A BIFURCATION PROBLEM GOVERNED BY THE BOUNDARY CONDITION I *

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Abstract

We deal with positive solutions of $\Delta u = a(x)u^p$ in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ subject to the boundary condition $\partial u/\partial\nu = \lambda u$, λ a parameter, $p > 1$. We prove that this problem has a unique positive solution if and only if $0 < \lambda < \sigma_1$ where, roughly speaking, σ_1 is finite if and only if $|\partial\Omega \cap \{a = 0\}| > 0$ and coincides with the first eigenvalue of an associated eigenvalue problem. Moreover, we find the limit profile of the solution as $\lambda \to \sigma_1$.

1. Introduction

It is the main concern of the present work the study of the following semilinear boundary value problem:

$$\begin{cases} \Delta u = a(x)u^p & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda u & x \in \partial\Omega\,, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain of class $C^{2,\alpha}$, $0 < \alpha < 1$, with outward unit normal ν on $\partial\Omega$ and where $\lambda \in \mathbb{R}$ is regarded as a perturbation parameter. It will be assumed that $a(x) \in C^\alpha(\overline{\Omega})$, $a \not\equiv 0$, is a nonnegative coefficient, while the exponent p will be kept in the range $p > 1$.

The main feature of problem (1.1) is its dependence on the parameter λ precisely in the boundary condition. Our main objective here is just to study the variation regimes of λ giving existence of positive solutions to (1.1) (so, we have avoided employing the more formal $|u|^p$ or $|u|^{p-1}u$ instead of u^p) analyzing their uniqueness, dependence on λ and covering the asymptotic behavior when $\lambda \to \infty$ in those cases where such solutions exist for large λ.

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Problem (1.1) may be considered in some sense as "sublinear" due to the sign of the nonlinearity, controlled by the condition $a(x) \geq 0$ in Ω. Accordingly, aside of considering the case $a(x) > 0$, $x \in \Omega$, it is also of much interest to ascertain the effect exerted on the existence and behavior of positive solutions to (1.1) by the vanishing of $a(x)$ somewhere in Ω. Especially, if $a \not\equiv 0$ is zero in a whole subdomain of Ω (see, for instance, [3], [20], [4], [7], [8], [18], [6] for this kind of features in the realm of Dirichlet or Robin boundary conditions which do not depend on parameters). In order to simplify the exposition and to avoid unnecessary technical complications it will be assumed that the set $\Omega \cap \{a = 0\}$ is either empty, i.e. $a(x) > 0$ for all $x \in \Omega$, or (its interior) constitutes a smooth subdomain $\Omega_0 \subset \Omega$. As will be opportunely remarked later (see Remark 9), many other more common possibilities can be handled as variations of this reference situation. Moreover, when searching for weak solutions, the requirements on Ω_0 can be further relaxed (see Theorem 2).

In order to state our main results we are describing with more precision our hypotheses on the vanishing subdomain $\Omega_0 \subset \Omega$. If $a(x_0) = 0$ at some $x_0 \in \Omega$ it will be assumed that $\Omega \cap \{a = 0\} = \Omega \setminus \{a > 0\} = \Omega \cap \overline{\Omega}_0$ where $\Omega_0 \subset \Omega$ is a $C^{2,\alpha}$ subdomain of Ω. As observed later, no essentially new phenomena arise if Ω_0 consists of several connected pieces (see Remark 11). Being both $\partial\Omega$, $\partial\Omega_0$ open and compact smooth $n-1$ dimensional manifolds, and hence locally connected, they can only exhibit finitely many connected pieces all of them also being pairwise disjoint closed $n-1$ dimensional manifolds. Since $a \not\equiv 0$ then $\Omega_0 \neq \emptyset$ implies $\Omega \cap \partial\Omega_0 \neq \emptyset$. Again for the sake of brevity, the following requirement on $\partial\Omega_0$ will be assumed in most part of the work:

(H) "Writing $\partial\Omega_0 = \Gamma_1 \cup \Gamma_2$, with $\Gamma_1 = \partial\Omega \cap \partial\Omega_0$ and $\Gamma_2 = \Omega \cap \partial\Omega_0$, Γ_2 satisfies $\overline{\Gamma}_2 \subset \Omega$"

(notice that $\Gamma_2 \neq \emptyset$ whenever $\Omega_0 \neq \emptyset$). We could equivalently ask that Γ_2 be a closed subset of $\partial\Omega$. Hypothesis (H) says that those possible connected components of $\partial\Omega_0$ touching $\partial\Omega$ are "separated" from the ones meeting Ω (which are required in (H) to lie entirely in Ω). See Figure 1. This is obviously the case if, for instance, either $\Omega_0 = \emptyset$ or $\Omega_0 \subset \overline{\Omega}_0 \subset \Omega$. Nevertheless, in the part of this work devoted to weak solutions we are also dealing with a more general setting allowing that a component of $\partial\Omega_0$ simultaneously meets $\partial\Omega$ and Ω.

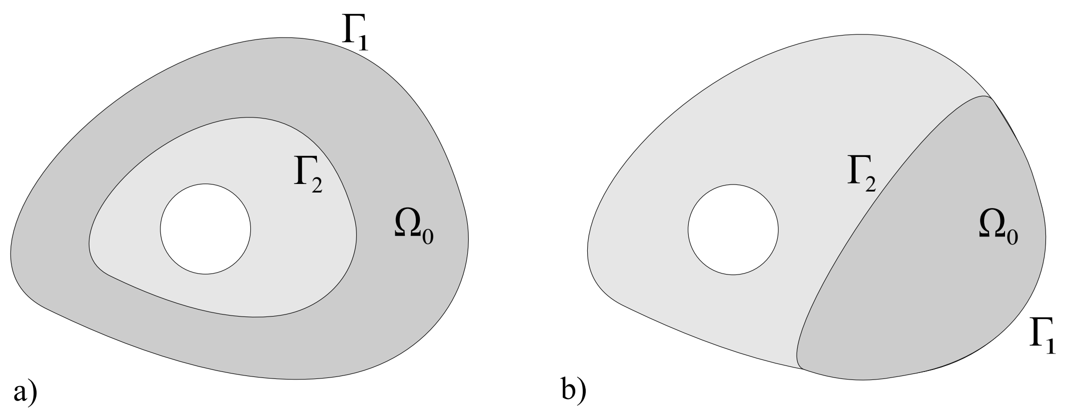

Figure 1. a) A valid configuration of Ω_0 in hypothesis (H). b) A domain Ω not satisfying (H). The domain Ω is the union of the two shaded regions.

In the λ-regime for the existence of positive solutions to (1.1) the relative position of $\partial\Omega$ with respect to the null set Ω_0 turns out to be crucial. If Ω_0 is far apart from $\partial\Omega$, i.e. $\Gamma_1 = \emptyset$ or plainly $\Omega_0 = \emptyset$, then positive solutions exist for λ arbitrarily large. On the contrary, such an existence is limited above for λ by a threshold value $\lambda = \sigma_1$ which is the principal eigenvalue of the mixed Dirichlet-Steklov eigenvalue problem:

$$\begin{cases} \Delta\varphi = 0 & x \in \Omega_0 \\ \dfrac{\partial\varphi}{\partial\nu} = \sigma\varphi & x \in \Gamma_1 \\ \varphi = 0 & x \in \Gamma_2. \end{cases} \tag{1.2}$$

The required discussion on the existence and properties of the principal eigenvalue σ_1 for this unusual problem and other eigenvalue problems will be provided in Section 2. For the moment, we are already in position of stating our main results.

Theorem 1. *Assume that $a \in C^\alpha(\overline{\Omega})$, $0 < \alpha < 1$ is nonnegative while either $a(x) > 0$ for all $x \in \Omega$ or $\Omega \cap \{a = 0\} = \Omega \cap \overline{\Omega}_0$ with Ω_0 a $C^{2,\alpha}$ subdomain of Ω satisfying* (H). *Then the following properties hold:*

i) Problem (1.1) *admits a positive solution if and only if:*

$$0 < \lambda < \sigma_1, \tag{1.3}$$

where $0 < \sigma_1 < \infty$ is the principal eigenvalue of (1.2) *provided Γ_1 is nonempty, $\sigma_1 = \infty$ otherwise. Moreover, the positive solution is* unique.

ii) If u_λ stands for the positive solution to (1.1) *then $u_\lambda \in C^{2,\alpha}(\overline{\Omega})$ for every λ satisfying* (1.3)*, being the mapping $\lambda \mapsto u_\lambda$ increasing and continuous regarded as valued in $C^{2,\alpha}(\overline{\Omega})$. Moreover:*

$$\lim_{\lambda \to 0+} u_\lambda = 0,$$

in $C^{2,\alpha}(\Omega)$, i.e. u_λ bifurcates from $u = 0$ at $\lambda = 0$.

iii) Observing (1.1) *as the stationary problem associated to:*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - a(x)u^p & x \in \Omega \\ \dfrac{\partial u}{\partial\nu} = \lambda u & x \in \partial\Omega\,, \end{cases} \tag{1.4}$$

then u_λ is asymptotically stable for all λ satisfying (1.3)*. Moreover, it is globally attractive for the positive solutions to* (1.4)*.*

Remark 1. It should be remarked that $a(x)$ in Theorem 1 is allowed to vanish somewhere on $\partial\Omega$ (even being identically zero on $\partial\Omega$).

If we look for weak solutions to (1.1) instead of classical, the conditions on $a(x)$, Ω and Ω_0 can be considerably relaxed. Regarding existence, next theorem describes a particular choice for the weak setting.

