]$ and the estimate for $\|\partial_{h_n} u\|$ imply that $\hat{v} \in \mathcal{F}_\delta$. On the other hand, set, in the fixed point expression (2.23) for $v_n := \partial_{h_n} u$ the functions $\xi_n := \xi_{h_n}$, $\eta_n = \eta_{h_n}$ and $\zeta_n := \zeta_{h_n}$. Since $\eta_n(\rho)$ is an intermediate function between $u_d(\rho)$ and $u_{d+h_n}(\rho)$ it follows that both $\eta_n \to u_d$ and $f'(\eta_n) \to f'(u_d)$ uniformly in $[0,\delta]$, and by the same reason $\xi_n \to -\varphi_p(u_d')$ uniformly in $[0,\delta]$. Thus taking limits in (2.23) we arrive at

$$\hat{v}(r) = -\frac{1}{p-1}\int_0^r \frac{1}{|u_d'(s)|^{p-2}}\int_0^s \left[\left(\frac{\rho+d}{s+d}\right)^{N-1} f'(u_d(\rho))\hat{v}(\rho) + (N-1)\frac{s-\rho}{(s+d)^2}\left(\frac{\rho+d}{s+d}\right)^{N-2} f(u_d(\rho))\right] d\rho\; ds, \tag{2.24}$$

since $\zeta_n \to d$. However transforming the last integral in ρ by means of an integration by parts we arrive at

$$\hat{v} = -\frac{1}{p-1}\int_0^r \frac{1}{|u_d'(s)|^{p-2}}\int_0^s \left(\frac{\rho+d}{s+d}\right)^{N-1}\left[f'(u_d(\rho))\hat{v}(\rho) - \frac{N-1}{(\rho+d)^2}\varphi_p(u_d'(\rho))\right] d\rho\; ds.$$

Since, as has been already seen, $\hat{v} \in \mathcal{F}_\delta$, this means that $\hat{v} = v_d$, the unique solution to (2.20) in $\mathcal{F}_\delta$. Thus, $\partial_h u \to v_d$ in $C[0,\delta]$. As before, using the fixed point equation (2.23) it follows that $(\partial_h u)' \to v_d'$ as $h \to 0$ in $C[0,\delta]$.

As for the continuity of v_d as a function of (u_0, d) and values in $C^1[0,\delta]$ observe that the estimate $\|\partial_h u\| \leq M$ leads to the same uniform estimate for $v_d(\cdot, u_0, d)$ with $(u_0, d) \in [\bar{u}_0 - \varepsilon, \bar{u}_0] \times [d_0, +\infty)$. Using the fixed point equation $v_d = \mathcal{T}(v_d)$ it follows that $\{v_d(\cdot, u_0, d)\}$ is equicontinuous and so precompact in $C[0,\delta]$. Standard arguments lead to the continuity of v_d in (u_0, d). More importantly, the limit value of $\mathcal{T}$ as $d \to +\infty$ leads to the operator,

$$\bar{\mathcal{T}}v = \frac{1}{p-1}\int_0^r \frac{1}{|u'(s)|^{p-2}}\int_0^s f'(u(\rho))v(\rho)\; d\rho\; ds\,,$$

where u stands for the unique (nontrivial if $u_0 = \bar{u}_0$) solution of (2.9). The unique fixed point of $\bar{\mathcal{T}}$ in $\mathcal{F}_\delta$ is $v = 0$. Therefore, $v_d(\cdot, u_0, d) \to 0$ in $C^1[0,\delta]$ as $d \to +\infty$. The proof of Theorem 2.6 is now finished. □

3. Some uniqueness results on simple homogeneization

In this section we are dealing with the radial positive solutions for λ large to

$$\begin{aligned} -\Delta_p u = \lambda f(\lambda, u) \quad &\text{in} \quad D \\ u = 0 \quad &\text{on} \quad \partial D, \end{aligned} \tag{1.4}$$

where D is a radially symmetric bounded domain in $\mathbb{R}^N$. It will be assumed that $f(\lambda, \cdot)$ admits a positive zero $\bar{u}_0$ of order $k \in \mathbb{R}$ for all λ (it may have other zeros) and we are examining in detail those families $\{u_\lambda\}_{\lambda \geq \lambda_0}$ of positive radial solutions to (1.4) which satisfy $\sup_B u_\lambda \leq \bar{u}_0$, together with

$$\lim_{\lambda \to +\infty} u_\lambda(x) = \bar{u}_0, \tag{3.1}$$

uniformly over compact sets $K \subset D$ (such $\{u_\lambda\}_{\lambda \geq \lambda_0}$ will be termed here as a *homogenizing* family). Theorems 3.4 and 3.5 below provide the existence of a unique such family when f satisfies quite "natural" conditions.

3.1. Preliminary results. Before proceeding further, some preliminary remarks concerning the regularity and geometry of radial solutions to (1.4), under hypotheses $(\mathrm{H})_\lambda$ on f, are in order.

Firstly, if $u \in W_0^{1,p}(D)$ is a nonnegative weak solution to (1.4), such that $0 \leq u(x) \leq \bar{u}_0$ in D, then $u \in C^{1,\beta}(\overline{D})$ for some $0 < \beta < 1$ (see §3.7 in [24] and [25]). If in addition u is radially symmetric and we set $u(x) = u(r)$, $r = |x|$, it can be checked that both u, and $r^{N-1}\varphi_p(u')$ are C^1 in $0 \leq r \leq R$, if $D = B$, while u, $\varphi_p(u')$ lie in $C^1[a, R]$ in case of $D = A$. In balls B, $u = u(r)$ defines a classical solution to the problem,

$$\begin{aligned} -(r^{N-1}\varphi_p(u'))' = \lambda r^{N-1} f(\lambda, u), \qquad & 0 < r < R \\ u'(0) = u(R) = 0, \qquad & \end{aligned} \tag{3.2}$$

the corresponding assertion being true in annuli A but $u(r)$ defined instead in $a \leq r \leq R$ with boundary conditions $u(a) = u(R) = 0$.

Secondly, any radial positive solution u to (1.4) in the case of balls B is so that $u_0 := u(0) = \sup_B u = \sup_{0 \leq r \leq 1} u(r)$. In addition, $u(r)$ is nonincreasing in $0 \leq r \leq R$. Indeed, both assertions follow from the fact that $H(\lambda, u, u') = |u'|^p + p'\lambda F(\lambda, u)$, where $F(\lambda, u) = \int_0^u f(\lambda, s) ds$, is a nonincreasing function of r (cf. §2). Thus positive radial solutions u to (1.4) are parametrized by their maxima u_0 through the initial value problem,

$$\begin{aligned} -(r^{N-1}\varphi_p(u'))' = \lambda r^{N-1} f(\lambda, u), \qquad & 0 < r < R \\ u(0) = u_0\,, \ u'(0) = 0\,. \qquad & \end{aligned} \tag{3.3}$$

Similarly, if D is an annulus A and $0 < u(r) \leq \bar{u}_0$ solves (1.4) with $u_0 := \sup_A u = u(r_0)$, for some $a < r_0 < R$, it can be shown that u does not increase (respectively, does not decrease) in $r_0 \leq r \leq R$ ($a \leq r \leq r_0$). Thus $u(r)$ can be observed as a solution to (3.3) in $a < r < R$ replacing the initial value $r = 0$ by $r = r_0$.

On the other hand, the behaviour of $H(\lambda, u, u')$ also entails that $F(\lambda, u) \leq F(\lambda, u_0)$ in $0 \leq u < u_0$. As we have in mind (3.1), $u_0 = u_0(\lambda) \to \bar{u}_0$ as $\lambda \to +\infty$ then $\bar{F}(u) \leq \bar{F}(\bar{u}_0)$ for $0 \leq u < \bar{u}_0$, $\bar{F}$ denoting the uniform limit of $F(\lambda, \cdot)$ as $\lambda \to +\infty$. We are further requiring the stronger condition

$$\bar{F}(u) < \bar{F}(\bar{u}_0) \qquad \text{for every } 0 \leq u < \bar{u}_0. \tag{3.4}$$

In fact, if f does not depend on λ and $N \geq 2$, a careful analysis permits showing (compare with the case $p = 2$ in [26]) that condition (3.4) is necessary in order to have a family verifying (3.1).

