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Stationary Patterns to Diffusion Problems

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In this paper we are giving a complete account, as $\lambda \to \infty$, of all possible solutions u to the problem $u_{xx} + \lambda f(u) = 0$, $u(0) = u(1) = 0$ which satisfy $0 \leqslant u(x) \leqslant M$ in $0 < x < 1$ for some fixed $M > 0$. A complete classification of them together with a detailed analysis of their limit profile, including exact location of 'inner layers', is given, their stability character is also determined. The analysis is completed by studying the cases where solutions u degenerate at zeros of f and considering also the p-Laplacian version of the problem where the diffusion term is replaced by the operator $(|u_x|^{p-2}u_x)_x$, $p > 1$.

KEY WORDS: singular perturbation; transition and boundary layers; flat patterns and cores; stability

1. Introduction

In the present work we are giving a detailed account of all possible families of positive stationary patterns of the problem

$$\begin{aligned} &u_t = u_{xx} + \lambda f(u), \quad 0 < x < 1 \\ &u(0) = u(1) = 0 \end{aligned} \tag{E}$$

which remain uniformly bounded when the positive parameter λ is large, a subject that has received considerable interest in the literature, including its n-dimensional version (cf. [17, 19, 2, 5, 4, 20, 7] just to quote only a few recent works). We are dealing with C^1 reaction terms $f(u)$ (see section (6) for weaker assumptions) which will be assumed to exhibit non-negative zeros with finite multiplicity. It is well known that (E) defines a gradient-type semiflow under quite general growth conditions on $f(u)$ (cf. [12, 10]). In addition, the asymptotic behaviour of bounded solutions to (E) is completely characterized by its stationary solutions. In fact, it is well known that the ω-limit set of every bounded semiorbit consists of a single equilibrium of (E) [13, 22]. It will be

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shown here that when $\lambda \to +\infty$ and f satisfies quite reasonable conditions, all stationary positive solutions u to (E) lying below certain positive constant M are non-degenerate (thus isolated), their stability being elucidated, while their limit profile will be determined in full detail. In particular, every family of positive solutions can be parametrized by means of their maxima for large λ (section 5).

The key phenomenon to classify all the possible stationary patterns to (E), i.e. the solutions $u = u_\lambda(x)$ to

$$\begin{aligned} &u_{xx} + \lambda f(u) = 0, \quad 0 < x < 1 \\ &u(0) = u(1) = 0 \end{aligned} \tag{1}$$

is the developing of 'flat patterns' as $\lambda \to \infty$. In other words, the validity of the limit

$$u_\lambda \to u_0 \quad \text{as } \lambda \to +\infty \tag{L}$$

uniformly over compact sets of a whole subinterval (a 'flat core') J of $(0, 1)$ (see [14] for an earlier account on the subject and chapter 3 in [15] for further references). Since u_0 must be a zero of f, we will be able to characterize the appearance and distribution of such cores J obtaining *exact* formulas giving the asymptotic location $x = \xi$ of their boundaries (the so-called 'inner' or 'transition' layers) as $\lambda \to \infty$. Stationary solutions u_λ undergo a progressively steeper jump between two zeros when x crosses such transition layers ξ. Much work has been done recently in the non-autonomous case where f is essentially the cubic $f(x, u) = u(u-1)(c(x)-u)$, $0 < c(x) < 1$. However, it should be stressed that in this case the location of inner layers ξ is 'forced' by the restriction $c(\xi) = 1/2$ (see [3] for the case of Neumann conditions, [11] for recent results in general boundary conditions and the general review [10]). In our case the non-linearity is simpler, however, the location of inner layers ξ is a more subtle 'free' phenomenon being their precise positions given in terms of f (section 4).

In sections 3 and 4 we are exhaustively describing the structure of the positive stationary patterns in terms of the accumulation values $\bar{u}_0$, as $\lambda \to +\infty$, of the maxima sequence $\{\bar{u}_\lambda\}$, $\bar{u}_\lambda = \max u_\lambda(x)$, together with the zeros $0 \leqslant u_i \leqslant \bar{u}_0$ of f with the same energy $F(u) = \int_0^u f$ as $\bar{u}_0$. In this regard, we are performing a fine asymptotic analysis of $T(\bar{u}_\lambda)$, T the so-called 'time-map' [17, 18], as $\bar{u}_\lambda \to \bar{u}_0$, together with getting exact estimates of the vanishing rate of $\bar{u}_\lambda - \bar{u}_0$ as $\lambda \to \infty$. This permits us to elucidate the rôle of the relative values of the multiplicities of such zeros u_i in the arising of the corresponding transition layers therefore sharpening previous results in the subject [14]. The stability analysis is developed in section 5. We are producing there an 'ad hoc' approach to the degeneracy issue which seems more general than the one in [16] and is directly based upon the knowledge of the asymptotic behaviour of $T'(\bar{u}_\lambda)$. This allows us in addition, to discriminate the stability character of all possible solutions as $\lambda \to \infty$ avoiding any use of topological techniques (cf. [21]). On the other hand, the work is completed in section 6 with a study of the case of solutions degenerating at zeros of f together with the non-linear diffusion version of problem (1). Namely,

$$\begin{aligned} &(|u_x|^{p-2}u_x)_x + \lambda f(u) = 0, \quad 0 < x < 1 \\ &u(0) = u(1) = 0 \end{aligned} \tag{1a}$$

with $p > 1$. It should be remarked that the flat core formation was studied in [9], for the context of (1a) and particular f's, in the elementary situation in which solutions u to (1a) meet the value u_0 in (L) at a full interval J (a 'dead core', see sections 6.1 and 6.2(d)). However, our results in section 6 analyse all the aspects of the singular perturbation problem by covering the case where dead cores are absent.

Finally, it should be pointed out that the techniques in sections 3 and 4 can be used in the same way to obtain similar results for non-homogeneous Dirichlet conditions $u(0) = b_0,\ u(1) = b_1$ or alternatively either Neumann or Robin boundary value problems: $-u'(0) + a_0 u(0) = b_0$, $u'(1) + a_1 u(1) = b_1$, $a_i \geqslant 0$, $i = 0, 1$.