Theorem 2. *Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz bounded domain, $a(x) \in L^\infty(\Omega)$ satisfying either $a(x) > 0$ a.e. in Ω, or $a(x) > 0$ a. e. in $\Omega \setminus \Omega_0$, $a(x) = 0$ a.e. in Ω_0, Ω_0 a C^1 subdomain of Ω. If $\Gamma_1 := \partial\Omega \cap \partial\Omega_0$ defines a C^1 $n-1$ dimensional submanifold of $\partial\Omega$ with boundary, when nonempty, then problem (1.1) admits a positive weak solution $u \in W^{1,2}(\Omega) \cap C^{1,\beta}(\Omega)$ for every $0 < \beta < 1$ provided that $0 < \lambda < \sigma_1$, where σ_1 stands for the weak principal eigenvalue of (1.2) if $\Gamma_1 \neq \emptyset$, $\sigma_1 = \infty$ otherwise.*

Remark 2. The meeting region $\Gamma_1 = \partial\Omega \cap \partial\Omega_0$ if nonempty is required to be a not too small part of $\partial\Omega$ (indeed a submanifold with boundary of $\partial\Omega$). For instance, Figure 1.b) provides a possible simple example of such situation. Cases of “smaller” intersections, for instance $\partial\Omega \cap \partial\Omega_0$ a $n-2$ submanifold of $\partial\Omega$ or even a smaller object) may be handled as a perturbation of the first case. However, that is beyond our objectives in this work.

The behavior of positive solutions u_λ to (1.1) described in Theorem 1 is similar to the corresponding solutions to the logistic problem

$$\begin{cases} -\Delta u = \lambda u - a(x)u^r & x \in \Omega \\ \quad u = 0 & x \in \partial\Omega, \end{cases} \tag{1.5}$$

$r > 1$, $a(x)$ being as in that theorem and $\lambda_1(\Omega) < \lambda < \lambda_1(\Omega_0)$ ($\lambda_1(\Omega), \lambda_1(\Omega_0)$ the first Dirichlet eigenvalues in Ω and Ω_0, respectively). See [8] for details.

Theorem 1 asserts that positive solutions u_λ to (1.1) abruptly cease to exist when λ crosses σ_1 if such a value is finite. This raises the question on what kind of singularization undergoes u_λ when $\lambda \to \sigma_1$ and where does it occur. Next result gives a full answer to this question (see [8], [18] for the corresponding case in the logistic problem (1.5)).

Theorem 3. *Under the hypotheses of Theorem 1 assume in addition that $a \in C^1(\overline{\Omega})$. Set $\Omega^+ := \Omega \cap \{a > 0\}$ and suppose that $\Gamma_1 \neq \emptyset$ and so $\sigma_1 < \infty$. The profile of the positive solution u_λ to (1.1) as $\lambda \nearrow \sigma_1$ is then described in the following terms.*

i) Firstly,

$$\lim_{\lambda \nearrow \sigma_1} u_\lambda(x) = \infty,$$

uniformly in $\overline{\Omega}_0$.

ii) If $\Gamma^+ := \partial\Omega^+ \cap \partial\Omega \neq \emptyset$ (Figure 2.a)) then,

$$u_\lambda(x) \to z_{\mathcal{M},\Omega^+}(x),$$

as $\lambda \nearrow \sigma_1$ in $C^{2,\alpha}(\Omega^+ \cup \Gamma^+)$, where $z_{\mathcal{M},\Omega^+} \in C^{2,\alpha}(\Omega^+ \cup \Gamma^+)$ stands for the minimum solution to the singular mixed boundary value problem:

$$\begin{cases} \Delta u = a(x)u^p & x \in \Omega^+ \\ u = \infty, & x \in \partial\Omega^+ \cap \Omega \\ \dfrac{\partial u}{\partial \nu} = \sigma_1 u, & x \in \Gamma^+. \end{cases} \tag{1.6}$$

iii) If, on the contrary, $\partial\Omega^+ \subset \Omega$ (Figure 2.b)) then

$$u_\lambda(x) \to z_{\mathcal{D},\Omega^+}(x),$$

as $\lambda \nearrow \sigma_1$ in $C^{2,\alpha}(\Omega^+)$, where $z_{\mathcal{D},\Omega^+}(x) \in C^{2,\alpha}(\Omega^+)$ is the minimum solution of the singular Dirichlet problem,

$$\begin{cases} \Delta u = a(x)u^p & x \in \Omega^+ \\ u = \infty, & x \in \partial\Omega^+. \end{cases} \tag{1.7}$$

Remark 3. Observe that from (H), $\Omega = \Omega_0 \cup \Gamma_2 \cup \Omega^+$ and so $\partial\Omega^+ = \Gamma^+ \cup \Gamma_2$, all the sets involved being pair-wise disjoint.

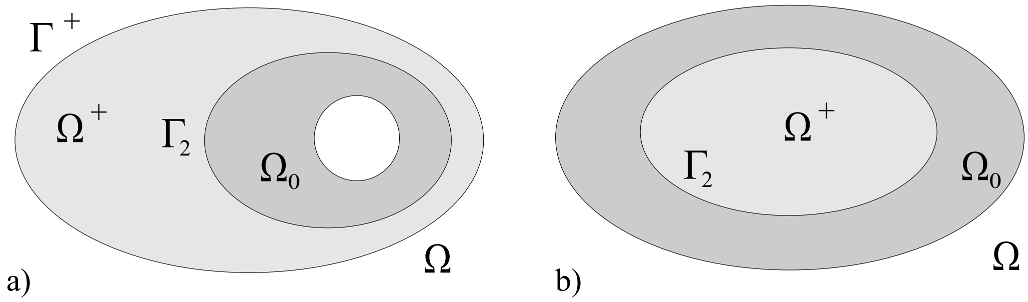

FIGURE 2. a) A simple configuration for Ω as in point ii). b) A possible Ω as in point iii). The domain Ω is the union of the two shaded regions.

As a counterpart of Theorem 3 the following result describes the asymptotic profile of u_λ if $\sigma_1 = \infty$.

Theorem 4. *Suppose that $\sigma_1 = \infty$ in Theorem 1. Then,*

$$u_\lambda(x) \to z_{\mathcal{D}}(x),$$

as $\lambda \to \infty$ in $C^{2,\alpha}(\Omega)$ where $z_{\mathcal{D}}(x) \in C^{2,\alpha}(\Omega)$ is now the minimum solution to the problem,

$$\begin{cases} \Delta u = a(x)u^p & x \in \Omega \\ u = \infty, & x \in \partial\Omega. \end{cases} \tag{1.8}$$

Moreover:

$$\liminf_{\lambda\to\infty} \left(\lambda^{-2/(p-1)} \sup_{x\in\Omega} u_\lambda(x) \right) \geq |a|_{\infty,\Omega}{}^{-1/(p-1)}. \tag{1.9}$$

Remark 4. It can be shown that the minimum solution to (1.8) coincides with the corresponding minimum solution of the singular Neumann problem,

$$\begin{cases} \Delta u = a(x)u^p & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \infty, & x \in \partial\Omega. \end{cases}$$

On the other hand, the uniqueness of a positive solution to the singular problem (1.6) holds if $a(x)$ suitably decays to zero at $\Gamma_2 = \partial\Omega^+ \setminus \partial\Omega$. Notice that a must be identically zero there. The uniqueness for (1.8) holds even under less restrictive assumptions on the behavior of a at $\partial\Omega$. See [9], [19] for further details and references.

This paper is organized as follows. Section 2 contains a detailed analysis of some auxiliary special eigenvalue problems. Existence and uniqueness results are treated in Section 3 while the discussion of asymptotic profiles of the positive solutions as $\lambda \to \sigma_1$ are considered in Section 4.

2. Preliminaries

In the present section two different kinds of –to some extent– non standard eigenvalue problems are considered. The first one was studied in detail in [15] (see Section 2 there) and the material required for the present work is thus collected in a single statement. An outline of its proof is enclosed for later use and also in order to properly highlight the smoothness properties of the relevant eigenfunctions.

Proposition 5. *Let $\Omega \subset \mathbb{R}^N$ be a smooth $C^{2,\alpha}$ domain, $0 < \alpha < 1$, and consider the eigenvalue problem:*

$$\begin{cases} \Delta\phi = \mu\phi & x \in \Omega \\ \dfrac{\partial\phi}{\partial\nu} = \lambda\phi & x \in \partial\Omega, \end{cases} \tag{2.1}$$

where μ stands for the eigenvalue while $\lambda \in \mathbb{R}$ has the "status" of a parameter. Then, the following properties hold.

i) For every $\lambda \in \mathbb{R}$, (2.1) admits a unique principal eigenvalue μ_1, i. e. an eigenvalue associated to a one-signed eigenfunction, which is simple with any associated eigenfunction $\phi_1 \in C^{2,\alpha}(\overline{\Omega})$.

ii) As a function of $\lambda \in \mathbb{R}$, μ_1 is smooth and increasing with $\mu_1 = 0$ at $\lambda = 0$ and $\mu_1 \to \infty$ as $\lambda \to \infty$. Moreover:

$$\liminf_{\lambda\to\infty} \frac{\mu_1}{\lambda^2} \geq 1. \tag{2.2}$$

Remark 5. Problem (2.1) becomes non standard when $\lambda > 0$ which is just the case of interest for the present work (with $\lambda < 0$, $\lambda = 0$ (2.1) is the Robin, Neumann eigenvalue problem, respectively).