On the other hand, problem (1.4) may exhibit, for certain solutions u, dead cores $\mathcal{O} = \{x \in D : u(x) = \bar{u}_0\}$ which may have nonempty interior (see §1). Of course, this requires that $\bar{u}_0$ is a zero for $f(\lambda, \cdot)$, while a necessary condition for the existence of such dead cores $\mathcal{O}$ is the balance $0 < k < p - 1$ between p and the order k of the zero $\bar{u}_0$ of $f(\lambda, \cdot)$ ([8, 7]).

As a summary of the preceding observations it turns out quite reasonable to assume that, aside $(\mathrm{H})_\lambda$ in §2.1, the following condition on f also holds,

$(\mathrm{H4})_\lambda$ $f(\lambda, \cdot)$ is C^1 in $\mathbb{R}$ while $\bar{F}(u) = \int_0^u \bar{f}(s)\, ds$ satisfies the inequality (3.4).

Remarks 3.1.
a) $(\mathrm{H4})_\lambda$ implies that the condition $F(\lambda, u) < F(\lambda, u_0)$ if $0 \leq u < u_0$ also holds for $u_0 < \bar{u}_0$, $u_0 \sim \bar{u}_0$ and large λ. This is coherent with the expected homogenizing behaviour (3.1).
b) In the one-dimensional case $N = 1$ of (1.4) explicit examples can be constructed where the weaker condition $\bar{F}(u) \leq \bar{F}(\bar{u}_0)$ in $0 \leq u < \bar{u}_0$ still allows the existence of infinitely many homogenizing families $\{u_\lambda\}$ (cf. [27]). This is in strong contrast with our uniqueness result contained in Theorem 3.4 below.

3.2. The case of balls. Our first result asserts that every positive radial solution to (1.4) in a homogenizing family $\{u_\lambda\}$ develops a dead core $\mathcal{O}_\lambda = \{x : \ u_\lambda(x) = \bar{u}_0\}$ for λ large enough when D is a ball.

Lemma 3.1 *Assume that f satisfies $(\mathrm{H})_\lambda$ and let $\{u_\lambda\}$ be a homogenizing family of positive radial solutions to (1.4) in B. Then there exists $\lambda_0 > 0$ such that the set $\{x : \ u_\lambda(x) = \bar{u}_0\}$ is nonempty for every $\lambda > \lambda_0$.*

Proof. Firstly observe that by setting $w(r) = u(\lambda^{-1/p} r)$, (3.3) can be written as

$$\begin{aligned} -(r^{N-1}\varphi_p(w'))' &= r^{N-1} f(\lambda, w) \\ w(0) = u_0\,, \ w'(0) &= 0. \end{aligned} \tag{3.5}$$

Thus $w(\lambda^{1/p} R) = 0$ if $u = u(r)$ is a radial solution to (1.4).

Let us next consider any sequence $\lambda_n \to +\infty$ and set $\overline{u}_{0n} = \sup_B u_{\lambda_n}$. Then $w_n(r) = u_{\lambda_n}(\lambda_n^{-1/p} r)$ solves the problem (3.5) with $u_0 = \overline{u}_{0n}$ while $w_n(r) > 0$ for $0 \leq r < \lambda_n^{1/p} R$ and $w_n(\lambda_n^{1/p} R) = 0$. Our objective is to prove that the homogeneization (3.1) is not compatible with the fact $\overline{u}_{0n} < \bar{u}_0$ for each n, and so dead cores $\{u_\lambda(x) = \bar{u}_0\}$ must arise for λ large enough.

By the homogeneization hypothesis (3.1), fixing any $r_0 > 0$ and $\varepsilon > 0$ so small as desired, we can achieve

$$\bar{u}_0 - \varepsilon \leq w_\lambda(r) \leq \bar{u}_0 \qquad \text{in } 0 \leq r \leq r_0\ , \tag{3.6}$$

provided λ is larger than a certain $\lambda_0 = \lambda_0(r_0, \varepsilon)$. Theorem 2.5 (see also §2.6) then implies that $w_n \to \bar{w}$ in $C^1[0, \delta]$ for a certain $\delta > 0$, where $\bar{w}$ is the unique nontrivial solution to the problem

$$\begin{aligned} -(r^{N-1}\varphi_p(w'))' &= r^{N-1} \bar{f}(w) \\ w(0) = \bar{u}_0\,, \ w'(0) &= 0. \end{aligned} \tag{3.7}$$

Since $\bar{w}(\delta) < \bar{u}_0$, $\varepsilon > 0$ can be taken so that $w_n(\delta) \leq \bar{u}_0 - 2\varepsilon$ for n large, what clearly contradicts (3.6). □

In order to state our next result let us introduce some notation. Let u be any positive radial solution to (1.4) exhibiting a dead core $\mathcal{O}$. Since u is nonincreasing in $|x|$ then $\mathcal{O} = B_{R_0}(0)$, for some $R_0 > 0$. This means that the alternative expression $w(r) = u(\lambda^{-1/p} r)$ as a solution of (3.5) in $0 \leq r \leq \lambda^{1/p} R$ satisfies $w = \bar{u}_0$ if $r \leq d$, while $0 < w < \bar{u}_0$ in $d < r < \lambda^{1/p} R$, d being given by $d = \lambda^{1/p} R_0$. Therefore, $v(r) = w(r + d)$ defines on the interval $[0, \lambda^{1/p} R - d]$ a nontrivial solution to the problem $(2.1)_\lambda$ with $u_0 = \bar{u}_0$, i.e., v solves

$$\begin{aligned} &-((r+d)^{N-1}\varphi_p(v'))' = (r+d)^{N-1} f(\lambda, v) \\ &v(0) = \bar{u}_0\,,\ \ v'(0) = 0\,. \end{aligned} \qquad (2.1)_\lambda$$

The next result gives a first insight of the behaviour of $d = d(\lambda)$ in a homogenizing family $\{u_\lambda\}$ of (1.4).

Lemma 3.2 *Let $\{u_\lambda\}$, be a family of positive radial solutions to (1.4) verifying (3.1). Then $d(\lambda) \to +\infty$ as $\lambda \to +\infty$.*

Proof. Assume the existence of a sequence $\lambda_n \to +\infty$ such that $d_n := d(\lambda_n)$ remains bounded. Then passing to a subsequence if necessary we find that $d_n \to d_0 \geq 0$. By setting $v_n(r) = u_{\lambda_n}(\lambda_n^{-1/p}(r + d_n))$, then Theorem 2.5 implies that $v_n \to \bar{v}$ in $C^1[0, \delta]$ for some small positive δ, where $\bar{v}$ is the nontrivial solution to (2.1) with $d = d_0$ and f replaced by $\bar{f}$. However $\bar{v}(\delta) < \bar{u}_0$, and this contradicts the fact that $v_n(\delta) \to \bar{u}_0$. Therefore $d(\lambda) \to +\infty$ as $\lambda \to +\infty$. □

A major conclusion of Lemmas 3.1 and 3.2 is the fact that the relevant behaviour of radial positive solutions $u(r)$ to (1.4) occurs in the interval $\lambda^{-1/p} d < r \leq R$. When such solutions u are expressed as $v(r) = u(\lambda^{-1/p}(r+d))$ their behaviour is dictated by the nontrivial solutions to $(2.1)_\lambda$ as $\lambda, d \to +\infty$ (cf. §2). Conversely, the problem of finding such families $\{u_\lambda\}$ can be read as constructing nontrivial solutions v to $(2.1)_\lambda$ such that $v(T) = 0$ for some suitable first positive $T = T(\lambda, d)$, as $\lambda, d \to +\infty$. Thus, let us proceed now to perform a detailed analysis of these solutions.