2. Preliminaries

We are assuming that $f(u)$ is a continuous function in $\bar{\mathbb{R}}^+$, of class C^1 except possibly at its zeros $u_0 \geqslant 0$ which will be supposed to be of finite multiplicity $k > 0$. Since real multiplicities $k \in \mathbb{R}^+$ will be allowed (moreover, $0 < k < 1$ in section 6.1) such zeros will be defined through the existence of a coefficient $\gamma \neq 0$, so that

$$\lim_{u \to u_0} \frac{f(u)}{\varphi_{k+1}(u_0 - u)} = \gamma$$

$\varphi_p(z) = |z|^{p-2} z$ being the odd extension of z^{p-1} to $\mathbb{R}$. In some places we will need a slightly stronger condition. Namely, $f(u_0) = 0$ together with the limit condition on the derivative:

$$\lim_{u \to u_0} \frac{f'(u)}{|u - u_0|^{k-1}} = -k\gamma$$

It will also be said that u_0 is an *even* zero of order k provided that $f(u) \sim \gamma |u - u_0|^k$ as $u \to u_0$ or the stronger condition $f(u_0) = 0$ together with $f'(u) \sim -k\gamma\varphi_k(u_0 - u)$ as $u \to u_0$. The finite-order hypothesis implies that f can only exhibit finitely many zeros on every bounded interval of $\bar{\mathbb{R}}^+$. On the other hand, it will be assumed in the sequel that f is C^1 in $\bar{\mathbb{R}}^+$ and thus $k \geqslant 1$ at all zeros. We are delaying the consideration of more general f until section 6.1 where degenerate cases will be separately analysed.

Suppose now that $\{u_\lambda\}$ is a family of positive solutions to (1) with $\bar{u}_\lambda := \max u_\lambda(x)$ bounded as $\lambda \to \infty$. Then, passing through a subsequence if necessary, some $\bar{u}_0 \geqslant 0$ must exist such that

$$\bar{u}_\lambda \to \bar{u}_0 \quad \text{as } \lambda \to +\infty \tag{2}$$

Let us examine some basic facts concerning $\bar{u}_0$. Firstly, any positive solution $u(x)$ to (1) can be represented in terms of its supremum $\bar{u} := \max u(x)$ as

$$\frac{1}{\sqrt{2}} \int_{u(x)}^{\bar{u}} \frac{ds}{\sqrt{(F(\bar{u}) - F(s))}} = \sqrt{\lambda}\left(x - \frac{1}{2}\right), \quad \frac{1}{2} \leqslant x \leqslant 1$$

where u is symmetric with regard to $x = 1/2$ and $F(u) = \int_0^u f$. Observe that $f(\bar{u}) \neq 0$ since we are assuming $f \in C^1$ (see section 6.1 where the possibility $f(\bar{u}) = 0$ is analysed).

In particular, $F(u) < F(\bar{u})$ for $0 \leqslant u < \bar{u}$ while $u'(0) = -u'(1) = \sqrt{(2F(\bar{u}))}$. It is quite convenient to set the notation (cf. [17])

$$T(\bar{u}) = \frac{1}{\sqrt{2}} \int_0^{\bar{u}} \frac{\mathrm{d}s}{\sqrt{(F(\bar{u}) - F(s))}} \tag{3}$$

Notice that,

$$T(\bar{u}) = \sqrt{(\lambda)}/2. \tag{4}$$

Reciprocally, every positive $\bar{u}$ such that $F(u) < F(\bar{u})$ for $0 \leqslant u < \bar{u}$ is easily seen to generate a solution $u = u(x)$ to (1) with $\bar{u} = \max u(x)$ provided $\bar{u}$ solves equation (4).

Therefore, if $\{u_\lambda(x)\}$ is any family of positive solutions satisfying (2) then

$$F(u) \leqslant F(\bar{u}_0) \quad \text{as} \quad 0 \leqslant u \leqslant \bar{u}_0 \tag{5}$$

In particular, $f(\bar{u}_0) \geqslant 0$. Condition (5) suggests to introduce the set

$$\mathcal{N} = \mathcal{N}(\bar{u}_0) := \{u \in [0, \bar{u}_0] \colon F(u) = F(\bar{u}_0)\} \tag{6}$$

Let us show next that $F(\bar{u}_0)$ is a critical value of F. Otherwise, since $\mathcal{N} \cap (0, \bar{u}_0)$ can only contain zeros of f then $\mathcal{N} = \{\bar{u}_0\}$ with $f(\bar{u}_0) > 0$ if $F(\bar{u}_0) > 0$ or $\mathcal{N} = \{0, \bar{u}_0\}$ and $f(0) < 0$, $f(\bar{u}_0) > 0$ provided $F(\bar{u}_0) = 0$. In both cases the integral in $T(\bar{u}_0)$ converges while $\bar{u}_\lambda \to \bar{u}_0$ as $\lambda \to +\infty$. But this is not possible since $T(\bar{u}_\lambda) = \sqrt{\lambda}/2 \to +\infty$ as $\lambda \to +\infty$. Therefore, $F(\bar{u}_0)$ is a critical value for $F(u)$ and, in particular, $\mathcal{N}$ must always contain some zero of f.

Summarizing, any family $\{u_\lambda\}$ of solutions to (1) satisfying $0 \leqslant u_\lambda(x) \leqslant M$ for every $\lambda \geqslant \lambda_1 \geqslant 0$ and some $M \geqslant 0$, possesses its accumulation values $\bar{u}_0$ among the finite set of inverse images $F^{-1}(c)$ of critical values c of F which additionally satisfy the energy condition (5). Finitely many possibilities in the configurations of $\mathcal{N}(\bar{u}_0)$ is what is permitting to trace a detailed classification of all such families $\{u_\lambda\}$ in next sections 3 and 4.

3. Simple flat patterns

3.1. *Simple patterns*

This is the kind of profiles developed by a family u_λ when $\bar{u}_\lambda \to \bar{u}_0$ and $\mathcal{N}$ consists of one single point $\mathcal{N} = \{\bar{u}_0\}$, hence a zero of f. The two only possible behaviours are characterized in the next theorems corresponding, respectively, to the cases where $\bar{u}_\lambda \to \bar{u}_0-$ ('simple lower flat' pattern) and $\bar{u}_\lambda \to \bar{u}_0+$ ('simple upper flat' pattern) as $\lambda \to \infty$. As will be seen later (Remark 2) no generality is lost by assuming that $\bar{u}_0 > 0$. Thus,

$$F(u) < F(\bar{u}_0) \quad \text{when} \quad 0 \leqslant u < \bar{u}_0 \tag{7}$$

becomes a necessary condition. Let us begin by considering the first kind of families: the 'simple lower flat' patterns (see Fig. 1(a)).