Sketch of the proof. We are only dealing with the anomalous sign $\lambda > 0$. Define $\mathcal{M} = \{u \in H^1(\Omega) : \int_\Omega u^2 = 1\}$ and consider $J : \mathcal{M} \to \mathbb{R}$ defined as:

$$J(u) = \int_\Omega |\nabla u|^2 - \lambda \int_{\partial\Omega} u^2.$$

The functional J is sequentially weakly lower semicontinuous. It is in addition coercive in $\mathcal{M}$. In fact, by using the equivalent norm $|u|^2_{H^1(\Omega)} = \int_\Omega |\nabla u|^2 + \int_{\partial\Omega} u^2$, the existence of a sequence $u_n \in \mathcal{M}$, $|u_n|_{H^1(\Omega)} \to \infty$ with $J(u_n) \leq K$ for some $K > 0$ implies that $\int_{\partial\Omega} u_n^2 \to \infty$. Then, after choosing a subsequence if necessary, $v_n := u_n/|u_n|_{L^2(\partial\Omega)}$ weakly converges to some $v \in H^1(\Omega)$ being $v_n \to v$ both in $L^2(\Omega)$ and $L^2(\partial\Omega)$. That is not possible since $|v_n|_{L^2(\Omega)} = |u_n|^{-1}_{L^2(\partial\Omega)} \to 0$ while $v \neq 0$. Thus, J is coercive and reaches an infimum in $\mathcal{M}$ ([22]) which must be negative. Setting $-\mu_1 = \inf_{\mathcal{M}} J$ then:

$$-\mu_1 = \inf_{u\in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 - \lambda \int_{\partial\Omega} u^2}{\int_\Omega u^2} := \inf_{u\in H^1(\Omega)\setminus\{0\}} Q(u),$$

and any $\phi \in H^1(\Omega)$ satisfying $Q(\phi) = -\mu_1$ defines a weak eigenfunction to (2.1) associated to μ_1, i.e.,

$$\int_\Omega \nabla\phi\nabla v + \mu_1 \int_\Omega \phi v = \lambda \int_{\partial\Omega} \phi v, \tag{2.3}$$

for all $v \in H^1(\Omega)$. To show that $\phi \in C^{2,\alpha}(\overline{\Omega})$ consider the auxiliary problem:

$$\begin{cases} \Delta v - Mv = f & x \in \Omega \\ \dfrac{\partial v}{\partial \nu} - \lambda v = 0 & x \in \partial\Omega, \end{cases} \tag{2.4}$$

with $M > 0$, $f := (\mu_1 - M)\phi$. Problem (2.4) can be transformed into a C^α coefficients Neumann problem by setting $w = e^{\lambda h(x)}v$, h being any $C^{2,\alpha}$ extension to $\overline{\Omega}$ of dist $(x, \partial\Omega)$ in a neighbourhood of $\partial\Omega$. As a first consequence, $\phi \in H^2(\Omega)$. On the other hand, it follows from [1] that (2.4) becomes uniquely solvable in $W^{2,p}(\Omega)$ for any $f \in L^p(\Omega)$ and $p \geq 1$ if M is selected conveniently large. Bootstrapping in ϕ we get $\phi \in W^{2,p}(\Omega)$ for all $p \geq 1$, hence $\phi \in C^{1,\beta}(\overline{\Omega})$ for all $0 < \beta < 1$. Being (2.4) also uniquely solvable in $C^{2,\alpha}(\overline{\Omega})$ for M large (see [1], [12]) we finally get the searched regularity $\phi \in C^{2,\alpha}(\overline{\Omega})$.

That any eigenfunction ϕ to μ_1 must be one-signed follows from the fact that, say ϕ^+ is also an eigenfunction provided $\phi^+ \not\equiv 0$ since $Q(\phi^+) = -\mu_1$. By the maximum principle $\phi^+ > 0$ in $\overline{\Omega}$, hence $\phi^- = 0$ and ϕ is positive.

Check [15] for a detailed account of the remaining and other additional interesting properties of μ_1. □

Remark 6. As a consequence of the estimate (2.2) some rough information on the behavior of the –normalized in some way– principal eigenfunction $\phi_{1,\lambda} > 0$ as $\lambda \to \infty$ can be given. In fact, observe that from the maximum principle $\phi_{1,\lambda}(x) < \sup_{\partial\Omega} \phi_{1,\lambda}$ for all $x \in \Omega$. Assume that $\phi_{1,\lambda}$ has been normalized with $\sup_{\partial\Omega} \phi_{1,\lambda} = 1$. Then by integrating in (2.1) we arrive at:

$$0 < \int_\Omega \phi_{1,\lambda} < \frac{\lambda}{\mu_1}|\partial\Omega|.$$

Thus from (2.2) it follows that $\phi_{1,\lambda} \to 0$ in $L^1(\Omega)$ as $\lambda \to \infty$. A more detailed account of this convergence will be given later in Section 4 (cf. Remark 10).

In our next result an eigenvalue problem similar to (1.2) is considered. In fact a Schrödinger potential term $q(x)$ is included for later use in Section 4 (see the proof of Theorem 3).

Theorem 6. *Let $\Omega_0 \subset \mathbb{R}^N$ be a class $C^{2,\alpha}$ bounded domain whose boundary splits up in two sets Γ_1, Γ_2 of connected pieces (thus, each of them being a closed $n-1$ dimensional manifold) while the potential $q(x) \in C^\alpha(\overline{\Omega}_0)$. Then, the eigenvalue problem,*

$$\begin{cases} \Delta\varphi - q(x)\varphi = 0 & x \in \Omega_0 \\ \dfrac{\partial\varphi}{\partial\nu} = \sigma\varphi & x \in \Gamma_1 \\ \varphi = 0 & x \in \Gamma_2 \end{cases} \tag{2.5}$$

admits a principal eigenvalue, i.e. an eigenvalue with a one-signed eigenfunction, if and only if

$$\lambda_1(q) > 0, \tag{2.6}$$

where $\lambda_1(q)$ stands for the principal Dirichlet eigenvalue of the operator $-\Delta + q(x)$ in the domain Ω_0. Moreover, such an eigenvalue $\sigma_1(q)$ is the unique principal eigenvalue, simple and the smaller of all possible eigenvalues. In addition, $\sigma_1(q)$ is increasing with respect to q. Finally, any associated eigenfunction φ_1 belongs to $C^{2,\alpha}(\overline{\Omega}_0)$.

Remark 7. As a consequence of Theorem 6 observe that the eigenvalue problem (1.2) admits a principal eigenvalue σ_1 (σ_1 will be used with the meaning of $\sigma_1(q)_{|q=0}$) since $\lambda_1(q)_{|q=0}$, the first Dirichlet eigenvalue of $-\Delta$ in Ω_0, is positive. Furthermore, σ_1 is positive. In fact, if φ_1 is a positive associated eigenfunction it follows from the maximum principle that $\varphi_1 > 0$ on Γ_1 while direct integration gives:

$$\sigma_1 = \frac{\int_{\Omega_0} |\nabla\varphi_1|^2}{\int_{\Gamma_1} \varphi_1^2}.$$

The same holds true for $\sigma_1(q)$ if either $q(x) \geq 0$ in Ω_0 or $\inf_{\Omega_0} q$ is not too large (in absolute value) in case of being negative.

Proof of Theorem 6. We begin by noticing that the existence of a principal eigenvalue σ with associated positive eigenfunction φ provides a positive strict supersolution to the equation,

$$-\Delta u + qu = 0,$$

in Ω_0 since φ must be positive on Γ_1. This is well known to imply that $\lambda_1(q) > 0$ (see [17]).

Let us now show the sufficiency of (2.6) to ensure the existence of a principal eigenvalue. Define now $H = H_{\Gamma_2}(\Omega_0) = \{u \in H^1(\Omega_0) : u_{|\Gamma_2} = 0\}$ equipped with the norm $|u|_H^2 = \int_{\Omega_0} |\nabla u|^2 + \int_{\Gamma_2} u^2 = \int_{\Omega_0} |\nabla u|^2$, which is equivalent to the standard one in $H^1(\Omega)$, $\mathcal{M} = \{u \in H : \int_{\Gamma_1} u^2 = 1\}$ being $J : \mathcal{M} \to \mathbb{R}$ defined as

$$J(u) = \int_{\Omega_0} |\nabla u|^2 + \int_{\Omega_0} qu^2.$$

We are next proving that (2.6) implies the coercivity of J. Assume on the contrary that there exists $u_n \in \mathcal{M}$ such that $|u_n|_H \to \infty$ with $J(u_n) \leq C$ for some constant C. Since q is bounded this implies that $t_n := |u_n|_{L^2(\Omega_0)} \to \infty$. Setting $v_n = u_n/t_n$ we find that v_n

is bounded in $H^1(\Omega_0)$ then $v_n \rightharpoonup v$ weakly in $H^1(\Omega_0)$ and strongly in $L^2(\Omega_0)$ and $L^2(\Gamma_1)$. Observe that $v = 0$ in Γ_1 (in the sense of traces) and hence $v \in H_0^1(\Omega_0)$ while $|v|_{L^2(\Omega_0)} = 1$. Finally, from the boundedness of $J(u_n)$ we obtain,

$$\lambda_1(q) \int_{\Omega_0} v^2 \leq \int_{\Omega_0} |\nabla v|^2 + \int_{\Omega_0} qv^2 \leq 0,$$

and this implies $v = 0$ which is impossible. Thus, J is coercive in $\mathcal{M}$.

On the other hand, since J is also sequentially weakly lower semicontinuous, J attains a minimum value σ_1 in $\mathcal{M}$ ([22]),

$$\sigma_1 = \inf_{u \in H \setminus \{u|_{\Gamma_1} \equiv 0\}} \frac{\int_{\Omega_0} |\nabla u|^2 + \int_{\Omega_0} qu^2}{\int_{\Gamma_1} u^2}. \tag{2.7}$$

This variational representation simultaneously ensures that σ_1 is an eigenvalue, and the smaller among all other possible eigenvalues σ to (2.5). The discussion of both the smoothness and the one sign character of any associated eigenfunction φ_1 is the same as in Proposition 5. In fact, observe that the analysis in [1] of smoothness up to the boundary can be separately performed on each Γ_i, $i = 1, 2$. On the other hand, the simplicity of σ_1 follows from the fact that two possible independent eigenfunctions lead to two L^2 orthogonal eigenfunctions what is impossible in this case (this also explains why σ_1 is the unique principal eigenvalue). Finally, the monotonicity of $\sigma_1(q)$ with respect q follows from the variational representation (2.7). □

Remark 8. The proof of existence of a unique principal eigenvalue to problem (2.5) can be extended without changes to more general domains Ω_0. For instance, Ω_0 a Lipschitz bounded domain where $\Gamma_1 \subset \partial\Omega_0$ is a C^1 $n-1$ dimensional compact manifold with boundary and $\Gamma_2 := \partial\Omega_0 \setminus \Gamma_1$ nonempty. Notice that Γ_1 and Γ_2 meet at the $n-2$ dimensional boundary of Γ_1.