Firstly, Theorems 2.3 and 2.5 ensure that a nontrivial positive solution $v(r, \lambda, d)$ to $(2.1)_\lambda$ exists such that $v(\cdot, \lambda, d)$ converges in $C^1[0, \delta]$ as $\lambda, d \to +\infty$ for some $\delta > 0$, to the unique nontrivial solution v_∞ of the one-dimensional problem (1.6). Due to $(\mathrm{H4})_\lambda$, v_∞ can be extended to a positive solution in $0 \leq r \leq T_0$, T_0 being given by the (convergent) improper integral $T_0 = p'^{-1/p} \int_0^{\bar{u}_0} 1/(\bar{F}(\bar{u}_0) - \bar{F}(s))^{1/p}\, ds$. Observe that v_∞ can be further continuated to $[0, T_0 + \varepsilon]$, $\varepsilon > 0$ so small as to have $v_\infty' < 0$ in $0 < r \leq T_0 + \varepsilon$.

We are next showing that the local nontrivial solution $v(r, \lambda, d)$ to $(2.1)_\lambda$ can be extended to the whole interval $0 \leq r \leq T_0 + \varepsilon$ for ε small and λ, d large enough. Otherwise some sequences $\lambda_n, d_n \to +\infty$ and $\delta \leq r_n \leq T_0 + \varepsilon$ exist such that $v_n'(r_n) = 0$ ($v_n = v(\cdot, \lambda_n, d_n)$). Passing through a subsequence we may assume $r_n \to r_0$. Define,

$$\hat{v}_n(r) = \begin{cases} v_n(r), & 0 \leq r \leq r_n \\ v_n(r_n), & r_n \leq r \leq T_0 + \varepsilon. \end{cases}$$

Ascoli-Arzelá's theorem then asserts that $\{\hat{v}_n\}$ is precompact in $C[0, T_0 + \varepsilon]$. Thus a subsequence $\hat{v}_{n_k}$ can be found so that $\hat{v}_{n_k} \to \hat{v}$ in $C[0, T_0 + \varepsilon]$. Hence the equality

$$\hat{v}(r) = \bar{u}_0 - \int_0^r \varphi_{p'} \left(\int_0^s \bar{f}(\hat{v}(\rho))\, d\rho \right) ds\,,$$

holds for every $0 \leq r < r_0$. Therefore $\hat{v} = v_\infty$ in $[0, r_0)$. However, that is not possible since $\hat{v}'(r_0) = v'_\infty(r_0) < 0$. Hence, the nontrivial solution $v(\cdot, \lambda, d)$ to $(2.1)_\lambda$ can be extended to the interval $0 \leq r \leq T_0 + \varepsilon$ for λ and d large.

Finally, since $v(\cdot, \lambda, d) \to v_\infty$ in $C^1[0, T_0 + \varepsilon]$ as $\lambda, d \to +\infty$ then both $v'(\cdot, \lambda, d) < 0$ in $0 < r \leq T_0 + \varepsilon$ and $v(T_0 + \varepsilon, \lambda, d) < 0$ for λ and d large enough. Therefore a unique $T = T(\lambda, d)$ exists such that $v(r, \lambda, d) > 0$ for $0 \leq r < T$ and $v(T, \lambda, d) = 0$, provided that λ and d are great enough. Moreover, $T(\lambda, d) \to T_0$ as $\lambda, d \to +\infty$.

In our next result more precise properties of $T = T(\lambda, d)$ are stated.

Lemma 3.3 *There exist $\lambda_0, d_0 > 0$ such that the function $T : (\lambda_0, +\infty) \times (d_0, +\infty) \to \mathbb{R}^+$ is differentiable with respect to d. More importantly, $\lim_{\lambda,d\to+\infty} T'_d(\lambda, d) = 0$.*

Proof. Since both $v'(\cdot, \lambda, d) < 0$ and $v'_\infty < 0$ in $(0, T_0 + \varepsilon]$ for λ, d large, then the family $\{v(\cdot, \lambda, d)\}$ falls within the standard ordinary differential equations uniqueness regime for the equation $(2.1)_\lambda$ in any interval of the form $\delta \leq r \leq T_0 + \varepsilon$ for $\delta > 0$ as small as desired. On the other hand, $v(\cdot, \lambda, d)$ is differentiable with respect to d in some fixed interval $0 \leq r \leq \delta$ (cf. Theorem 2.6). Therefore, standard uniqueness in ordinary differential equations imply that $v(\cdot, \lambda, d)$ is differentiable with respect to d and values in $C^1[0, T_0 + \varepsilon]$ for λ and d large.

The discussion preceding the Lemma implies that $v'(T, \lambda, d) < 0$ for large λ, d and T close to T_0. Thus the implicit function theorem can be applied to solve the equation

$$v(T, \lambda, d) = 0 \tag{3.8}$$

for λ and d large and T close to T_0. Hence the uniqueness of solutions given by such theorem provides that $T = T(\lambda, d)$ is differentiable with respect to d. Moreover, differentiation with respect to d in (3.8) gives $T'_d(\lambda, d) = -\partial v/\partial d(T(\lambda, d), \lambda, d)/v'(T(\lambda, d), \lambda, d)$. On the other hand, Theorem 2.6 and the globalizing argument above lead to the convergence $\partial v/\partial d(T(\lambda, d), \lambda, d) \to 0$ in $C^1[0, T_0 + \varepsilon]$ as $\lambda, d \to +\infty$. This finishes the proof of Lemma 3.3. □

As a summary of the features on homogenizing families obtained in the present section, our existence and uniqueness result can already be stated.

Theorem 3.4 *Suppose f verifies hypotheses $(H)_\lambda$. Then, the problem*

$$\begin{aligned} -\Delta_p u = \lambda f(\lambda, u) \quad & \text{in} \quad B \\ u = 0 \quad & \text{on} \quad \partial B \end{aligned}$$

B a ball of $\mathbb{R}^N$, admits a unique family of positive radial solutions u_λ with the property that

$$\lim_{\lambda\to+\infty} u_\lambda(x) = \bar{u}_0$$

uniformly on compact subsets of B. *Moreover, every solution* u_λ *exhibits a dead core* $\mathcal{O}_\lambda = \{x : u_\lambda(x) = \bar{u}_0\}$ *for* λ *large enough. An exact estimate of the distance of* $\mathcal{O}_\lambda$ *to* ∂B *is given by*

$$\lim_{\lambda\to+\infty} \lambda^{1/p}\text{dist } (\mathcal{O}_\lambda, \partial B) = \left(\frac{p-1}{p}\right)^{1/p} \int_0^{\bar{u}_0} \frac{ds}{(\bar{F}(\bar{u}_0) - \bar{F}(s))^{1/p}}. \tag{3.9}$$

On the other hand, a further exact estimate of the boundary layer exhibited by u_λ *near* ∂B *is provided by*

$$\lim_{\lambda\to+\infty} \lambda^{-1/p}\frac{\partial u_\lambda}{\partial\nu}(x) = -\left(p'\bar{F}(\bar{u}_0)\right)^{1/p}, \tag{3.10}$$

uniformly on ∂B, *where* $\bar{F}(\bar{u}_0) = \int_0^{\bar{u}_0} \bar{f}(s)\, ds$ *and* ν *is the outward unit normal to* ∂B.

Remarks 3.2.
a) Theorem 3.4 considerably extends the results on dead cores in [8] by allowing a more general class of nonlinearities f. However, its scope is restricted in the present section to balls and annuli (see §5 for further results).
b) If the order k of the zero $\bar{u}_0$ of $f(\lambda,\cdot)$ in $(H2)_\lambda$ satisfies the complementary inequality $k \geq p-1$, then the existence of a homogenizing family $\{u_\lambda\}$ of positive radial solutions to (1.4) can be also shown (compare with [26] for the case $p=2$). However the appearance of dead cores is no more valid while a boundary layer near ∂B still arises, being its thickness exactly estimated by (3.10). All these results and the uniqueness feature will be considered in detail in a forthcoming paper ([27]).
c) Notice that, since $u_\lambda(r) = v(\lambda^{1/p}r - d)$, where v is a solution to the singular initial value problem $(2.1)_\lambda$, Theorem 2.3 implies $u'_\lambda(r) \sim C\left(r - \lambda^{-1/p}d\right)^{p/p-k-1}$ as $r \to \lambda^{-1/p}d+$. Thus the relation $p \leq 2(k+1)$ turns out to be a necessary and sufficient condition so that $u_\lambda \in C^2(\overline{B})$.