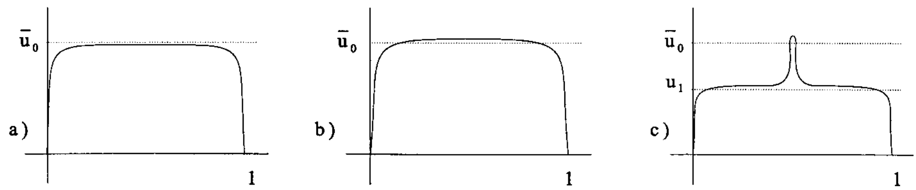

Fig. 1. Simple flat patterns: (a) lower, (b) upper, (c) spike

Theorem 1. *Let $\bar{u}_0 > 0$ be a zero of f of order k, which satisfies the energy condition* (7). *Then* (1) *admits a family $\{u_\lambda\}$ of positive solutions such that*

(i) *There exist $\varepsilon > 0$, $\lambda_0 > 0$ so that u_λ is the unique positive solution to* (1) *with $\bar{u}_0 - \varepsilon \leqslant \bar{u}_\lambda = \max u_\lambda < \bar{u}_0$ for each $\lambda \geqslant \lambda_0$.*

(ii) *For a multiplicity $k = 1$,*

$$\bar{u}_\lambda \sim \bar{u}_0 - \mathrm{e}^{-\sqrt{(\gamma\lambda)/2}}, \quad \lambda \to +\infty$$

meanwhile if $k > 1$ then,

$$\bar{u}_\lambda \sim \bar{u}_0 - \left(\frac{2B_k^2}{(k+1)\gamma}\right)^{1/(k-1)} \lambda^{-1/(k-1)}, \quad \lambda \to +\infty$$

where $B_k = B(1/2 - 1/(k+1), 1/2)$, B being the eulerian beta function.

(iii) *The family $\{u_\lambda\}$ homogenizes toward $\bar{u}_0$ in the sense that $u_\lambda \to \bar{u}_0$ uniformly over compact sets of* (0,1).

(iv) $u'_\lambda(0) = -u'_\lambda(1) \sim \sqrt{(2F(\bar{u}_0))}\lambda^{1/2}, \quad \lambda \to +\infty.$

Remark 1. In the case $k \in \mathbb{N}$ and f smooth enough we have $\gamma = (-1)^k f^{(k)}(\bar{u}_0)/k!$.

Proof of Theorem 1. Condition (7) and the structure of zero $\bar{u}_0$ imply the existence of $\delta_0 > 0$ such that $F(s) < F(u)$ for $0 \leqslant s < u$ and each $\bar{u}_0 - \delta_0 \leqslant u \leqslant \bar{u}_0$.

On the other hand, we claim that the integral $T(\bar{u})$ satisfies $T(\bar{u}) \to +\infty$ as $\bar{u} \to \bar{u}_0-$ while $T'(\bar{u}) > 0$ when $\bar{u} < \bar{u}_0$ is close enough to $\bar{u}_0$. These facts and section 2 provide the existence and uniqueness assertions in (i) since a unique solution $\bar{u} = \bar{u}_\lambda$ to (4) for λ large is ensured.

The boundary layer estimate in (iv) follows from the identity, $u'_\lambda(0) = -u'_\lambda(1) = \sqrt{(2F(\bar{u}_\lambda))}$.

The convergence assertion in (iii) is implicit in the fact $\{x: \bar{u}_0 - \varepsilon \leqslant u_\lambda(x) < \bar{u}_0\} = [x_\lambda(\varepsilon), 1 - x_\lambda(\varepsilon)]$, where $x_\lambda(\varepsilon) \sim (2\lambda)^{-1/2}\int_0^{\bar{u}_0 - \varepsilon}(F(\bar{u}_0) - F(s))^{-1/2}\,\mathrm{d}s$, as $\lambda \to +\infty$. Notice that the convergence of the integral implies that $x_\lambda(\varepsilon) \to 0$ as $\lambda \to +\infty$.

Accordingly, we are next showing the claim by estimating how $T(\bar{u})$ diverges as $\bar{u} \to \bar{u}_0-$. Fix $\varepsilon > 0$ small. Then $(\gamma - \varepsilon)(\bar{u}_0 - u)^k \leqslant f(u) \leqslant (\gamma + \varepsilon)(\bar{u}_0 - u)^k$ for $\bar{u}_0 - \delta \leqslant u \leqslant \bar{u}_0$ and a certain $\delta > 0$.

On the other hand, since the integral in (3) remains bounded between 0 and $\bar{u}_0 - \delta$ as $\bar{u} \to \bar{u}_0-$, the behaviour of $T(\bar{u})$ is dictated by

$$T_\delta(\bar{u}) = \frac{1}{\sqrt{2}}\int_{\bar{u}_0 - \delta}^{\bar{u}} \frac{\mathrm{d}s}{\sqrt{(F(\bar{u}) - F(s))}}.$$

Observe now that

$$\sqrt{\left(\frac{k+1}{2(\gamma+\varepsilon)}\right)}(\bar{u}_0-\bar{u})^{1-(k+1)/2}I(\bar{u})\leqslant T_\delta(\bar{u})\leqslant\sqrt{\left(\frac{k+1}{2(\gamma-\varepsilon)}\right)}(\bar{u}_0-\bar{u})^{1-(k+1)/2}I(\bar{u}) \tag{8}$$

where $I(\bar{u})$ stands for the integral $I(\bar{u})=\int_1^{\delta/(\bar{u}_0-\bar{u})}1/\sqrt{(\sigma^{k+1}-1)}\,\mathrm{d}\sigma$.