In our next result, the dependence of the eigenvalue problem (2.5) with respect to certain perturbations of the domain Ω_0 (Ω_0 under the hypotheses of Theorem 6) and the potential q is considered. The perturbed domains are,

$$\Omega_{0,t} = \Omega_0 \cup \{x \in \mathbb{R}^N : \operatorname{dist}(x, \Gamma_2) < |t|\},$$

$0 < |t| < \varepsilon$ being the perturbation parameter, $\varepsilon > 0$ small. Notice that Ω_0 is deformed an amount $|t|$ in the outward normal field *only* with respect the boundary Γ_2. Remark also that $\partial\Omega_{0,t} = \Gamma_1 \cup \Gamma_{2,t}$. The perturbed version of (2.5) under consideration is given by

$$\begin{cases} \Delta\varphi = q(t)\varphi & x \in \Omega_{0,t} \\ \dfrac{\partial\varphi}{\partial\nu} = \sigma\varphi & x \in \Gamma_1 \\ \varphi = 0 & x \in \Gamma_{2,t}, \end{cases} \tag{2.8}$$

where $|t| < \varepsilon$, $q \in C^1(-\varepsilon, \varepsilon)$, $q(0) = 0$. As observed in Remark 7, problem (2.8) admits a principal eigenvalue $\sigma_{1,t}$. By means of the holomorphic families of type (A) approach (see

[13]) and arguing as in [15, 18] –coefficient q in addition real analytic in this case– or by the direct variational approach used in [11] it can be shown that $\sigma_{1,t}$ is differentiable with respect to t. More importantly, an expression for its derivative can be produced and is enclosed in the next statement. For the sake of brevity its proof is omitted. Nevertheless it can be reconstructed following the arguments in either [11] or [18].

Proposition 7. *Under the conditions of Theorem 6 let σ_1 be the principal eigenvalue of* (1.2) *with associated eigenfunction φ_1. Then the derivative of the principal eigenvalue $\sigma_{1,t}$ to the perturbed problem* (2.8) *at $t = 0$ is given by the expression*

$$\frac{d}{dt}(\sigma_{1,t})_{|t=0} = \frac{q_1 \int_{\Omega_0} \varphi_1^2 - \int_{\Gamma_2} |\nabla \varphi_1|^2}{\int_{\Gamma_1} \varphi_1^2}, \tag{2.9}$$

where $q_1 = \dfrac{dq}{dt}_{|t=0}$.

3. Existence and uniqueness results

This section is devoted to the proof of our results concerning existence and uniqueness of positive solutions to the problem (1.1).

3.1. Proof of Theorem 2

Proof of Theorem 2. Firstly, we are introducing the concept of weak solution to be used. It is said that $u \in H^1(\Omega)$ is a weak solution of (1.1) provided:

$$\int_\Omega a|u|^{p+1} < \infty,$$

together with the usual relation:

$$\int_\Omega \nabla u \nabla v + \int_\Omega a|u|^{p-1} uv = \lambda \int_{\partial\Omega} uv, \tag{3.1}$$

for all $v \in C^1(\overline{\Omega})$. In the case $u \geq 0$ a.e. in Ω, the test function v may be supposed to belong to $H^1(\Omega)$ by a standard approximation argument. In particular, notice that positive weak solutions $u \in H^1(\Omega)$ are only possible provided $\lambda > 0$ since from (3.1) with $v = u$:

$$\int_\Omega |\nabla u|^2 + \int_\Omega a u^{p+1} = \lambda \int_{\partial\Omega} u^2.$$

Let us consider in $H^1(\Omega)$ the equivalent norm $|u|_{H^1(\Omega)} = (\int_\Omega |\nabla u|^2 + \int_{\partial\Omega} |u|^2)^{1/2}$ and introduce the functional,

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{p+1}\int_\Omega a(x)|u|^{p+1} - \frac{\lambda}{2}\int_{\partial\Omega} |u|^2,$$

which may be infinity at some $u \in H^1(\Omega)$. We will find a weak solution of (1.1) by showing that Φ reaches its minimum in $H^1(\Omega)$.

As a preliminary remark notice that for a small positive constant K we have,

$$\Phi(K) = \frac{K^{p+1}}{p+1}\int_\Omega a(x) - \frac{\lambda|\partial\Omega|}{2}K^2 < 0,$$

and a possible minimum u of Φ will satisfy $\Phi(u) < 0$, being such minimum nontrivial. Replacing u by $|u|$, we may assume in addition that u is nonnegative.

To see that there exists a minimum of Φ in $H^1(\Omega)$ let us prove that Φ is coercive and weakly lower semicontinuous (cf. [3], [22]). Let us first check that Φ is coercive. Proceeding by contradiction, assume that there exists a sequence $u_n \in H^1(\Omega)$ such that

$$|u_n|_{H^1(\Omega)} \to \infty \qquad \text{and} \qquad \Phi(u_n) \le C.$$

Hence

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2 + \frac{1}{p+1}\int_\Omega a(x)|u_n|^{p+1} \le C + \frac{\lambda}{2}\int_{\partial\Omega}|u_n|^2, \tag{3.2}$$

and therefore, since $|u_n|_{H^1(\Omega)} \to \infty$ we have that

$$\int_{\partial\Omega}|u_n|^2 \to \infty.$$

Setting $v_n = u_n/|u_n|_{L^2(\partial\Omega)}$ it follows from (3.2) that

$$\frac{1}{2}\int_\Omega |\nabla v_n|^2 + \frac{1}{(p+1)(\int_{\partial\Omega}|u_n|^2)^{1-p}}\int_\Omega a(x)|v_n|^{p+1} \le \frac{C}{\int_{\partial\Omega}|u_n|^2} + \frac{\lambda}{2}. \tag{3.3}$$

Therefore v_n is a bounded sequence in $H^1(\Omega)$ and thus there exists a subsequence (that we still call v_n) such that $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$ and $v_n \to v$ strongly in $L^2(\partial\Omega)$. Hence $|v|_{L^2(\partial\Omega)} = 1$. On the other hand, it follows from (3.3) that

$$\frac{1}{(p+1)(\int_{\partial\Omega}|u_n|^2)^{1-p}}\int_\Omega a(x)|v_n|^{p+1} \le \frac{C}{\int_{\partial\Omega}|u_n|^2} + \frac{\lambda}{2}.$$

Therefore

$$\frac{1}{(p+1)}\int_\Omega a(x)|v_n|^{p+1} \le C\left(\int_{\partial\Omega}|u_n|^2\right)^{1-p} \to 0, \qquad n \to \infty.$$

Hence

$$\int_\Omega a(x)|v|^{p+1} = 0,$$

and we conclude that $v \equiv 0$ in $\Omega^+ = \{a > 0\}$. Notice that in the case $\sigma_1 = +\infty$ this implies $v = 0$ on $\partial\Omega$, which is not possible since $|v|_{L^2(\partial\Omega)} = 1$.

If, on the contrary, $\sigma_1 < +\infty$, we get, using (3.3) again, that

$$\frac{1}{2}\int_\Omega |\nabla v|^2 \le \frac{\lambda}{2},$$

with $|v|_{L^2(\partial\Omega)} = 1$ and $v = 0$ on Γ_2, and taking into account (2.7) that is not compatible with the assumption $\lambda < \sigma_1$.

Finally, let us see that Φ is weakly lower semicontinuous. Assume that $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$. By standard compactness results we have that $u_n \to u$ both strongly in $L^2(\partial\Omega)$ and $L^2(\Omega)$. Thus to show that

$$\Phi(u) \le \liminf_{n\to\infty} \Phi(u_n),$$

it is only necessary to prove that

$$\int_\Omega a|u|^{p+1} \le \liminf_{n\to\infty} \int_\Omega a|u|_n^{p+1}. \tag{3.4}$$

However, Fatou's lemma implies that from any subsequence of u_n a new one can be selected so that (3.4) holds. Thus the relation holds true for the whole sequence.

We have just shown that Φ achieves an absolute minimum at some nonnegative $\tilde{u} \in H^1(\Omega)$. It can be checked now that Φ can be differentiated at any $u \in H^1(\Omega)$, $\Phi(u) < \infty$, in any direction $v \in C^1(\overline{\Omega})$ with:

$$\frac{d}{dt}(\Phi(u+tv))_{t=0} = \int_\Omega \nabla u \nabla v + \int_\Omega a|u|^p uv - \lambda \int_{\partial\Omega} uv. \tag{3.5}$$

This means that $\tilde{u}$ defines a weak solution to (1.1).

To conclude the proof observe that any weak nonnegative solution to (1.1) satisfies $-\Delta u \le 0$ weakly in Ω. Thus $u \in L^\infty_{\text{loc}}(\Omega)$ (see Chapter 8 in [12]) and a bootstrapping argument gives $u \in C^{1,\beta}(\Omega)$, $0 < \beta < 1$. □

3.2. Proof of Theorem 1

We are next providing a proof of Theorem 1. In the following lemma a proof of uniqueness for classical solutions is given. For later use we are dealing with slightly more general problems than (1.1). Then we will find the range for existence in a second lemma to finally proceed with the main course of the proof.