Proof of Theorem 3.4. No generality is lost if the radius of B is supposed to be $R=1$. As proved before, any positive radial solution u_λ in a homogenizing family must exhibit a dead core $\{u_\lambda(x) = \bar{u}_0\} = \{x : |x| \leq \lambda^{-1/p}d\}$, where $d = d(\lambda)$ (cf. Lemma 3.1). Thus, $u_\lambda(r) = v(\lambda^{1/p}r - d, \lambda, d)$, $v(\cdot,\lambda,d)$ being the unique nontrivial solution to $(2.1)_\lambda$, defined and positive in $0 \leq r < T(\lambda,d)$ and satisfying $v(T(\lambda,d),\lambda,d) = 0$, provided λ and d are large. This is certainly the case for large λ by virtue of Lemma 3.2.

Therefore, the problem of finding homogenizing families $\{u_\lambda\}$ to (1.4) reduces to solve the equation in d:

$$T(\lambda,d) + d = \lambda^{1/p}, \tag{3.11}$$

for large λ. Since Lemma 3.3 ensures that $\lim_{\lambda,d\to+\infty} T'_d(\lambda,d) = 0$, (3.11) can be solved and furnishes d as a function of λ, $d = d(\lambda)$ for $\lambda \geq \lambda^*$, λ^* being a large enough positive number. In particular, the uniqueness in solving (3.11) for $\lambda \geq \lambda^*$ entails the uniqueness of the homogenizing family so constructed. In order to obtain $\{u_\lambda\}$ if suffices with setting

$$u_\lambda(r) := \begin{cases} \bar{u}_0 & 0 \leq r \leq \lambda^{-1/p}d(\lambda) \\ v(\lambda^{1/p}r - d(\lambda), \lambda, d(\lambda)) & \lambda^{-1/p}d(\lambda) < r \leq 1\,. \end{cases}$$

To obtain estimate (3.9) notice that via (3.11) $\lambda^{1/p}\text{dist }(\mathcal{O}_\lambda, \partial B) = T(\lambda, d(\lambda))$, and hence (3.9) follows from $\lim_{\lambda,d\to+\infty} T(\lambda,d) = T_0$, $T_0 = (p')^{-1/p}\int_0^{\bar{u}_0} 1/(\bar{F}(\bar{u}_0) - \bar{F}(s))^{1/p}\, ds$.

Finally, to show (3.10), set $r(\lambda) = \lambda^{-1/p} d(\lambda)$. If (3.2) is multiplied by $u'_\lambda(r)$, an integration over the interval $[r(\lambda), r]$ leads to the "energy identity"

$$\frac{p-1}{p} |u'_\lambda(r)|^p + (N-1) \int_{r(\lambda)}^r \frac{|u'_\lambda(s)|^p}{s} ds = \lambda(F(\lambda, \bar{u}_0) - F(\lambda, u_\lambda(r))). \qquad (3.12)$$

This implies that the term $|u'_\lambda(r)|^p/\lambda$ remains bounded by $C := 2p/(p-1) \sup_{0\le s\le \bar{u}_0, \lambda\ge\lambda^*} |F|$ in $0 \le r \le 1$ for large λ. Therefore

$$\frac{1}{\lambda} \int_{r(\lambda)}^1 \frac{|u'_\lambda(s)|^p}{s} ds \le \frac{C}{r(\lambda)}(1 - r(\lambda)) \to 0$$

as $\lambda \to +\infty$. Therefore, by setting $r = 1$ in (3.12) we obtain $\lim_{\lambda\to+\infty} |u'_\lambda(1)|^p|/\lambda = p'\bar{F}(\bar{u}_0)$, which is the desired estimate. This concludes the proof of Theorem 3.4. □

3.3. Annuli. We are next showing that Theorem 3.4 is also valid when the domain D is any annulus A. More precisely.

Theorem 3.5 *Assume that f satisfies (H)$_\lambda$. Then, there exists a unique family $\{u_\lambda\}$ of positive radial solutions to (1.4) in $A = \{x : a < |x| < R\}$ which satisfies*

$$\lim_{\lambda\to+\infty} \sup_A u_\lambda = \bar{u}_0 \, . \qquad (3.13)$$

Moreover, if $\mathcal{O}_\lambda := \{x \in A : u_\lambda = \bar{u}_0\}$ then the following exact estimates hold,

$$\lim_{\lambda\to+\infty} \lambda^{1/p} \text{dist } (\{|x| = a\}, \mathcal{O}_\lambda) = \lim_{\lambda\to+\infty} \lambda^{1/p} \text{dist } (\{|x| = R\}, \mathcal{O}_\lambda) = T_0 \, , \qquad (3.14)$$

where $T_0 = p'^{-1/p} \int_0^{\bar{u}_0} 1/(\bar{F}(\bar{u}_0) - \bar{F}(s))^{1/p} \, ds$. In addition,

$$\lim_{\lambda\to+\infty} \lambda^{-1/p} \frac{du_\lambda}{dr}(a) = - \lim_{\lambda\to+\infty} \lambda^{-1/p} \frac{du_\lambda}{dr}(R) = (p'\bar{F}(\bar{u}_0))^{1/p} \, , \qquad (3.15)$$

where $\bar{F}(\bar{u}_0) = \int_0^{\bar{u}_0} \bar{f}(s) \, ds$.

Proof. As in the case D a ball, let us begin by proving that every solution u_λ in the family develops a dead core $\mathcal{O}_\lambda = \{x : u_\lambda(x) = \bar{u}_0\}$ when λ is large enough. In fact, and as in the proof of Lemma 3.1 assume that, on the contrary, a sequence $\lambda_n \to +\infty$ exists so that $\bar{u}_{0n} = u_{\lambda_n}(r_n) = \sup_A u_{\lambda_n}$, $a < r_n < R$ satisfies $\bar{u}_{0n} < \bar{u}_0$ for each n.

Defining $v_n(r) = u_{\lambda_n}(\lambda_n^{-1/p}(r + \hat{r}_n))$, where $\hat{r}_n = \lambda_n^{1/p} r_n$, $0 \le r \le \rho_n$, $\rho_n := \lambda_n^{1/p} R - \hat{r}_n$ then $v_n(\cdot)$ is the solution to

$$\begin{aligned} -((r + \hat{r}_n)^{N-1} \varphi_p(v'))' &= (r + \hat{r}_n)^{N-1} f(\lambda_n, v) \\ v(0) = \bar{u}_{0n} \;\; v'(0) &= 0 \, . \end{aligned} \qquad (3.16)$$

Since $\lambda_n^{1/p} a < \hat{r}_n < \lambda_n^{1/p} R$ then $\hat{r}_n \to +\infty$. Arguing as in §3.2, v_n can be continuated, for n large, over the interval $[0, T_0 + \varepsilon]$, $\varepsilon > 0$, so that $v_n \to v_\infty$ in $C^1[0, T_0 + \varepsilon]$, where v_∞ is the unique nontrivial solution to (1.6) which is defined in $[0, T_0 + \varepsilon]$.

Now observe that ε was elected in §3.2 in such a way that $v_\infty > 0$ for $r < T_0$ while $v_\infty < 0$ if $T_0 < r \le T_0 + \varepsilon$. It follows from this fact that $\liminf \rho_n \ge T_0$ and symmetrically $\limsup \rho_n \le T_0$. Therefore $\rho_n \to T_0$. In other words $r_n = R - \lambda_n^{-1/p} \rho_n \to R$.