If $k>1$, it follows from (8),

$$\sqrt{\left(\frac{k+1}{2(\gamma+\varepsilon)}\right)}\frac{B_k}{k+1}\leqslant\varliminf_{\bar{u}\to\bar{u}_0}(\bar{u}_0-\bar{u})^{(k-1)/2}T_\delta(\bar{u})$$
$$\leqslant\varlimsup_{\bar{u}\to\bar{u}_0}(\bar{u}_0-\bar{u})^{(k-1)/2}T_\delta(\bar{u})\leqslant\sqrt{\left(\frac{k+1}{2(\gamma-\varepsilon)}\right)}\frac{B_k}{k+1}$$

Since $T=T_\delta+O(1)$ as $\bar{u}\to\bar{u}_0$, T_δ can be replaced by T, in such inequalities and by further doing $\varepsilon\to 0+$ we obtain

$$T(\bar{u})\sim\frac{B_k}{\sqrt{(2\gamma(k+1))}}\frac{1}{(\bar{u}_0-\bar{u})^{(k-1)/2}}\quad\text{as }\bar{u}\to\bar{u}_0+ \tag{9}$$

By proceeding in the same way, if $k=1$, but noticing now that $\int_1^{\delta/(\bar{u}_0-\bar{u})}1/\sqrt{(\sigma^2-1)}\,\mathrm{d}\sigma\sim-\log(\bar{u}_0-\bar{u})$ as $\bar{u}\to\bar{u}_0-$ we alternatively obtain

$$\frac{1}{\sqrt{(\gamma+\varepsilon)}}\leqslant\varliminf_{\bar{u}\to\bar{u}_0}\frac{T_\delta(\bar{u})}{-\log(\bar{u}_0-\bar{u})}\leqslant\varlimsup_{\bar{u}\to\bar{u}_0}\frac{T_\delta(\bar{u})}{-\log(\bar{u}_0-\bar{u})}\leqslant\frac{1}{\sqrt{(\gamma-\varepsilon)}}$$

Replacing again T_δ by T in the limits and letting $\varepsilon\to 0+$ lead to

$$T(\bar{u})\sim-\frac{1}{\sqrt{(\gamma)}}\log(\bar{u}_0-\bar{u})\quad\text{as }\bar{u}\to\bar{u}_0- \tag{10}$$

At this level, the existence of at least a solution $\bar{u}_\lambda$ of $T(\bar{u})=\sqrt{\lambda/2}$ for $\lambda\geqslant\lambda_0$ is ensured. Note that every such family $\bar{u}_\lambda$ must satisfy the estimates in (ii) by the virtue of asymptotic relations (9), (10).

As for the assertion that $T'(\bar{u})>0$ for $\bar{u}<\bar{u}_0$, $\bar{u}\sim\bar{u}_0$, it is convenient to write $T(\bar{u})=(\sqrt{2})^{-1}\int_0^1\bar{u}/\sqrt{(F(\bar{u})-F(\bar{u}s))}\,\mathrm{d}s$. Taking derivatives gives

$$T'(\bar{u})=\frac{1}{\bar{u}}\left\{T(\bar{u})-\frac{1}{2\sqrt{2}}\int_0^{\bar{u}}\frac{\bar{u}f(\bar{u})-sf(s)}{(F(\bar{u})-F(s))^{3/2}}\,\mathrm{d}s\right\} \tag{11}$$

The divergence of $T(\bar{u})$ permits replacing the lower limit in the integral by some fixed $\bar{u}_0-\delta_1$ suitably close to $\bar{u}_0$ so that $sf(s)-\bar{u}f(\bar{u})>0$ for $\bar{u}_0-\delta_1\leqslant u<\bar{u}_0$. Therefore, $\lim_{\bar{u}\to\bar{u}_0-}T'(\bar{u}_0)=+\infty$.

This completes the proof. □

Let us study now the formation and properties of 'simple upper flat' patterns to (1) (see Fig. 1(b)). Observe that $\bar{u}_0$ must be now an even zero while, $\mathcal{N}=\{\bar{u}_0\}$ with $\bar{u}_0>0$, then $F(\bar{u}_0)>0$. The limit Lipschitz case $k=1$ has been also included for the sake of completeness.

Theorem 2. *Let* $\bar{u}_0 > 0$ *be an even zero of f with order k and assume that condition* (7) *holds. Then problem* (1) *admits a family of positive solutions* $\{u_\lambda\}$ *such that,*

(i) *For certain positive* ε, $\lambda_0 > 0$, u_λ *is the unique positive solution to* (1) *corresponding to* $\lambda > \lambda_0$ *and with* $\bar{u}_0 < \bar{u}_\lambda = \max u_\lambda(x) \leqslant \bar{u}_0 + \varepsilon$.

(ii)

$$\bar{u}_\lambda \sim \begin{cases} \bar{u}_0 + \mathrm{e}^{-\sqrt{(\gamma\lambda)}/2} \\ \quad \text{if } k = 1 \\ \bar{u}_0 + \left(\dfrac{2}{(k+1)\gamma}\right)^{1/(k-1)} \left(\left(\csc\left(\dfrac{\pi}{k+1}\right) + \cot\left(\dfrac{\pi}{k+1}\right)\right) B_k\right)^{2/(k-1)} \lambda^{-1/(k-1)} \\ \quad \text{if } k > 1 \end{cases}$$

as $\lambda \to +\infty$, *being* B_k *as in Theorem* 1.

(iii) *The limit* $u_\lambda \to \bar{u}_0$ *holds uniformly on compact sets of* (0, 1).

(iv) *The family* u_λ *develops a boundary layer at the points* $x = 0, 1$ *estimated as* $u'_\lambda(0) = -u'_\lambda(1) \sim \sqrt{(2F(\bar{u}_0))}\lambda^{1/2}$ *when* $\lambda \to +\infty$.

Proof of Theorem 2. The proofs of (i), (iii) and (iv) follow from the study of the behaviour of $T(\bar{u})$ and $T'(\bar{u})$ as $\bar{u} \to \bar{u}_0 +$, what in turn provides the estimates in (ii).