Lemma 8. *Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded $C^{2,\alpha}$ domain such that $\partial\Omega$ splits in two disjoint groups of connected pieces $\Gamma_{\mathcal{N}}$ and $\Gamma_{\mathcal{D}}$. Consider the problem,*

$$\begin{cases} \Delta u = a(x)u^p & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda u & x \in \Gamma_{\mathcal{N}} \\ u = g(x) & x \in \Gamma_{\mathcal{D}}, \end{cases}$$

where $g \in C^1(\Gamma_{\mathcal{D}})$, and let $u_1, u_2 \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be two classical nonnegative and nontrivial solutions. Then $u_1 = u_2$.

Proof. It can be shown by means of Hopf's maximum principle that any nonnegative classical solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, $u \not\equiv 0$, satisfies $\inf_\Omega u > 0$.

We next use ideas from [3]. Assume that u_1, u_2 are positive solutions and consider the relation,

$$\int_\Omega \left(-\frac{\Delta u_1}{u_1} + \frac{\Delta u_2}{u_2} \right) (u_1^2 - u_2^2) = -\int_\Omega a(x)(u_1^{p-1} - u_2^{p-1})(u_1^2 - u_2^2) \le 0. \tag{3.6}$$

Integrating by parts we get,

$$-\int_\Omega \Delta u_1 \frac{(u_1^2-u_2^2)}{u_1} = -\int_{\Gamma_\mathcal{N}} \frac{\partial u_1}{\partial \nu}\frac{(u_1^2-u_2^2)}{u_1} + \int_\Omega \nabla u_1 \nabla\left(\frac{u_1^2-u_2^2}{u_1}\right)$$
$$= -\lambda\int_{\Gamma_\mathcal{N}} (u_1^2-u_2^2) + \int_\Omega \nabla u_1 \nabla\left(\frac{u_1^2-u_2^2}{u_1}\right).$$

Similarly

$$-\int_\Omega \Delta u_2 \frac{(u_1^2-u_2^2)}{u_2} = -\lambda\int_{\Gamma_\mathcal{N}} (u_1^2-u_2^2) + \int_\Omega \nabla u_2 \nabla\left(\frac{u_1^2-u_2^2}{u_2}\right).$$

Thus,

$$\int_\Omega \left(-\frac{\Delta u_1}{u_1} + \frac{\Delta u_2}{u_2}\right)(u_1^2-u_2^2) = \int_\Omega \left|\nabla u_1 - \frac{u_1}{u_2}\nabla u_2\right|^2 + \left|\nabla u_2 - \frac{u_2}{u_1}\nabla u_1\right|^2$$
$$= \int_\Omega u_1^2\left|\nabla\left(\frac{u_2}{u_1}\right)\right|^2 + u_2^2\left|\nabla\left(\frac{u_1}{u_2}\right)\right|^2 \geq 0.$$

In conclusion, both integrals in (3.6) vanish. From the first one we get $u_2 = cu_1$ for some constant c, while the second implies $u_1 = u_2$ in $\{a > 0\}$. Therefore $u_1 = u_2$. □

Lemma 9. *Positive classical solutions to* (1.1) *are only possible if* λ *satisfies* (1.3)*:*

$$0 < \lambda < \sigma_1.$$

Proof. If u is a positive solution of (1.1), direct integration gives:

$$\int_\Omega au^p = \lambda\int_{\partial\Omega} u.$$

As already shown, u is bounded away from zero in $\overline{\Omega}$ and hence $\lambda > 0$.

As for the complementary estimate assume again that u is a positive solution. From Green's identity:

$$\int_{\partial\Omega_0} \varphi_1\frac{\partial u}{\partial\nu} - u\frac{\partial\varphi_1}{\partial\nu} = 0,$$

where φ_1 is a principal positive eigenfunction to (1.2). Thus,

$$(\lambda - \sigma_1)\int_{\Gamma_1} u\varphi_1 = \int_{\Gamma_2} u\frac{\partial\varphi_1}{\partial\nu}.$$

Since

$$\frac{\partial\varphi_1}{\partial\nu}(x) < 0, \qquad x \in \Gamma_2,$$

we then obtain that necessarily $\lambda < \sigma_1$. □

Proof of Theorem 1. To obtain a classical positive solution we are employing the method of sub and supersolutions (cf. [2]). A nonnegative $\underline{u} \in C^2(\Omega)\cap C^1(\overline{\Omega})$ is said to be a subsolution of (1.1) if

$$\Delta\underline{u} \geq a(x)\underline{u}^p \qquad x \in \Omega,$$

together with $\dfrac{\partial \underline{u}}{\partial \nu} \le \lambda \underline{u}$ on $\partial\Omega$ (a supersolution is defined by reversing the inequalities).

For every λ in the existence range (1.3) if ϕ_1 is the positive eigenfunction associated to the principal eigenvalue $\mu_1 = \mu_1(\lambda)$ of (2.1), normalized as $\sup_\Omega \phi_1 = 1$, then it can be checked that

$$\underline{u} := A\phi_1,$$

defines a subsolution of (1.1) provided,

$$0 < A \le \left(\frac{\mu_1}{|a|_{\infty,\Omega}}\right)^{1/(p-1)}, \tag{3.7}$$

that is, provided A is small enough. Finding a comparable supersolution is however a more subtle task. We are considering by turn the cases $a > 0$ in $\partial\Omega$ and $a = 0$ at some parts of $\partial\Omega$ (possibly at the whole $\partial\Omega$).

Under the first assumption, if $a > 0$ in $\overline{\Omega}$ (in particular $\Omega_0 = \emptyset$),

$$\overline{u} := B\phi_1,$$

is readily seen to be a supersolution if B is large enough as to satisfy:

$$B \ge \left(\frac{\mu_1}{\inf_\Omega a\ (\inf_\Omega \phi_1)^{p-1}}\right)^{1/(p-1)}.$$

Recall that $\inf_\Omega \phi_1 > 0$. Thus assume $a > 0$ on $\partial\Omega$ but $\Omega_0 \ne \emptyset$. To construct a supersolution an approach from [16] is used. Let $\delta > 0$ be chosen so that $B_\delta = B(\Omega_0, \delta) := \{x \in \Omega : \operatorname{dist}(x, \Omega_0) < \delta\} \subset\subset \Omega$. Let ψ_1 be the principal positive eigenfunction of the Dirichlet problem:

$$\begin{cases} -\Delta\psi = \sigma\psi & x \in B_\delta \\ \psi = 0 & x \in \partial B_\delta, \end{cases}$$

associated to the principal eigenvalue $\sigma = \sigma_1(B_\delta)$, satisfying $\sup_{B_\delta} \psi_1 = 1$ (recall that $\sigma_1(B_\delta) > 0$). The restriction of ψ_1 to $\overline{B(\Omega_0, \delta/2)}$ can be extended to the whole of $\overline{\Omega}$ as a positive $C^{2,\alpha}$ function $\hat{\psi}_1$ such that,

$$\frac{\partial \hat{\psi}_1}{\partial \nu} = \lambda \hat{\psi}_1,$$

on $\partial\Omega$. For instance, it suffices that $\hat{\psi}_1 = ce^{\lambda d(x)}$ near $\partial\Omega$ with $d(x) = \operatorname{dist}(x, \partial\Omega)$ and c a positive constant. Then,

$$\overline{u} := B\hat{\psi}_1,$$

defines a $C^{2,\alpha}$ supersolution to (1.1) if $B > 0$ is so large as to have:

$$\left\{\frac{\sup_{D_\delta} |\Delta\hat{\psi}_1|}{\inf_{D_\delta} a\ \{\inf_{D_\delta} \hat{\psi}_1\}^p}\right\}^{1/(p-1)} \le B,$$

where $D_\delta = \overline{\Omega} \setminus B(\Omega_0, \delta/2)$.

As a final conclusion, in the case $a > 0$ at $\partial\Omega$ a pair $\underline{u}, \overline{u}$ of sub and supersolutions can be constructed so that $\underline{u} \le \overline{u}$, hence a positive solution $u \in C^{2,\alpha}(\overline{\Omega})$ exists in the functional interval $[\underline{u}, \overline{u}]$ (cf. [2]).

Let us deal now with the case $a = 0$ at some points of $\partial\Omega$. Under hypothesis (H) we will first consider the very particular case $a(x) = 0$ for all $x \in \partial\Omega$. To construct a classical supersolution $\overline{u}$ consider,

$$\begin{aligned} \Omega_\delta &:= [B(\Omega_0, \delta) \cup \{\text{dist } (x, \partial\Omega) < \delta\}] \cap \Omega \\ &= \{x \in \Omega : \text{dist } (x, \Omega_0) < \delta \ \text{ or } \ \text{dist } (x, \partial\Omega) < \delta\}. \end{aligned}$$

Observe that $\partial\Omega_\delta = \partial\Omega \cup [\partial\Omega_\delta \cap \Omega]$ and that Ω_δ approaches Ω_0 and $\partial\Omega_\delta \cap \Omega$ approaches $(\partial\Omega \setminus \Gamma_1) \cup \Gamma_2$ as $\delta \to 0+$ (although Ω_δ may not be connected, the results in Section 2 concerning the eigenvalue problems still hold true). The auxiliary problem:

$$\begin{cases} \Delta\varphi = 0 & x \in \Omega_\delta \\ \dfrac{\partial\varphi}{\partial\nu} = \sigma\varphi & x \in \partial\Omega \\ \varphi = 0 & x \in \partial\Omega_\delta \cap \Omega \end{cases}$$

admits (Theorem 6) a principal eigenvalue $\sigma_{1,\delta}$ with a positive associated eigenfunction φ_1 (its dependence on δ being omitted at this moment). We claim that $\sigma_{1,\delta} \le \sigma_1$ while in addition $\lim_{\delta \searrow 0} \sigma_{1,\delta} = \sigma_1$. Assumed this fact, if $\lambda < \sigma_1$ a construction similar as the previous one can be performed with the positive eigenfunction φ_1 associated to $\sigma_{1,\delta}$, provided $\delta > 0$ is small as to verify,

$$\lambda < \sigma_{1,\delta} \le \sigma_1.$$

Namely, to extend its restriction to $\Omega_{\delta/2}$ to the whole $\overline{\Omega}$ as a positive smooth function $\hat{\varphi}_1$. Remark that now we define $\hat{\varphi}_1$ away from $\partial\Omega$ and have no interference with the boundary condition in (1.1). The desired supersolution is thus provided by,

$$\overline{u} := B\hat{\varphi}_1,$$

with $B > 0$ large enough. Once more we get the existence of a positive solution to (1.1) in this case.