Symmetrically, v_n can be extended, for n conveniently large, to the interval $-T_0 - \varepsilon \leq r \leq 0$ such that $v_n = v_n(r) \to v_\infty(-r)$ in $C^1[0, T_0 + \varepsilon]$. Now observe that v_n is defined and positive in $\rho_n^- < r \leq 0$, $\rho_n^- := -\hat{r}_n - \lambda_n^{1/p}$, with $v_n(\rho_n^-) = 0$. This easily leads again to $\lim \rho_n^- = -T_0$ and thus implying $r_n \to a$, what is not possible. Thus, every solution u_λ generates a dead core $\mathcal{O}_\lambda = \{x : u_\lambda = \bar{u}_0\} = \{x : a < r_1(\lambda) \leq |x| \leq r_2(\lambda) < R\}$, for λ large.

Setting $d_i(\lambda) = \lambda^{1/p} r_i(\lambda)$, $i = 1, 2$, it is now immediate that $d_i \to +\infty$ as $\lambda \to +\infty$ (compare with the case of the ball in Lemma 3.2). In particular $u_\lambda < \bar{u}_0$ if $r > \lambda^{-1/p} d_2(\lambda)$ (respectively $r < \lambda^{-1/p} d_1(\lambda)$). Therefore, to construct the family $\{u_\lambda\}$ is equivalent to finding the precise values of d_i as functions of λ. In fact, once this is done u_λ will be defined as,

$$u_\lambda(r) = \begin{cases} u^-(\lambda^{1/p} r, \lambda, d_1(\lambda)-) & \lambda^{1/p} a \leq \lambda^{1/p} r < d_1(\lambda) \\ \bar{u}_0 & d_1(\lambda) \leq r \leq d_2(\lambda) \\ u^-(\lambda^{1/p} r, \lambda, d_2(\lambda)+) & d_2(\lambda) \leq \lambda^{1/p} r < \lambda^{1/p} R\,, \end{cases} \tag{3.17}$$

where the terminology of Corollary 2.4 has been used.

Finally the existence of d_2 comes from solving, for large λ, the equation $d_2 + T(\lambda, d_2) = \lambda^{1/p} R$, where T is the first positive zero of $v(\cdot, \lambda, d_2)$, and $v(\cdot, \lambda, d_2)$ the unique nontrivial solution to $(2.1)_\lambda$ with $d = d_2$ (cf. the properties of $T = T(\lambda, d)$ in §3.2). As for d_1 observe that $(2.1)_\lambda$ exhibits a unique nontrivial solution $v(\cdot, \lambda, d)$ defined in $r < 0$ and satisfying $v(\cdot, \lambda, d) < \bar{u}_0$ (cf. Corollary 2.4). An analysis, completely similar to that in §3.2, shows that $v(\cdot, \lambda, d)$ exhibits, for d large, a first negative simple zero $\hat{T} = \hat{T}(\lambda, d)$, so that $\hat{T}_d'(\lambda, d) \to 0$ as $\lambda, d \to +\infty$. That is why d_1 is characterized as the unique solution to $d_1 + \hat{T}(\lambda, d_1) = \lambda^{1/p} a$, when λ is large.

Once the family $\{u_\lambda\}$ has been constructed by inserting d_i in (3.17), the proofs of (3.14) and (3.15) follow word for word the arguments of the corresponding assertions in Theorem 3.4. □

4. Perturbed logistic type problems

In this section we will analyze the uniqueness and asymptotic profile for large λ of positive solutions to the following class of perturbations of the logistic problem,

$$\begin{array}{rcll} -\Delta_p u & = & \lambda u^{p-1} - u^q + g(u) & \text{in } D \\ u & = & 0 & \text{on } \partial D, \end{array} \tag{1.2}$$

where p, q are positive constants, $p > 2$, $q > p - 1$, $\lambda > 0$ is a real positive parameter and D is a bounded radially symmetric domain of $\mathbb{R}^N$.

The perturbation term $g = g(u) \in C^1$ will be supposed to satisfy,

$$\lim_{u \to 0+} \frac{g(u)}{u^{p-1}} = 0, \qquad \lim_{u \to +\infty} \frac{g'(u)}{u^{q-1}} = 0\,. \tag{Hg}$$

Observe that (Hg) implies in particular that $g(0) = 0$ together with $g = o(u^q)$ as $u \to +\infty$.

To ascertain the existence and smoothness of possible positive solutions (1.2), we are first obtaining global estimates of such solutions. To this proposal set $g(\lambda, u) := \lambda u^{p-1} - u^q + g(u)$. As as consequence of (Hg), it can be proven that $g(\lambda, \cdot)$ has a maximum positive zero $\bar{u}_0(\lambda)$ which is simple for λ large, and verifies

$$\lim_{\lambda \to +\infty} \frac{\bar{u}_0(\lambda)}{\lambda^{1/(q-p+1)}} = 1\,. \tag{4.1}$$

Concerning problem (1.2) in a general bounded domain $\Omega \subset \mathbb{R}^N$, we have the following preliminary result (cf. Theorem 8 and Corollary 9 in [8], and [12] for the case where Ω is convex).

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{2,\beta}$ for some $0 < \beta < 1$. Assume $p > 1$, $q > p - 1$ and g satisfies (Hg). Then*

i) For every $\lambda > \lambda_{1,p}$ there exists at least a weak positive solution $u_\lambda \in W_0^{1,p}(\Omega)$ to (1.2). Furthermore, $u \in C^{1,\beta_0}(\bar{\Omega}) \cap C^{2,\beta_1}(\{x \in \Omega : \text{dist } (x, \partial\Omega) \leq \varepsilon\})$ for some $0 < \beta_0 < 1$, $\beta_1 = \min\{\beta, p-1\}$ and where $\varepsilon = \varepsilon(\lambda) > 0$.

ii) Every family of positive solutions $\{u_\lambda\}$, $\lambda \geq \lambda_0$, to (1.2) satisfies $0 < u_\lambda(x) \leq \bar{u}_0(\lambda)$ for each $x \in \Omega$. Moreover, $u_\lambda(x) \sim \bar{u}_0(\lambda)$ uniformly on compacts of Ω as $\lambda \to +\infty$. In other words (see (4.1)),

$$\lim_{\lambda \to +\infty} \lambda^{-\frac{1}{q-p+1}} u_\lambda(x) = 1$$

uniformly in compact subsets of Ω.

iii) Every such family $\{u_\lambda\}$ develops a boundary layer near $\partial\Omega$ which is precisely estimated as

$$\lim_{\lambda \to +\infty} \lambda^{-\frac{q+1}{p(q-p+1)}} \frac{\partial u_\lambda}{\partial \nu} = \left(\frac{q-p+1}{(q+1)(p-1)} \right)^{1/p} \tag{4.2}$$

uniformly on $\partial\Omega$, where ν stands for the outward unit normal on $\partial\Omega$.

Remarks 4.1.
a) Notice that the scope of Theorem 4.1 covers the full range $p > 1$. However the arising of dead cores $\mathcal{O}_\lambda = \{x \in \Omega : u = \bar{u}_0\}$ is only possible in the regime $p > 2$ (see §1). The existence of that phenomenom for (1.2) in general domains Ω was left unsolved in [8] (see next section for a complete answer).
b) It is convenient to fit now Theorem 4.1 in the framework of §§2 and 3. In view of ii) if u is changed in (1.2) by $u/\bar{u}_0(\lambda)$ (the unknown being written again as u) we find the equivalent problem,

$$\begin{array}{rcll} -\Delta_p u &=& \lambda f(\lambda, u) & \text{in } D \\ u &=& 0 & \text{on } \partial D\,, \end{array} \tag{4.3}$$

where $f(\lambda, u) = u^{p-1} - \bar{u}_0(\lambda)^\delta u^q/\lambda + g(\bar{u}_0(\lambda)u)/(\lambda \bar{u}_0(\lambda)^{p-1})$ with $\delta = q - p + 1$. From the analysis at the beginning of the section it follows that $f(\lambda, \cdot)$ admits $\bar{u}_0 = 1$ as a simple zero for λ large. From (Hg) we can assert that both $g(\bar{u}_0(\lambda)u)/(\lambda \bar{u}_0(\lambda)^{p-1})$ and $g'(\bar{u}_0(\lambda)u)/(\lambda \bar{u}_0(\lambda)^{p-2})$ converge to zero uniformly in $0 \leq u \leq 1$ as $\lambda \to +\infty$. Hence, $f(\lambda, \cdot) \to \bar{f}$ and $\partial f/\partial u(\lambda, \cdot) \to \bar{f}'$ uniformly in $0 \leq u \leq 1$ as $\lambda \to +\infty$ with $\bar{f}(u) = u^{p-1} - u^q$. Therefore, $f(\lambda, u)$ satisfies the hypotheses $(\text{H})_\lambda$ of §2.1 with $k = 1$, $p > 2$ and $\bar{\gamma} = \delta$ for the asymptotic case $\lambda \to +\infty$, and the results in §3 are applicable.