To estimate the behaviour of $T(\bar{u})$ observe that the integral in (3) remains bounded between the limits 0 and $\bar{u}_0 - \delta$, $\delta > 0$, $\delta \sim 0$. Thus $T(\bar{u})$ differs $O(1)$ from

$$T_\delta(\bar{u}) = \frac{1}{\sqrt{2}} \int_{\bar{u}_0 - \delta}^{\bar{u}_0} \frac{\mathrm{d}s}{\sqrt{(F(\bar{u}) - F(s))}} + \frac{1}{\sqrt{2}} \int_{\bar{u}_0}^{\bar{u}} \frac{\mathrm{d}s}{\sqrt{(F(\bar{u}) - F(s))}} := I_{1,\delta}(\bar{u}) + I_{2,\delta}(\bar{u})$$

as $\bar{u} \to \bar{u}_0$. Taking into account that $f(u) = \gamma|\bar{u}_0 - \bar{u}|^k + o(|\bar{u}_0 - \bar{u}|^k)$ we find

$$I_{1,\delta}(\bar{u}) \sim \sqrt{\left(\frac{k+1}{2\gamma}\right)} (\bar{u} - \bar{u}_0)^{-(k-1)/2} \int_0^{\delta/(\bar{u} - \bar{u}_0)} \frac{\mathrm{d}\sigma}{(\sigma^{k+1} + 1)^{1/2}}$$

Thus

$$I_{1,\delta}(\bar{u}) \sim \begin{cases} -\dfrac{1}{\sqrt{\gamma}} \log(\bar{u} - \bar{u}_0) & \text{if } k = 1 \\ \dfrac{1}{\sqrt{(2\gamma(k+1))}} (\bar{u} - \bar{u}_0)^{-(k-1)/2} B\left(\dfrac{1}{2} - \dfrac{1}{k+1}, \dfrac{1}{k+1}\right) & \text{if } k > 1 \end{cases}$$

as $\bar{u} \to \bar{u}_0 +$, $B(p, q)$ being the beta function.

As for $I_{2,\delta}$ we find similarly,

$$I_{2,\delta}(\bar{u}) \sim \begin{cases} \dfrac{\pi}{\sqrt{2\gamma}} & \text{if } k = 1 \\ \dfrac{1}{\sqrt{(2\gamma(k+1))}} (\bar{u} - \bar{u}_0)^{-(k-1)/2} B\left(\dfrac{1}{2}, \dfrac{1}{k+1}\right) & \text{if } k > 1 \end{cases}$$

On the other hand, formula 6.1.17 in [1] permits writing, respectively, $B(1/2 - 1/(k+1),\ 1/(k+1)) = \csc(\pi/(k+1))B_k$, $B(1/2, 1/(k+1)) = \cot(\pi/(k+1))B_k$. Therefore, we finally achieve

$$T(\bar{u}) \sim \begin{cases} -\dfrac{1}{\sqrt{\gamma}}\log(\bar{u}-\bar{u}_0) & \text{if } k=1 \\ \dfrac{1}{\sqrt{(2\gamma(k+1))}}(\bar{u}-\bar{u}_0)^{-(k-1)/2}\left(\csc\left(\dfrac{\pi}{k+1}\right)+\cot\left(\dfrac{\pi}{k+1}\right)\right)B_k & \text{if } k>1 \end{cases}$$

as $\bar{u} \to \bar{u}_0 +$.

To ascertain the behaviour of $T'(\bar{u})$ as $\bar{u} \to \bar{u}_0+$ first observe that (11) implies

$$T'(\bar{u}) = -\frac{1}{2\sqrt{2\bar{u}}}\int_0^{\bar{u}} \frac{g(\bar{u})-g(s)}{\{F(\bar{u})-F(s)\}^{3/2}}\,\mathrm{d}s \tag{12}$$

where $g(u) := uf(u) - 2F(u) = -2F(\bar{u}_0) + \bar{u}_0\gamma|u-\bar{u}_0|^k + o(|u-\bar{u}_0|^k)$ as $u \to \bar{u}_0+$. The integral in (12) is equivalent to the corresponding one with limits $\bar{u}_0 - \delta$, $\bar{u}$ as $\bar{u} \to \bar{u}_0+$, for any small enough δ. A careful analysis—following the lines of the previous ones—of such integral reveals that

$$T'(\bar{u}) \sim -\frac{(k+1)^{3/2}}{2\sqrt{2\gamma}}C_k(\bar{u}-\bar{u}_0)^{-(k+1)/2} \to -\infty \quad \text{as } \bar{u} \to \bar{u}_0+ \tag{13}$$

where $C_k = \int_0^1 (2 - z^{(k-1)/2})(1-z^k)/(1+z^{k+1})^{3/2}\,\mathrm{d}z$. This yields the uniqueness of the solutions as desired. □

Remarks 2. (a) If $\bar{u}_0 = 0$ and $k=1$ no family of positive solutions u_λ satisfying $\bar{u}_\lambda \to 0$ exists. If fact, $f(u) \sim \gamma u^k$, $u \to 0+$ in this case, and so $T(\bar{u}) \sim \pi/\sqrt{2\gamma}$ as $\bar{u} \to 0+$ what implies that (4) has no solutions if λ is large.

(b) If $\bar{u}_0 = 0$ but $k > 1$ the situation is just as described in Theorem (2) with the exception that the term in *cot* does not appear in (ii). In addition, $\{u_\lambda\}$ develops a boundary layer as $x = 0, 1$ although $\lim_{\lambda\to+\infty} u'_\lambda(0) = \lim_{\lambda\to+\infty} u'_\lambda(1) = 0$. In fact, by the virtue of (ii),

$$u'_\lambda(0)^2 = 2\lambda F(\bar{u}_\lambda) \sim 2\lambda\frac{\gamma}{k+1}\bar{u}_\lambda^{k+1} \sim C\lambda^{-2/(k-1)}$$

where, $C = (2\gamma/(k+1))(2/(k+1)\gamma)^{1/(k-1)}(\csc(\pi/(k+1)) + \cot(\pi/(k+1))B_k)^{2/(k-1)}$.

3.2. *Spike patterns*

When a family $\{u_\lambda\}$ of positive solutions satisfies $\bar{u}_\lambda \to \bar{u}_0$ as $\lambda \to +\infty$ but, unlike the previous case, the set $\mathscr{N}$ in (6) only contains a zero u_1 which is different from $\bar{u}_0$ (so $f(\bar{u}_0) > 0$), then a central pike is developed by the solutions u_λ (see Fig. 1(c)) as $\lambda \to +\infty$ giving rise, as described in our next result, to the 'simple spike' pattern.