For the general case of a introduce $a_\delta(x) = \eta(x)a(x)$ where η is smooth, supp $\eta = \{\text{dist } (x, \partial\Omega) \ge \delta\}$, $\eta = 1$ in $\{\text{dist } (x, \partial\Omega) \ge 2\delta\}$, $0 \le \eta \le 1$. Assuming $\delta > 0$, the problem

$$\begin{cases} \Delta u = a_\delta(x)u^p & x \in \Omega \\ \\ \dfrac{\partial u}{\partial\nu} = \lambda u & x \in \partial\Omega\,, \end{cases}$$

as just seen before admits a positive solution $u_{\lambda,\delta}$ if $\lambda < \sigma_{1,\delta} < \sigma_1$ and defines in turn a supersolution to (1.1). It can be also enlarged by multiplying by a constant, i.e.,

$$\overline{u} = Bu_{\lambda,\delta},$$

with $B > 0$. This concludes the proof of existence.

Let us come back to show the claim regarding $\sigma_{1,\delta}$. To fix notation set $\Gamma_\delta = \partial\Omega_\delta \cap \Omega = \{x \in \Omega : \text{dist}\,(x, \partial\Omega) = \delta \;\text{ or }\; \text{dist}\,(x, \Omega_0) = \delta\}$, $H^1_{\Gamma_\delta} = \{u \in H^1(\Omega_\delta) : u_{|\Gamma_\delta} = 0\}$. Then, if for $u \in H^1_{\Gamma_\delta}$ and $\delta > 0$ we define,

$$Q_\delta(u) = \frac{\int_{\Omega_\delta} |\nabla u|^2}{\int_{\partial\Omega} u^2},$$

whereas, for $u \in H^1_{\Gamma_2}$

$$Q_0(u) = \frac{\int_{\Omega_0} |\nabla u|^2}{\int_{\Gamma_1} u^2},$$

then, for every $\delta \ge 0$ small

$$\sigma_{1,\delta} = \inf_{u \in H^1_{\Gamma_\delta} \setminus \{0\}} Q_\delta(u),$$

wherein we understand $\Gamma_\delta = \Gamma_2$ and $\sigma_{1,\delta} = \sigma_1$ when $\delta = 0$. Select now any eigenfunction associated to σ_1, $\varphi_1 \in H^1_{\Gamma_2}(\Omega_0)$. Its extension $\bar\varphi_1$ as zero to Ω_δ belongs to $H^1_{\Gamma_\delta}(\Omega_\delta)$ while:

$$\sigma_1 = Q_0(\varphi_1) = Q_\delta(\bar\varphi_1) \ge \sigma_{1,\delta}.$$

If, similarly φ_{1,δ_i} are eigenfunctions associated to σ_{1,δ_i}, $i = 1, 2$, $0 < \delta_1 < \delta_2$, φ_{1,δ_1} extended as zero to Ω_{δ_2} ($\bar\varphi_{1,\delta_1}$ the extension) lies in $H^1_{\Gamma_{\delta_2}}$ and,

$$\sigma_{1,\delta_1} = Q_{\delta_1}(\varphi_{1,\delta_1}) = Q_{\delta_2}(\bar\varphi_{1,\delta_1}) \ge \sigma_{\delta_2}.$$

In conclusion $\sigma_{1,\delta}$ does not decrease as $\delta \searrow 0$ and $\sigma_{1,\delta} \le \sigma_1$.

We are next showing that $\liminf_{\delta\searrow 0} \sigma_{1,\delta} = \sigma_1$ (i.e., that $\sigma_{1,\delta} \to \sigma_1$ as $\delta \searrow 0$). In fact, for each $\delta > 0$ small normalize $\varphi_{1,\delta}$ so that $\int_{\partial\Omega} \varphi_{1,\delta}^2 = 1$, designate as $\bar\varphi_{1,\delta} \in H^1(\Omega)$ its extension by zero to Ω and use $|u|_1 = \left(\int_\Omega |\nabla u|^2 + \int_{\partial\Omega} u^2\right)^{1/2}$ as a equivalent norm for $H^1(\Omega)$. Since,

$$|\varphi_{1,\delta}|_1^2 = 1 + \sigma_{1,\delta} \le 1 + \sigma_1,$$

$\bar\varphi_{1,\delta} \rightharpoonup \bar\varphi$ weakly in $H^1(\Omega)$ as $\delta \searrow 0$, hence $\bar\varphi_{1,\delta} \to \bar\varphi$ in $L^2(\Omega)$, $L^2(\partial\Omega)$ and $L^2(\Gamma_2)$. As a first conclusion, $\bar\varphi = 0$ a. e. in $\Omega \setminus \Omega_0$ and thus $\bar\varphi = 0$ both on Γ_2 and $\partial\Omega \setminus \Gamma_1$ in the sense of traces. In particular,

$$\int_{\Gamma_1} \bar\varphi^2 = 1.$$

On the other hand,

$$\sigma_1 \le \int_{\Omega_0} |\nabla\bar\varphi|^2 = \int_\Omega |\nabla\bar\varphi|^2 \le \liminf_{\delta\to 0} \int_\Omega |\nabla\bar\varphi_{1,\delta}|^2 = \liminf_{\delta\to 0} \sigma_{1,\delta} \le \sigma_1.$$

This implies both that $\bar\varphi$ as observed in Ω_0 defines an eigenfunction associated to σ_1 and the desired limit. The proof of the claim is concluded.

We are next showing the remaining parts of Theorem 1. As for *ii)* observe that problem $(1.1)_\lambda$ always admits subsolutions as smaller as desired (see (3.7)) while the positive solution u_{λ_1} to (1.1) with λ_1 replacing λ defines a strict supersolution to $(1.1)_\lambda$ if $\lambda_1 > \lambda$. Thus,

$$u_\lambda(x) < u_{\lambda_1}(x) \qquad x \in \Omega,$$

for $\lambda < \lambda_1$. To achieve the continuity of u_λ in λ assume that $\lambda_n \to \lambda$, $\lambda > 0$, and set $\underline{\lambda} = \inf \lambda_n$ (it can be assumed $\underline{\lambda} > 0$), $\bar{\lambda} = \sup \lambda_n$. Then $u_{\underline{\lambda}} \le u_{\lambda_n} \le u_{\bar{\lambda}}$ and u_{λ_n} is bounded in L^∞. Now the relevant estimate in [2] (see Lemma 3.2 there) implies the boundedness of u_{λ_n} in $W^{1,p}(\Omega)$ for all $p > 1$ what, in turn and by standard arguments, successively provides its boundedness in $W^{2,p}(\Omega)$ for all $p > 1$ and in $C^{1,\beta}(\overline{\Omega})$ for all $0 < \beta < 1$. Next, the Schauder estimates ([12]) provide a subsequence $u_{\lambda_{n'}} \to u_\lambda$ in $C^{2,\alpha}(\overline{\Omega})$. Finally, the uniqueness and the same argument permit extending such convergence to the whole sequence u_{λ_n}. By the same token, if $\lambda_n \to 0$ in $\mathbb{R}^+$, u_{λ_n} converges in $C^{2,\alpha}(\overline{\Omega})$ to a nonnegative solution of (1.1) with $\lambda = 0$ what means that this limit is zero.

To show *iii)* let $u_0 \in C^{1,\alpha}(\overline{\Omega})^+$, $u_0 \not\equiv 0$ be an initial datum for (1.4) (due to the parabolic regularization effect, this smoothness can be considerably relaxed). Then, (1.4) admits a nonnegative and globally defined for $t > 0$ solution $u(x,t)$, $u(\cdot,t) \in C^{2,\alpha}(\overline{\Omega})$ for all t. In fact $u_-(x,t) = 0$ serves as a subsolution while $u_+(x,t) = c\phi_1 e^{\mu_1(\lambda)t}$, with ϕ_1 a positive principal eigenfunction of (2.1), defines a positive supersolution for a suitably chosen constant $c > 0$. On the other hand, the parabolic strong maximum principle implies that for all $t > 0$, $u(x,t) > 0$ in $\overline{\Omega}$. This means that for $t_0 > 0$ small as desired,

$$\underline{u}(x) \le u(x,t_0) \le \overline{u}(x), \qquad x \in \Omega,$$

if the constants A, B, modulating respectively the subsolution $\underline{u}$ and the supersolution $\overline{u}$ are properly chosen. It is then well known (cf. [21]) that this implies

$$u(x,t) \to u_\lambda(x) \qquad x \in \Omega,$$

as $t \to \infty$ (exponentially) in $C^{2,\alpha}(\overline{\Omega})$ and the proof of Theorem 1 is finished. □

Remark 9. It is implicit in the proof of Theorem 1 the existence for all $\lambda > 0$ of a unique positive classical solution $u \in C^{2,\alpha}(\overline{\Omega})$ to (1.1) if, for instance, dist $(\{a = 0\} \cap \Omega, \partial\Omega) > 0$ even being $a = 0$ at points of $\partial\Omega$. Thus $a \in C^\alpha(\overline{\Omega})$ may vanish in Ω provided there is a gap between $\{a = 0\}$ and the boundary $\partial\Omega$.