On the other hand, a major conclusion in Theorem 4.1 -ii) is the fact that *every* family $\{u_\lambda\}$ of positive solutions to (4.3) *homogenizes* towards $\bar{u}_0 = 1$ as $\lambda \to +\infty$. In other words, condition (3.1)

$$\lim_{\lambda \to +\infty} u_\lambda(x) = 1\,, \qquad x \in D\,, \tag{4.4}$$

holds uniformly on compacts of D for arbitrary such families $\{u_\lambda\}$. However, such positive solutions u_λ need not be radial in general (cf. §1) as admitted in §3. Thus, to obtain a uniqueness result some more work has to be done.

We are collecting all these reflections in our next uniqueness result. The domain D will be either a ball $B = \{x : |x| < R\}$ or an annulus $A = \{x : 0 < a < |x| < R\}$, for certain positive constants a and R.

Theorem 4.2 *Assume that p, q satisfy $q > p-1$, $p > 2$ while the perturbation term $g = g(u)$ verifies (Hg). Then there exists some $\lambda_0 > 0$ such that for every $\lambda \geq \lambda_0$ problem (1.2) exhibits a unique (and thereof radially symmetric) positive solution $u = u_\lambda(x)$ in D. Moreover, the set $\mathcal{O}_\lambda = \{x : \ u_\lambda(x) = \bar{u}_0(\lambda)\}$ is nonempty and*

$$\lim_{\lambda\to+\infty} \lambda^{1/p}\text{dist }(\mathcal{O}_\lambda, \partial B) = I(p,q)\,, \tag{4.5}$$

where $I(p,q) = ((q+1)/(p-1))^{1/p} \int_0^1 1/(q-p+1-(q+1)s^p + ps^q)^{1/p}\ ds$.

Proof. To achieve the existence and uniqueness assertions it suffices with noting that every family of positive solutions to (1.2) is homogenizing, as was precised in *ii*) of Theorem 4.1. Thus, by virtue of Theorems 3.4 and 3.5, the normalized problem (4.3) admits for λ large enough, a *unique* radial positive solution $u = u_\lambda(r)$. Now assume that (4.3) exhibits another family $\hat{u}_\lambda = \hat{u}_\lambda(x)$, $\lambda \geq \lambda_2$ of non radial positive solutions in D. Theorem A in Appendix on radial subsolutions (repectively, supersolutions) will allow us to conclude that $\tilde{u}_\lambda = u_\lambda$ for λ large. In fact, since $\underline{u} = \tilde{u}_\lambda$ defines a subsolution and $\tilde{u}_\lambda \leq \bar{u} := 1$, which can be observed as a radial supersolution to (4.3), then necessarily $\tilde{u}_\lambda \leq u_\lambda \leq 1$ in D, $\lambda \geq \lambda_2$. Similarly, the positivity of $\tilde{u}_\lambda$ in D and the strong maximum principle (cf. [7]) permit finding $\varepsilon > 0$ so small as to have $0 < \varepsilon\phi(x) < \tilde{u}_\lambda(x)$ for $x \in D$, ϕ the first positive Dirichlet eigenfunction of $-\Delta_p$ in D, which is known to be radial ([28]). As $\underline{u} := \varepsilon\ \phi$ defines a subsolution to (4.3) if $\lambda > \lambda_{1,p}$ (ε smaller if necessary), then the complementary inequality $\varepsilon\ \phi \leq u_\lambda \leq \tilde{u}_\lambda$ leads to the desired assertion $\tilde{u}_\lambda = u_\lambda$ in D for $\lambda > \max\{\lambda_2, \lambda_{1,p}\}$. This concludes the proof of Theorem 4.2. □

5. Dead core formation in general domains

Our next result states that all possible positive solutions $u = u(x)$ to the perturbed problem (1.2) necessarily exhibit a dead core $\mathcal{O}_\lambda = \{x \in \Omega : u = \bar{u}_0(\lambda)\}$ for λ large enough, satisfying the distance *dist* $(\mathcal{O}_\lambda, \partial\Omega)$ the same exact asymptotic estimate (4.5) proven in [8] for the logistic case (1.1).

Theorem 5.1 *Let Ω be a bounded $C^{2,\beta}$ domain of $\mathbb{R}^N$, $0 < \beta < 1$, and $\{u_\lambda\}_{\lambda\geq\lambda_0}$ an arbitrary family of positive solutions to*

$$\begin{array}{rcll} -\Delta_p u &=& \lambda u^{p-1} - u^q + g(u) & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega, \end{array} \tag{1.2}$$

where $p > 2$, $q > p-1$ while the perturbation term satisfies (Hg) in §4. If $\bar{u}_0(\lambda)$ stands for the unique positive zero of $g(\lambda, u) = \lambda u^{p-1} - u^q + g(u)$ for λ large, then there exists λ^* such that for $\lambda \geq \lambda^*$ the sets $\mathcal{O}_\lambda = \{x : \ u_\lambda(x) = \bar{u}_0(\lambda)\}$ are nonempty and satisfy,

$$\lim_{\lambda\to+\infty} \lambda^{1/p}\text{dist }(\mathcal{O}_\lambda, \partial\Omega) = I(p,q)\,,$$

where $I(p,q)$ is the expression given in Theorem 4.2.

Proof. Let us choose any positive solution u_λ to (1.2) for λ large. Fix $x_0 \in \partial\Omega$ and take a ball $B \subset \Omega$ tangent to $\partial\Omega$ at x_0. It is clear that u_λ is a supersolution to

$$\begin{aligned} -\Delta_p u &= \lambda u^{p-1} - u^q + g(u) && \text{in } B \\ u &= 0 && \text{on } \partial B, \end{aligned} \tag{5.1}$$

because $u_\lambda \geq 0$ on ∂B. According to Theorem 4.2, designate by $u_{\lambda,B}$ the unique positive solution to (5.1). It follows from ii) in Theorem 4.1 and Theorem A in Appendix that $u_{\lambda,B} \leq u_\lambda \leq \bar{u}_0(\lambda)$ in B, which implies that $\mathcal{O}_\lambda$ is nonempty for λ large. Moreover, if we set $\mathcal{O}_{\lambda,B} = \{x \in B : \ u_{\lambda,B}(x) = \bar{u}_0(\lambda)\}$, we have $\mathcal{O}_{\lambda,B} \subset \mathcal{O}_\lambda$ and thus *dist* $(\mathcal{O}_\lambda, x_0) \leq$ *dist* $(\mathcal{O}_{\lambda,B}, x_0)$. Therefore, $\lambda^{1/p}$*dist* $(\mathcal{O}_\lambda, \partial\Omega) \leq \lambda^{1/p}$*dist* $(\mathcal{O}_{\lambda,B}, \partial B)$ and hence

$$\limsup_{\lambda\to+\infty} \lambda^{1/p} \text{dist } (\mathcal{O}_\lambda, \partial\Omega) \leq I(p,q)\,,$$