Theorem 3. *Let* $0 \leqslant u_1 < \bar{u}_0$ *be a zero of order* k *of* f *such that* $F(u_1) = F(\bar{u}_0)$ *and* $F(u) < F(\bar{u}_0)$ if $0 < u < \bar{u}_0$, $u \neq u_1$. *In addition, assume that* $f(0) < 0$ *if* $F(\bar{u}_0) = 0$ *and* $u_1 > 0$. *Then, there exists a family* $\{u_\lambda\}$ *of positive solutions to* (1) *such that*

(i) u_λ *is the unique positive solution with* $\bar{u}_0 < u_\lambda = \max u_\lambda \leqslant \bar{u}_0 + \varepsilon$ *for* $\lambda \geqslant \lambda_0$ *and certain positive* ε, λ_0

(ii)

$$\bar{u}_\lambda \sim \begin{cases} \bar{u}_0 + e^{-\sqrt{(2\gamma\lambda)}/2\beta} & \text{if } k = 1 \\ \bar{u}_0 + \dfrac{1}{f(\bar{u}_0)}\left(\dfrac{\beta\sqrt{2}}{k+1}\csc\left(\dfrac{\pi}{k+1}\right)B_k\right)^{2(k+1)/(k-1)}\left(\dfrac{k+1}{\gamma}\right)^{2/(k-1)}\lambda^{-(k+1)/(k-1)} & \text{if } k > 1 \end{cases}$$

as $\lambda \to +\infty$, *where* $\beta = 1, 2$ *if, respectively*, $u_1 = 0, u_1 \neq 0$ *and* B_k *is as in Theorem* 1.

(iii) $u_\lambda \to u_1$ *uniformly on compact sets of* $(0,1)\backslash\{1/2\}$.

(iv) $u'_\lambda(0) = -u'_\lambda(1) \sim \sqrt{(2F(\bar{u}_0))}\lambda^{1/2}$ *as* $\lambda \to +\infty$ *provided* $u_1 > 0$.

Remark 3. If $F(\bar{u}_0) = 0$ the appearance of a boundary layer is still compatible with $\lim_{\lambda \to +\infty} u'_\lambda(0) = 0$ (cf. Remark 2).

Proof of Theorem 3. Assuming that $u_1 > 0$ let us study $T(\bar{u})$ as $\bar{u} \to \bar{u}_0$. Firstly, notice that

$$T(\bar{u}) = T_\delta(\bar{u}) + O(1) := \frac{1}{\sqrt{2}}\int_{u_1-\delta}^{u_1+\delta} \frac{ds}{(F(\bar{u}) - F(s))^{1/2}} + O(1)$$

for every positive small δ as $\bar{u} \to \bar{u}_0$. It is convenient to write $F(\bar{u}) - F(s) = F(\bar{u}) - F(\bar{u}_0) + F(u_1) - F(s)$. Following the strategy of Theorem 1 we find that

$$\frac{\sqrt{2}}{(\Delta F)^{(k-1)/2(k+1)}} J_{+,\delta}(\bar{u}) \leqslant T_\delta(\bar{u}) \leqslant \frac{\sqrt{2}}{(\Delta F)^{(k-1)/2(k+1)}} J_{-,\delta}(\bar{u}) \tag{14}$$

where $\Delta F = F(\bar{u}) - F(\bar{u}_0)$ and

$$J_{\pm,\delta}(\bar{u}) = \left(\frac{k+1}{\gamma \pm \varepsilon}\right)^{1/(k+1)} \int_0^{\{(\gamma\pm\varepsilon)/(k+1)\Delta F\}^{1/(k+1)\delta}} \frac{d\sigma}{(1+\sigma^{k+1})^{1/2}}$$

From (14) and using $\Delta F \sim f(\bar{u}_0)(\bar{u} - \bar{u}_0)$ as $\bar{u} \to \bar{u}_0$ we get the estimate

$$T(\bar{u}) \sim \begin{cases} -\dfrac{\sqrt{2}}{\sqrt{\gamma}} \log(\bar{u} - \bar{u}_0) & \text{if } k = 1 \\ \dfrac{\sqrt{2}}{k+1}(f(\bar{u}_0)(\bar{u} - \bar{u}_0))^{-(k-1)/2(k+1)}\left(\dfrac{k+1}{\gamma}\right)^{1/(k+1)}\csc\left(\dfrac{\pi}{k+1}\right)B_k & \text{if } k > 1 \end{cases} \tag{15}$$

as $\bar{u} \to \bar{u}_0 +$. In the case $u_1 = 0$ only an integral between 0 and δ is involved, both coefficients in (15) must be divided by two.

Regarding the uniqueness of solutions it suffices with showing that $T'(\bar{u}) \to -\infty$ as $\bar{u} \to \bar{u}_0+$. In fact, we achieve from (12) that, for any $\delta > 0$ small,

$$T'(\bar{u}) = -\frac{1}{2\sqrt{2\bar{u}}}\int_{u_1-\delta}^{u_1+\delta} \frac{g(\bar{u}) - g(s)}{\{F(\bar{u}) - F(s)\}^{3/2}}\,\mathrm{d}s + O(1)$$

as $\bar{u} \to \bar{u}_0$. Since $g(u) - g(s) = \bar{u}_0 f(\bar{u}_0) + o(1)$ in $|s - u_1| < \delta$ when $\delta \to 0+$ and $\bar{u} \to \bar{u}_0$, some computations lead to,

$$T'(\bar{u}) \sim -\frac{1}{\sqrt{2}}\left(\frac{k+1}{\gamma}\right)^{1/(k+1)} I_k f(\bar{u}_0)^{-(k-1)/2(k+1)}(\bar{u} - \bar{u}_0)^{-(3k+1)/2(k+1)}\ \bar{u} \to \bar{u}_0 \qquad (16)$$

where $I_k = \int_0^{\infty}(1 + s^{k+1})^{-3/2}\,\mathrm{d}s = [(k-1)/(k+1)^2]B_k \csc(\pi/(k+1))$. This shows the assertion.

Finally, let us show that $u_\lambda \to u_1$ uniformly on compact sets of $(0,1)\backslash\{1/2\}$. In fact, if $x^{\pm}_{\lambda,\delta}$ stands for the unique points in (0,1/2) where $u_\lambda = u_1 \pm \delta$ then

$$x^{-}_{\lambda,\delta} \sim \frac{1}{\sqrt{2\lambda}}\int_0^{u_1-\delta} \frac{\mathrm{d}s}{(F(\bar{u}_0) - F(s))^{1/2}}, \quad x^{+}_{\lambda,\delta} \sim \frac{1}{2} - \frac{1}{\sqrt{2\lambda}}\int_{u_1+\delta}^{\bar{u}} \frac{\mathrm{d}s}{(F(\bar{u}_0) - F(s))^{1/2}}$$

as $\lambda \to +\infty$. Since the involved integrals are convergent then $x^{-}_{\lambda,\delta} \to 0$, $x^{+}_{\lambda,\delta} \to 1/2$ as $\lambda \to +\infty$. This shows the desired assertion. □

Remark 4. With plain uniqueness aside, few cases are available in the literature where the exact number of positive solutions to (1) for *all* λ is known. For the very particular case $f(u) = -u(u-b)(u-c)$, $0 < b < c$, $F(c) > 0$ it has been shown in [17] that such number is two (cf. [21] for a more general f's).