4. Behavior as $\lambda \to \sigma_1$

In this section we are providing the proofs of Theorems 3 and 4. We are first considering the latter.

Proof of Theorem 4. First of all, it is well known that for a function $a \in C^\alpha(\overline{\Omega})$, *positive* in Ω, problem (1.8)

$$\begin{cases} \Delta u = a(x)u^p & x \in \Omega \\ u = \infty & x \in \partial\Omega, \end{cases}$$

admits a minimum positive solution $z_D(x) \in C^{2,\alpha}(\Omega)$ attaining the boundary condition in the sense $\lim_{d(x)\searrow 0} u(x) = \infty$, $d(x) = \text{dist }(x, \partial\Omega)$ (see the references in [5, 9, 19]). It can be also shown that,

$$z_D(x) = \sup u(x) \qquad x \in \Omega,$$

being the supremum extended to all classical nonnegative solutions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of $\Delta u = au^p$. We claim that the same facts *also* hold in the case where a vanishes in a whole subdomain $\Omega_0 \subset\subset \Omega$ just under the conditions of the present statement (see a brief proof below). Assumed this and if $u_\lambda \in C^{2,\alpha}(\overline{\Omega})$ stands for the positive solution to (1.1) then,

$$u_\lambda(x) < z_D(x) \qquad x \in \Omega,$$

for all $\lambda > 0$. Being u_λ increasing in λ this implies that,

$$z(x) := \sup_{\lambda>0} u_\lambda(x) = \lim_{\lambda\to\infty} u_\lambda(x) \le z_D(x) \qquad x \in \Omega.$$

Moreover, employing the L^p estimates and bootstrap in the standard way (check the proof of Theorem 1) we get $z \in C^{2,\alpha}(\Omega)$ being the limit valid in $C^{2,\alpha}(\Omega)$. Therefore, z solves the equation in (1.8).

Our next issue is elucidating the boundary behavior of z. It will be in fact shown that z and z_D coincide. As a first remark observe that,

$$z(x) = \sup_{m>0} v_m(x) \qquad x \in \Omega,$$

where for $m > 0$, $v_m \in C^{2,\alpha}(\overline{\Omega})$ is the unique positive solution to,

$$\begin{cases} \Delta u = a(x)u^p & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = m & x \in \partial\Omega. \end{cases}$$

In fact, notice that v_m increases in m while, for $\lambda > \lambda_1 > 0$,

$$\frac{\partial u_\lambda}{\partial \nu} = \lambda u_\lambda > \lambda u_{\lambda_1} \to \infty,$$

uniformly on $\partial\Omega$ as $\lambda \to \infty$.

On the other hand, in order to work with the constant coefficients case observe that for each $m > 0$ we get by comparison $\tilde{v}_m < v_m$ where $\tilde{v}_m$ is the positive solution of,

$$\begin{cases} \Delta u = |a|_\infty u^p & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = m & x \in \partial\Omega. \end{cases}$$

Thus, being $\tilde{z} = \sup_m \tilde{v}_m$ we have $\tilde{z} \le z$ and so to get $z = z_D$ we only need to show that $\tilde{z} \to \infty$ as $d \to 0+$. Using now ideas from [5, 9] it is not too hard to prove the following assertions. Consider,

$$\gamma = \frac{2}{p-1} \qquad\qquad A = \left\{ \frac{\gamma(\gamma+1)}{|a|_\infty} \right\}^{1/(p-1)},$$

together with,

$$h(d) = (A-\eta)d^{-\gamma} \qquad d(x) = \text{dist }(x, \partial\Omega),$$

$0 < \eta < A$. Then, for η as small as desired positive constants δ, τ_0 can be found such that,

$$w_\tau(x) = h(d(x) + \tau),$$

defines a $C^{2,\alpha}$ subsolution of $\Delta u = |a|_\infty u^p$ in the region $D_\delta = \{x \in \Omega : 0 < d(x) < \delta\}$ for every $0 \le \tau < \tau_0$. In addition, a positive constant k can be chosen so that,

$$0 < w_\tau(x) - k < \tilde{z}(x), \tag{4.1}$$

for every x with $d(x) = \delta$. Notice that $w_\tau - k$ is again a subsolution under the same conditions as w_τ. On the other hand, $\tilde{v}_m$ solves the next mixed problem in D_δ,

$$\begin{cases} \Delta u = |a|_\infty u^p & x \in D_\delta \\ \dfrac{\partial u}{\partial \nu} = m & x \in \partial\Omega \\ u = \tilde{v}_m & d = \delta. \end{cases}$$

For $0 < \tau < \tau_0$ fixed we conclude in view of (4.1) and the finiteness of $\dfrac{\partial w_\tau}{\partial \nu}$ at $\partial\Omega$ that for m large enough,

$$w_\tau(x) - k \le \tilde{v}_m(x) \qquad x \in D_\delta.$$

Making first $m \to \infty$ and then $\tau \searrow 0$ we get,

$$h(d) - k \le \tilde{z},$$

which gives $\tilde{z} \to \infty$ as $d \searrow 0$, as desired.

Let us prove now the claim. Namely, that the singular boundary value problem (1.8) has a minimum classical positive solution $z \in C^{2,\alpha}(\Omega)$ if $a \in C^\alpha(\overline{\Omega})$ vanishes in a $C^{2,\alpha}$ subdomain $\Omega_0 \subset \overline{\Omega}_0 \subset \Omega$ (see [10] for a more general setting). Following a standard approach (cf. [9], [14]) z is given by the limit, whose existence must be proved,

$$z(x) = \lim u_m(x) \qquad x \in \Omega, \tag{4.2}$$

where $u_m \in C^{2,\alpha}(\Omega)$ is the positive solution to the problem,

$$\begin{cases} \Delta u = a(x) u^p & x \in \Omega \\ u = m & x \in \partial\Omega. \end{cases}$$

The existence of u_m is achieved, say by sub and supersolutions, while u_m increases in m and the limit (4.2) is actually a supremum.

If $z_{\mathcal{D},\Omega^+} \in C^{2,\alpha}(\Omega^+)$, $\Omega^+ = \{a(x) > 0\} \cap \Omega$, stands for the minimum positive solution to

$$\begin{cases} \Delta u = a(x) u^p & x \in \Omega^+ \\ u = \infty & x \in \partial\Omega^+, \end{cases}$$

then,

$$u_m(x) < z_{\mathcal{D},\Omega^+}(x) \qquad x \in \Omega^+,$$

and so by the bootstrap argument invoked before we get the existence and validity of the limit (4.2) in $C^{2,\alpha}(\Omega^+)$. In order to cover $\overline{\Omega}_0$ observe that u_m solves the Dirichlet problem,

$$\begin{cases} \Delta u = a(x) u^p & x \in B(\Omega_0, \delta) \\ u = u_m & x \in \partial B(\Omega_0, \delta), \end{cases}$$

where $B(\Omega_0,\delta)=\{x\in\Omega : \text{dist }(x,\Omega_0)<\delta\}$, $\delta>0$ small. On the other hand, since $\partial B(\Omega_0,\delta)\subset\Omega^+$ then,

$$u_m(x)<z_{\mathcal{D},\Omega^+}(x) \qquad x\in\partial B(\Omega_0,\delta).$$

Being every u_m subharmonic this means that u_m is bounded in $B(\Omega_0,\delta)$ and again, this implies the existence of the limit (4.2) in $C^{2,\alpha}(B(\Omega_0,\delta))$, therefore in $C^{2,\alpha}(\Omega)$. This finishes the proof of the claim.

Finally, the asymptotic rate (1.9) follows from (see (3.7)),

$$\sup_\Omega u_\lambda \geq \left(\frac{\mu_1}{|a|_\infty}\right)^{1/(p-1)}$$

together with the estimate (2.2) for μ_1 as $\lambda\to\infty$. □

Remark 10. It was pointed out in Remark 6 that the principal positive eigenfunction $\phi_{1,\lambda}$, normalized as $\sup_\Omega\phi_{1,\lambda}=1$, decays to zero a. e. in Ω as $\lambda\to\infty$. As a consequence of the last part of the proof of Theorem 4 it further follows that $\phi_{1,\lambda}(x)\to 0$ as $\lambda\to\infty$ for all $x\in\Omega$, faster than any polynomial. In fact (3.7) implies, using $p>1$ as a parameter, that

$$\lambda^{2/(p-1)}\phi_{1,\lambda}(x)\leq\left(\frac{\lambda^2}{\mu_1}\right)^{1/(p-1)}|a|_{\infty,\Omega}^{1/(p-1)}u_\lambda(x) \qquad x\in\Omega.$$

Hence, from (2.2):

$$\limsup_{\lambda\to\infty}\left(\lambda^{2/(p-1)}\phi_{1,\lambda}(x)\right)\leq|a|_{\infty,\Omega}^{1/(p-1)}z(x),$$

and the conclusion follows.