where $I(p,q)$ is the expression in the statement of Theorem 4.2.
As for the complementary estimate, choose $x_0 = x_0(\lambda) \in \partial\Omega$ such that *dist* $(\mathcal{O}_\lambda, \partial\Omega) =$ *dist* $(\mathcal{O}_\lambda, x_0)$. Then, positive numbers $a, R > 0$ and a point $z \in \mathbb{R}^N$ can be found such that the annulus $A = \{x : \ a < |x - z| < R\}$ both contains Ω and keeps tangent to $\partial\Omega$ at x_0. Based upon Theorem 4.2, let $u_{\lambda,A}$ be the unique positive solution to the problem

$$\begin{aligned} -\Delta_p u &= \lambda u^{p-1} - u^q + g(u) && \text{in } A \\ u &= 0 && \text{on } \partial A, \end{aligned} \tag{5.2}$$

for λ large, while u_λ is the fixed positive solution to (1.2). Define $\underline{u} = \underline{u}_\lambda(x) \in W_0^{1,p}(A)$ as $\underline{u}_\lambda(x) = u_\lambda(x)$ in Ω and $\underline{u}_\lambda = 0$ for $x \in A \setminus \Omega$. We claim that $\underline{u}_\lambda$ is a subsolution to (5.2). If this is assumed, we find that $u_\lambda \leq u_{\lambda,A}$ in Ω and so $\mathcal{O}_\lambda \subset \mathcal{O}_{\lambda,A} := \{x : \ u_{\lambda,A}(x) = \bar{u}_0(\lambda)\}$. Thus, *dist* $(\mathcal{O}_\lambda, x_0) \geq$ *dist* $(\mathcal{O}_{\lambda,A}, x_0)$ what implies,

$$\liminf_{\lambda\to+\infty} \lambda^{1/p} \text{dist } (\mathcal{O}_\lambda, \partial\Omega) \geq I(p,q)\,,$$

and (4.5) is shown.

To finish the proof let us show the claim. By definition it must be achieved that,

$$\int_\Omega |\nabla u_\lambda|^{p-2} \nabla u_\lambda \nabla\varphi \leq \int_\Omega g(\lambda, u_\lambda)\ \varphi\,, \tag{5.3}$$

for arbitrary nonnegative $\varphi \in C_0^1(A)$ (while we know that equality in (5.3) holds for every $\varphi \in C_0^1(\Omega)$).

Let us fix $\varepsilon > 0$ so small as in Theorem 4.1 - i). Set $\Omega_\varepsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon\}$ and $\Gamma_\varepsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) < \varepsilon\}$. . In addition, let $\zeta = \zeta(x) \in C^\infty(\mathbb{R}^N)$ be so that $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ in Ω_ε while $\zeta \equiv 0$ outside Ω. Then, any $\varphi \in C_0^1(A)$ can be decomposed as $\varphi = \zeta\varphi + (1 - \zeta)\varphi := \varphi_0 + \varphi_1$, and so,

$$\int_\Omega |\nabla u_\lambda|^{p-2} \nabla u_\lambda \nabla\varphi = \int_{\Omega_\varepsilon} g(\lambda, u_\lambda)\ \varphi_0 + \int_{\Gamma_\varepsilon} |\nabla u_\lambda|^{p-2} \nabla u_\lambda \nabla\varphi_1\,, \tag{5.4}$$

since $\varphi_0 \in C_0^1(\Omega)$ and $\varphi_1 \equiv 0$ in Ω_ε. Now observe that u_λ is C^{2,β_1}, $0 < \beta_1 < 1$, in the integration domain of the last term (see Theorem 4.1). So, if one integrates by parts and observes that $\varphi_1 \equiv 0$ on the boundary $\partial\Omega_\varepsilon$ of Ω_ε then arrives at

$$\int_{\Gamma_\varepsilon} |\nabla u_\lambda|^{p-2} \nabla u_\lambda \nabla\varphi_1 = \int_{\Gamma_\varepsilon} g(\lambda, u_\lambda)\ \varphi_1 + \int_{\partial\Omega} |\nabla u_\lambda|^{p-2} \frac{\partial u_\lambda}{\partial\nu} \varphi_1\,,$$

ν being the outward unit normal at $\partial\Omega$. However, from the maximum principle (see [7]), $\partial u_\lambda/\partial\nu < 0$ on $\partial\Omega$. Thus, coming back to (5.4) we achieve,

$$\int_\Omega |\nabla u_\lambda|^{p-2}\nabla u_\lambda \nabla\varphi \le \int_{\Omega_\varepsilon} g(\lambda,u_\lambda)\ \varphi_0 + \int_{\Gamma_\varepsilon} g(\lambda,u_\lambda)\ \varphi_1 = \int_\Omega g(\lambda,u_\lambda)\ \varphi\,,$$

as desired. This concludes the proof. □

Appendix

In this appendix we are going to provide some details about the method of sub and supersolutions for the p-Laplacian under radial symmetry. More precisely, consider the problem

$$\begin{array}{lll} -\Delta_p u = f(u) & \text{in} & D \\ u = 0 & \text{on} & \partial D, \end{array} \tag{A.1}$$

where D is a rotationally invariant bounded domain of $\mathbb{R}^N$, i.e. D is a ball or an annulus, and $f \in C^\alpha(\mathbb{R})$ is a locally Hölder function with exponent α, $0 < \alpha < 1$. By a subsolution of (A.1) it will be meant a function $\underline{u} \in W_0^{1,p}(D) \cap L^\infty(D)$ such that

$$\begin{array}{lll} -\Delta_p \underline{u} \le f(\underline{u}) & \text{in} & D \\ u \le 0 & \text{on} & \partial D, \end{array}$$

in the weak sense, that is

$$\int_D |\nabla\underline{u}|^{p-2}\nabla\underline{u}\nabla\phi \le \int_D f(\underline{u})\phi,$$

for every nonnegative $\phi \in C_0^1(D)$. A supersolution is defined by reversing the above inequalities. Then we have the following theorem (compare with [29]).

Theorem A *Suppose there exist $\underline{u}$, $\bar{u} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ sub and supersolution respectively of equation (A.1), such that $\underline{u} \le \overline{u}$, and that f is locally $\alpha-$Hölder continuous. Then there exist $\underline{u} \le u_- \le u_+ \le \bar{u}$, u_-, $u_+ \in C^{1,\beta_0}(\bar{D})$ for some $0 < \beta_0 < 1$, the minimal, respectively the maximal, weak solution to (A.1) in the interval $[\underline{u}, \bar{u}]$. Moreover, if $\underline{u}$ (respectively $\bar{u}$) is a radially symmetric function, then so is u_- (resp. u_+).*

Proof. Since f is locally $\alpha-$Hölder continuous, we can find a constant $M > 0$ such that $-M(u-v)^\alpha \le f(u) - f(v) \le M(u-v)^\alpha$ for every $a \le v < u \le b$, where $a = \inf \underline{u}$ and $b = \sup \overline{u}$, the infimum and supremum being understood as essential. It can be also assumed, without loss of generality, that $\alpha \le p-1$.

We define u_1 to be the unique solution in $W_0^{1,p}(D)$ to

$$\begin{array}{lll} -\Delta_p u + M(u-\underline{u})^\alpha = f(\underline{u}) & \text{in} & D \\ u = 0 & \text{on} & \partial D. \end{array} \tag{A.2}$$

The uniqueness of solutions to (A.2) is a consequence of the weak comparison principle of Tolksdorf (Lemma 3.1 in [24]), while the existence follows from the fact that the functional

$$J(u) = \int_D \frac{1}{p}|\nabla u|^p + \frac{M}{\alpha+1}(u-\underline{u})^{\alpha+1} - f(\underline{u})u$$

is sequentially weakly lower-semicontinuous and coercive in $W_0^{1,p}(D)$ (observe that J is well defined since $\alpha+1 \leq p$ and $f(\underline{u}) \in L^\infty(D)$). Thus, a well known theorem in the calculus of variations provides a global minimizer of J in $W_0^{1,p}(D)$, which is a solution to (A.2). Similarly a sequence $\{u_n\}$ is defined by taking u_n to be the unique solution to (A.2), replacing $\underline{u}$ by u_{n-1}.