4. Multiple flat patterns

To complete our analysis, let us focus now our attention in the case where a family $\{u_\lambda\}$ satisfies (2), $\bar{u}_\lambda = \max u_\lambda \to \bar{u}_0$, but $\mathcal{N}$ in (6) exhibits more than one zero of f (and thus necessarily $\bar{u}_0 > 0$). In this case, it will be seen that when λ becomes large, flat cores corresponding to different zeros arise and asymptotically fill the whole interval (0, 1) as $\lambda \to +\infty$. A point ξ separating the neighbouring flat cores defines an internal layer. The more relevant feature is that exact expressions for such points can be produced (cf. Theorems 4 and 5 below).

Accordingly, we are assuming in the sequel that $\bar{u}_0 > 0$ satisfies (5) while the set $\mathcal{N}$ in (6) contains $0 < u_1 < u_2 < \cdots < u_m$ zeros of f with multiplicities $\gamma_i > 0$. It may happen that $\bar{u}_0$ be (a necessarily even) zero of f of order γ. In addition, if $F(\bar{u}_0) = 0$ it could also be $f(0) = 0$ with $f(u) \sim -\gamma u^k$ if $u \to 0+$. Whenever some of these cases occur we will set, respectively, $u_{m+1} = \bar{u}_0$ and $\gamma_{m+1} = \gamma$ or $u_0 = 0$ and $\gamma_0 = \gamma$.

4.1. *Equal order case*

Our next results consider the case where all zeros in $\mathcal{N}$ share exactly the same multiplicity k. We are beginning with the 'multiple flat' case where $f(\bar{u}_0) = 0$ (see Fig. 2(a)).

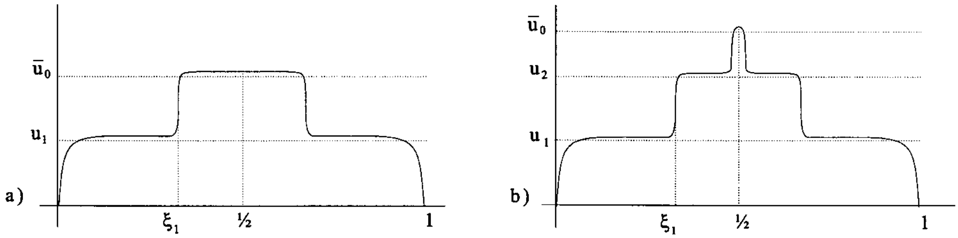

Fig. 2. (a) Multiple flat. (b) Multiple flat + spike

Theorem 4. *Assume that* $\mathcal{N} = \{u_1, \dots, u_m, \bar{u}_0\}$ *and so* $F(u_i) = F(\bar{u}_0)$, *with* $F(\bar{u}_0) > 0$, *where every* u_i, $1 \leqslant i \leqslant m$, *has order k while* $u_{m+1} = \bar{u}_0$ *is even with multiplicity k. Then there exists a family* $\{u_\lambda\}$ *of positive solutions to* (1) *such that*

(i) u_λ *is the unique positive solution to* (1) *with* $\bar{u}_0 < \bar{u}_\lambda = \max u_\lambda \leqslant \bar{u}_0 + \varepsilon$ *if* $\lambda \geqslant \lambda_0$ *for certain positive* ε, λ_0.

(ii) *The estimates,*

$$\bar{u}_\lambda \sim \begin{cases} \bar{u}_0 + \exp\left\{-\dfrac{\sqrt{2\lambda}}{4\left[\sum_{i=1}^m (\sqrt{\gamma_i})^{-1} + (\sqrt{(2\gamma_{m+1})})^{-1}\right]}\right\} \\ \bar{u}_0 + \left(\dfrac{8B_k^2 \csc^2(\pi/(k+1))}{\gamma_{m+1}(k+1)}\right)^{1/(k-1)} \left[\sum_{i=1}^m \left(\dfrac{\gamma_{m+1}}{\gamma_i}\right)^{1/(k+1)} + \cos^2\left(\dfrac{\pi}{2(k+1)}\right)\right]^{2/(k-1)} \lambda^{-1/(k-1)} \end{cases}$$

hold, respectively, for $k = 1$, $k > 1$ *when* $\lambda \to +\infty$ *with* B_k *as in Theorem* (1).

(iii) *Define*

$$\xi_j = \begin{cases} \dfrac{1}{2}\dfrac{\sum_{i=1}^j (\sqrt{(\gamma_i)})^{-1}}{\sum_{i=1}^m (\sqrt{(\gamma_i)})^{-1} + (\sqrt{(2\gamma_{m+1})})^{-1}}, & k = 1 \\ \dfrac{1}{2}\dfrac{\sum_{i=1}^j \gamma_i^{-1/(k+1)}}{\sum_{i=1}^m \gamma_i^{-1/(k+1)} + \gamma_{m+1}^{-1/(k+1)} \cos^2\left(\dfrac{\pi}{2(k+1)}\right)}, & k > 1 \end{cases}$$

for $1 \leqslant j \leqslant m$ *and* $\xi_0 = 0$, $\xi_{m+1} = 1/2$. *Then*, $u_\lambda \to u_{j+1}$ *uniformly on compacts of* (ξ_j, ξ_{j+1}) *as* $\lambda \to +\infty$.

(iv) $u'_\lambda(0) \sim \sqrt{(2F(\bar{u}_0))}\lambda^{1/2}$ *as* $\lambda \to +\infty$.

Remark 5. The case $F(\bar{u}_0) = 0$ together with $f(0) = 0$ (no changes arise in the complementary case $f(0) < 0$) leads exactly to the same results with slight changes in the constants. Specifically, if $f(u) \sim -\gamma_0 u^k$ as $u \downarrow 0+$ then the factors $1/(2\sqrt{\gamma_0})$ and $(1/2)(\gamma_{m+1}/\gamma_0)^{1/(k+1)}$ must be added, respectively, to the constants into brackets in (ii). Similarly, the factors $(2\sqrt{\gamma_0})^{-1}$ and $(2\gamma_0^{1/(k+1)})^{-1}$ must be added, respectively, to both numerator and denominator of the fractions in (iii). In the new expressions $\xi_0 > 0$ and we must set in addition $\xi_{-1} := 0$.