Proof of Theorem 3. Let us first deal with the behavior of u_λ in Ω_0 with special emphasis on the boundary behavior. To construct a suitable subsolution to (1.1) consider an special choice of the perturbed eigenvalue problem (2.5). Namely,

$$\begin{cases}\Delta\varphi=c_1\delta\ \varphi & x\in B(\Omega_0,\delta)\\ \dfrac{\partial\varphi}{\partial\nu}=\sigma\varphi & x\in\Gamma_1\\ \varphi=0 & x\in\Gamma_{2,\delta},\end{cases}\tag{4.3}$$

where $B(\Omega_0,\delta)=\{x\in\Omega : \text{dist }(x,\Omega_0)<\delta\}$, $\Gamma_{2,\delta}=\{x\in\Omega : \text{dist }(x,\Omega_0)=\delta\}$ and $\delta>0$ is small (notice that $\partial B(\Omega_0,\delta)=\Gamma_1\cup\Gamma_{2,\delta}$). If φ_1 is any principal eigenfunction to (1.2), c_1 is a positive constant chosen so that

$$c_1\int_{\Omega_0}\varphi_1^2<\int_{\Gamma_2}|\nabla\varphi_1|^2.$$

Thus, by using the expression (2.9) for the derivative $d\sigma_{1,t}/dt_{|t=0}$ (now $t=\delta$), the principal eigenvalue $\sigma_{1,\delta}>0$ to (4.3) increases when δ decays to zero. Moreover, with the same arguments as in the proof of Theorem 1 (now in a simpler case) it is shown that $\lim_{\delta\searrow 0}\sigma_{1,\delta}=\sigma_1$. On the other hand, if $\tilde\varphi_\delta$ stands for the principal eigenfunction to (4.3) normalized according to $\sup_{\Omega_0}\tilde\varphi_\delta=1$, then the L^∞ boundedness of $\{\tilde\varphi_\delta\}$ and the estimates in [2], [1] yield $\tilde\varphi_\delta \to \varphi_1$ in $C^{2,\alpha}(\overline{\Omega}_0)$, where now φ_1 is the principal eigenfunction of (1.2) normalized as $\tilde\varphi_\delta$.

For $\lambda \nearrow \sigma_1$ we take $\delta = \delta(\lambda) \searrow 0$ such that $\sigma_{1,\delta} < \lambda < \sigma_1$ and look for a weak subsolution to (1.1) in the form

$$\underline{u} = A\ \tilde\varphi_\delta,$$

where $\tilde\varphi_\delta$ is extended as zero to $\overline{\Omega}\setminus B(\Omega_0,\delta)$ and $A = A(\delta)$. We claim that such a subsolution can be constructed so that,

$$\delta\ A(\delta) \to \infty \qquad \delta \to 0+\,. \tag{4.4}$$

Assumed this fact and since as large as desired positive supersolutions $\overline{u}$ to (1.1) can be found, then the positive solution u_λ satisfies

$$\underline{u}(x) \le u_\lambda(x) \le \overline{u}(x) \qquad x \in \Omega.$$

Being $\tilde\varphi_\delta \to \varphi_1$ in $C^{2,\alpha}(\overline{\Omega}_0)$ and $\varphi_1 > 0$ in $\Omega_0 \cup \Gamma_1$, (4.4) implies that $u_\lambda \to \infty$ in $\Omega_0 \cup \Gamma_1$. Moreover as,

$$\inf_{\Gamma_2} \varphi_\delta \sim C_1\delta \qquad \delta \searrow 0, \tag{4.5}$$

(see [18]) by (4.4) we conclude that $u_\lambda \to \infty$ as $\lambda \nearrow \sigma_1$ uniformly on Γ_2. This proves *i)*.

To show the claim we only need to check that an A satisfying (4.4) can be found with the additional requirement,

$$A\ \tilde\varphi_\delta(x) \le \left\{\frac{c_1}{\sup_{0<d(x)<\delta}(a(x)/\delta)}\right\}^{1/(p-1)} \qquad 0 < d(x) < \delta, \tag{4.6}$$

with $d(x) = \text{dist}\ (x,\Omega_0)$. This election is actually possible due to (4.5) and the fact $a(x) = o(d(x))$ as $d(x) \searrow 0$.

Observe now that *iii)* is an immediate consequence of the previous discussion. In fact, $\Omega^+ \subset\subset \Omega$ means that $\partial\Omega^+ = \Gamma_2$. Since $u_{\lambda|\Gamma_2} \to \infty$ uniformly on Γ_2 we then conclude that $u_\lambda \to z_{\mathcal{D},\Omega^+}$ in $C^{2,\alpha}(\Omega^+)$ (see the proof of Theorem 4).

As for *ii)* we are proving the existence of the minimum solution to (1.6). Accordingly, assume that $z_1(x) \in C^{2,\alpha}(\Omega^+ \cup \Gamma^+)$ is any of its possible positive solutions. For each $m \in \mathbb{N}$ the auxiliary problem,

$$\begin{cases} \Delta u = a(x)u^p & x \in \Omega^+ \\[2mm] \dfrac{\partial u}{\partial \nu} = \sigma_1 u & x \in \Gamma^+ \\[2mm] u = m & x \in \Gamma_2, \end{cases} \tag{4.7}$$

admits at most a unique positive solution in $C^{2,\alpha}(\overline{\Omega^+})$ (Lemma 8) and has supersolutions as large as required (proof of Theorem 1). Therefore it possesses a positive solution $u_m \in C^{2,\alpha}(\overline{\Omega^+})$ which by comparison is seen to satisfy,

$$u_m(x) \le z_1(x).$$

In fact, being $\Omega^{+,\delta} := \Omega \setminus \overline{B(\Omega_0, \delta)}$, such inequality holds on $\Omega^{+,\delta} \cup \Gamma^+$, for every $\delta \to 0+$. On the other hand u_m is increasing in m while $u_m(x) \le z_{\mathcal{D},\Omega^+}(x)$ in Ω^+. Hence,

$$\lim u_m(x) = \sup u_m(x) := z(x) \le z_{\mathcal{D},\Omega^+}(x) \qquad x \in \Omega^+, \tag{4.8}$$

and this limit is valid in $C^{2,\alpha}(\Omega^+)$. Hence, z solves $\Delta u = a(x)u^p$ in Ω^+. To cover Γ^+ observe that $\partial\Omega^{+,\delta} = \Gamma^+ \cup \Gamma_{2,\delta}$, $\Gamma_{2,\delta} = \{x \in \Omega : \text{dist }(x, \Omega_0) = \delta\}$ while for $\delta > 0$ small the auxiliary problem,

$$\begin{cases} \Delta u = a(x)u^p & x \in \Omega^{+,\delta} \\ \\ \dfrac{\partial u}{\partial \nu} = \sigma_1 u & x \in \Gamma^+ \\ \\ \quad u = z_{\mathcal{D},\Omega^+} & x \in \Gamma_{2,\delta}, \end{cases} \tag{4.9}$$

also admits a unique positive solution $u_\delta(x) \in C^{2,\alpha}(\overline{\Omega^{+,\delta}})$ (just the same reasons as in (4.7)). Since every u_m a is subsolution to (4.9), and that problem possesses large supersolutions, we obtain

$$u_m(x) \le u_\delta(x) \qquad x \in \overline{\Omega^{+,\delta}}.$$

By using the estimates in [2] and [1] we achieve that the limit (4.8) holds in $C^{2,\alpha}(\overline{\Omega^{+,\delta}})$ for $\delta > 0$ arbitrarily small. Thus the limit is actually valid in $C^{2,\alpha}(\Omega^+ \cup \Gamma^+)$ and in particular

$$\frac{\partial z}{\partial \nu} = \sigma_1 z \qquad x \in \Gamma^+.$$

Since in addition $z(x) \le z_1(x)$, z is the minimum solution to (1.6) which is denoted as $z_{\mathcal{M},\Omega^+}$.

On the other hand, the solution u_λ to (1.1) satisfies,

$$u_\lambda(x) \le z_{\mathcal{M},\Omega^+}(x) \qquad x \in \Omega^+ \cup \Gamma^+,$$

since $u_\lambda(x) \le u_m(x)$ in $\overline{\Omega^+}$ for m large. Reasoning as before

$$z_2(x) := \sup_{\lambda \nearrow \sigma_1} u_\lambda(x) = \lim_{\lambda \nearrow \sigma_1} u_\lambda(x) \le z_{\mathcal{M},\Omega^+}(x) \qquad x \in \Omega^+ \cup \Gamma^+,$$

defines a classical solution of (1.6) being the limit valid in $C^{2,\alpha}(\Omega^+ \cup \Gamma^+)$. Therefore, $z_2 = z_{\mathcal{M},\Omega^+}$ and the proof is concluded. □

Remark 11. Let us briefly discuss what happens if $\Omega_0 \subset \Omega$ consists of M connected pieces $\Omega_{0,1}, \dots, \Omega_{0,M}$, all of them $C^{2,\alpha}$ and satisfying (H) in the sense that for each $1 \le i \le M$, $\Gamma_{1,i} = \partial\Omega_{0,i} \cap \partial\Omega$ is nonempty and consists of connected pieces of $\partial\Omega$, $\Gamma_{2,i} = \partial\Omega_{0,i} \cap \Omega$.

If $\Omega_{0,i} \subset\subset \Omega$ for each $i \in \{1, \dots, M\}$ then a unique positive classical solution u_λ to (1.1) exists for all $\lambda > 0$ which exhibits all the features of both Theorems 1 and 4.

On the contrary, if $\Gamma_{1,j} \neq \emptyset$ for some $1 \le j \le M$ define $\sigma^* = \min\{\sigma_{1,i}\}$ where $\sigma_{1,i}$ stands for the principal eigenvalue to problem (2.1) in $\Omega_{0,i}$, assumed that $\Gamma_{1,i} \neq \emptyset$. Then, we also achieve the existence of a unique positive solution u_λ if and only if $0 < \lambda < \sigma^*$ together with the remaining properties of Theorem 1 with σ_1 replaced by σ^*. As for Theorem 3, $u_\lambda(x) \to \infty$ uniformly in $\cup_i \overline{\Omega_{0,i}}$ as $\lambda \nearrow \sigma^*$, the union being extended to those connected pieces with exactly $\sigma_{1,i} = \sigma^*$. Regarding *ii)* and *iii)*, Ω^+ must be now replaced by $\Omega^* = \Omega \setminus \cup_i \overline{\Omega_{0,i}}$, Γ^+ by $\Gamma^* = \partial\Omega^* \cap \partial\Omega$ and σ_1 by σ^*. Then the same conclusions hold again. In particular, observe that u_λ remains finite as $\lambda \nearrow \sigma^*$ on the closure $\overline{\Omega_{0,l}}$ of every component, if any, with $\sigma_{1,l} > \sigma^*$.

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