By using the weak comparison principle it is proved that $\underline{u} \leq u_{n-1} \leq u_n \leq \bar{u}$ for each n, so $u_n \in L^\infty(D)$ and $u_- := \sup u_n$ is also a bounded function which verifies $\underline{u} \leq u_- \leq \bar{u}$. Now let us show that u_- is in fact a solution to (A.1). Since $|f(u_{n-1}) - M(u_n - u_{n-1})^\alpha|_\infty$ is uniformly bounded by $C_0 := \sup_{a\leq s\leq b} |f(s)| + 2^\alpha M|s|^\alpha$, it follows by Lemma 3.7 in [24] that

$$|u_n|_{C^{1,\beta}} \leq C,$$

for some $0 < \beta < 1$ and $C > 0$, depending only on N, p, C_0, a and b. Thus for every $0 < \beta_0 < \beta$, there exists a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \to u_-$ in $C^{1,\beta_0}(\overline{D})$. Since $u_n \to u_-$ pointwise in D, limits can be taken in

$$\int_D |\nabla u_{n_k}|^{p-2}\nabla u_{n_k}\nabla\phi + M(u_{n_k} - u_{n_k-1})^\alpha\phi = \int_D f(u_{n_k-1})\phi \ ,$$

to obtain that u_- is a solution to (A.1). Indeed a stronger result holds: the whole sequence $\{u_n\}$ tends to u_- in $C^{1,\beta_0}(\overline{D})$. This is an easy consequence of the uniqueness of the limit.

The maximal solution u_+ can be obtained as the limit in $C^{1,\beta_0}(\overline{D})$, for every $0 < \beta_0 < \beta$, of the decreasing sequence $\{v_n\}$, $\underline{u} \leq u_n \leq v_n \leq \bar{u}$, obtained by setting v_n the solution to the problem

$$\begin{array}{lll} -\Delta_p v - M(v_{n-1} - v)^\alpha = f(v_{n-1}) & \text{in} & D \\ v = 0 & \text{on} & \partial D\,, \end{array}$$

with $v_0 = \bar{u}$. It finally follows from the preceding discussion that u_- is the minimal and u_+ the maximal solution to (A.1) in the function interval $[\underline{u}, \overline{u}]$.

To finish the proof of the Theorem suppose for instance that $\underline{u}$ is radially symmetric. Since $u_n \in C^1(\overline{D})$ solves a rotationally invariant problem, uniqueness then yields that u_n must also be radially symmetric. Therefore u_- must be radial. □

Remark.
The assertion of Theorem A giving the minimal and maximal solutions $\underline{u} \leq u_- \leq u_+ \leq \bar{u}$ (the so-called "method of sub and supersolutions") holds true for arbitrary $C^{1,\beta}$ bounded domains Ω of $\mathbb{R}^N$. In fact, the proof remains unchanged with the sole exception that the more general version of the $C^{1,\beta}$ estimates in Theorem 1 of [30] must now be used.

Final Remarks

We would like to acknowledge Professor Walter who informed us, after this research was written, about the work [21]. The existence and uniqueness assertions in Theorem 2.3 and Corollary 2.4 here can be derived from Corollary h p. 54 of [21], in the first case by producing a suitable supersolution. He also kindly pointed out an error in the proof of Theorem A in the Appendix.

References

[1] J. I. Díaz, J. E. Saa, Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires, *C. R. Acad. Sci. Paris Sér. I Math.* **305** (1987), 521-524.

[2] S. B. Angenent, Uniqueness of the solution of a semilinear boundary value problem, *Math. Ann.* **272** (1985), 129-138.

[3] N. Dancer, On the number of positive solutions of weakly nonlinear elliptic equations when a parameter is large, *Proc. London Math. Soc.* **53** (1986), 429-452.

[4] P. Clément and G. Sweers, Existence and multiplicity results for a semilinear elliptic eigenvalue problem, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **14** (1987), 97-121.

[5] J. Fraile, J. López-Gómez and J. Sabina de Lis, On the global structure of the set of positive solutions of some semilinear elliptic equations, *J. Differential Equations* **123** (1995), 180-212.

[6] G. Sweers, On the maximum of solutions for a semilinear elliptic problem, *Proc. Roy. Soc. Edinburgh Sect. A* **108** (1988), 357-370.

[7] J. L. Vázquez, A strong maximum principle for some quasilinear elliptic equations, *Appl. Math. Optim.* **12**(1984), 191-202.

[8] J. García Melián and J. Sabina de Lis, Stationary profiles of degenerate problems when a parameter is large, to appear in *Differential Integral Equations* (1998).

[9] J. I. Díaz, "Nonlinear Partial Differential Equations and Free Boundaries. Volume I: Elliptic Equations", Pitman Res. Notes in Math. Ser. 106, Pitman, London (1985).

[10] S. Kamin and L. Véron, Flat core properties associated to the p-Laplace operator, *Proc. Amer. Math. Soc.* **118** (1993), 1079-1085.

[11] M. Guedda and L. Véron, Bifurcation phenomena associated to the p-Laplace operator, *Trans. Amer. Math. Soc.* **310**: 8 (1988), 419-431.

[12] J. García Melián and J. Sabina de Lis, The behaviour of stationary solutions to some nonlinear diffusion problems, *Nonlinear Anal.* **30**: 8 (1997), 5499-5504.

[13] B. Gidas, W. M. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* **68** (1979), 209-243.

[14] C. V. Coffman, A non-linear boundary value problem with many positive solutions, *J. Differential Equations* **54** (1984), 429-437.

[15] M. Badiale and E. Nabana, A note on radiality of solutions of p-Laplacian equation, *Appl. Anal.* **52** (1994), 35-43.

[16] S. Kesavan and F. Pacella, Symmetry of positive solutions of a quasilinear elliptic equation via isoperimetric inequalitites, *Appl. Anal.* **54** (1994), 27-37.

[17] W. A. Coppel, "Stability and Asymptotic Behaviour of Differential Equations", Heath Mathematical Monographs, Boston, 1965.

[18] J. K. Hale, "Ordinary Differential Equations", Wiley Interscience, New York 1969.

[19] B. Franchi, E. Lanconelli and J. Serrin, Existence and uniqueness of nonnegative solutions of quasilinear equations in $\mathbb{R}^n$, *Adv. Math.* **118** (1996), 177-243.

[20] W. Walter, Sturm-Liouville theory for the radial Δ_p operator, *Math. Z.* **227** (1998), pp. 175-185.

[21] W. Reichel and W. Walter, Radial Solutions of equations and Inequalities Involving the p-Laplacian, *J. of Inequal. & Appl.* **1** (1997), pp. 47-71.

[22] P. J. McKenna, W. Reichel and W. Walter, Symmetry and multiplicity for nonlinear elliptic differential equations with boundary blow-up, *Nonlinear Anal.* **28** (1997), 1213-1225.

[23] W. Walter, Differential Inequalities, in "Inequalities, Fifty years on from Hardy, Littlewood and Pólya", Proc. of the Int. Conference London Math. Soc. (W. N. Everitt, ed.), Marcel Dekker Inc., New York, 1991.

[24] P. Tolksdorf, On the Dirichlet problem for quasilinear equations in domains with conical boundary points, *Comm. Partial Differential Equations* **8** (1983), 773-817.

[25] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, *J. Differential Equations* **51** (1984), 126-150.

[26] J. Smoller and A. Wasserman, An existence theorem for positive solutions of semilinear elliptic equations , *Arch. Rational Mech. Anal.* **95** (1986), 211-216.

[27] J. García Melián and J. Sabina de Lis, work in preparation.

[28] T. Battacharya, Radial symmetry of the first eigenfunction for the p-Laplacian in the ball, *Proc. Amer. Math. Soc.* **104** (1988), 169-174.

[29] H. Amann, On the existence of positive solutions of nonlinear elliptic boundary value problems, *Indiana Univ. Math. J.* **21** (1971), 125-146.

[30] G. M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, *Nonlinear Anal.* **12** (1988), 1203-1219.