Regarding the boundary layer in (iv) the corresponding observations as those in Remarks 2 and 3 are in order.

Proof of Theorem 4. Let us begin by characterizing the divergence rate of $T(\bar{u})$ as $\bar{u}\downarrow\bar{u}_0$. Thus,

$$T(\bar{u})=\frac{1}{\sqrt{2}}\left(\sum_{i=1}^{m}\int_{u_i-\delta}^{u_i+\delta}+\int_{\bar{u}_0-\delta}^{\bar{u}}\right)\frac{\mathrm{d}s}{(F(\bar{u})-F(s))^{1/2}}+O(1)$$

Proceeding as in the proof of Theorem 3, each of the first m integrals $I_i(\bar{u})$ can be estimated as

$$I_i\sim\begin{cases}-\dfrac{\sqrt{2}}{\sqrt{\gamma_i}}\log(\bar{u}-\bar{u}_0) & \text{if } k=1\\ \dfrac{\sqrt{2}}{k+1}\Delta F^{-(k-1)/2(k+1)}\left(\dfrac{k+1}{\gamma_i}\right)^{1/(k+1)}\csc\left(\dfrac{\pi}{k+1}\right)B_k & \text{if } k>1\end{cases}$$

when $\bar{u}\to\bar{u}_0+$, with $\Delta F=F(\bar{u})-F(\bar{u}_0)\sim(\gamma_{m+1}/(k+1))(\bar{u}-\bar{u}_0)^{k+1}$. As for the last one, $I_{m+1}(\bar{u})$, we have similarly

$$I_{m+1}\sim\begin{cases}-\dfrac{1}{\sqrt{(\gamma_{m+1})}}\log(\bar{u}-\bar{u}_0) & \text{if } k=1\\ \dfrac{B_k}{\sqrt{(2\gamma_{m+1}(k+1))}}\left(\csc\left(\dfrac{\pi}{k+1}\right)+\cot\left(\dfrac{\pi}{k+1}\right)\right)(\bar{u}-\bar{u}_0)^{-(k-1)/2} & \text{if } k>1\end{cases}$$

Combining such expressions we find

$$T(\bar{u})\sim\begin{cases}-\sqrt{2}\left(\sum_{i=1}^{m}\dfrac{1}{\sqrt{\gamma_i}}+\dfrac{1}{\sqrt{(2\gamma_{m+1})}}\right)\log(\bar{u}-\bar{u}_0)\\ \dfrac{\sqrt{2}}{\sqrt{(\gamma_{m+1}(k+1))}}\csc\left(\dfrac{\pi}{k+1}\right)B_k\left[\sum_{i=1}^{m}\left(\dfrac{\gamma_{m+1}}{\gamma_i}\right)^{1/(k+1)}+\cos^2\left(\dfrac{\pi}{2(k+1)}\right)\right](\bar{u}-\bar{u}_0)^{-(k-1)/2}\end{cases}$$

as $\bar{u}\to\bar{u}_0$. The estimates in (ii) follow at once from the identity $T(\bar{u}_\lambda)=\sqrt{\lambda/2}$.

For the uniqueness we are showing that $T'(\bar{u})\to-\infty$ as $\bar{u}\downarrow\bar{u}_0$. To see this observe that

$$T'(\bar{u})=-\frac{1}{2\bar{u}\sqrt{2}}\left(\sum_{i=1}^{m}\int_{u_i-\delta}^{u_i+\delta}+\int_{u_0-\delta}^{\bar{u}}\right)\frac{g(\bar{u})-g(s)}{(F(\bar{u})-F(s))^{3/2}}+O(1),\quad \bar{u}\to\bar{u}_0+$$

for every small fixed $\delta>0$, with $g(\bar{u})=uf(u)-2F(u)$. Each of the first m integrals J_i can be estimated as

$$J_i\sim\frac{2\bar{u}_0(k-1)B_k}{\sqrt{(\gamma_{m+1}(k+1))}}\csc\left(\frac{\pi}{k+1}\right)\left(\frac{\gamma_{m+1}}{\gamma_i}\right)^{1/(k+1)}(\bar{u}-\bar{u}_0)^{-(k+1)/2},$$

when $\bar{u} \to \bar{u}_0 +$. Taking into account estimate (13) for the last integral we achieve the desired conclusion.

To show (iii) assume that, say, $k > 1$ and fix $\delta > 0$ so that $u_j < u_j - \delta < u_{j+1}$ and let $x_{\lambda,\delta}$ be the unique point in (0, 1/2) satisfying $u_\lambda = u_{j+1} - \delta$. Then, $\sqrt{\lambda}\, x_{\lambda,\delta} = (\sqrt{2})^{-1} \int_0^{u_{j+1}-\delta} (F(\bar{u}_\lambda) - F(s))^{-1/2}\, \mathrm{d}s$, and so

$$\sqrt{\lambda} x_{\lambda,\delta} \sim \frac{\sqrt{2} \csc(\pi/(k+1)) B_k}{\sqrt{(\gamma_{m+1}(k+1))}} \left\{ \sum_{i=1}^{j} \left(\frac{\gamma_{m+1}}{\gamma_i} \right)^{1/(k+1)} \right\} (\bar{u} - \bar{u}_0)^{-(k-1)/2}$$

when $\lambda \to +\infty$. Since $T(\bar{u}_\lambda) \sim \sqrt{\lambda}/2$, then the obtained estimates for T implies that $x_{\lambda,\delta} \to \xi_j$ as $\lambda \to +\infty$. □

We are considering now the situation where all zeros in $\mathcal{N}$ have the same order but $f(\bar{u}_0) \neq 0$. In this case, aside of several flat cores, u_λ exhibits a central pike at $\xi = 1/2$ as $\lambda \to +\infty$ giving rise to a multiple flat pattern with spike (see Fig. 2(b)).

For the sake of completeness we are also including in the statement the case where $F(\bar{u}_0) = 0$ together with $f(0) = 0, f(u) \sim -\gamma_0 u^k$ as $u \to 0$ (cf. Remark 5).

Theorem 5. *Assume that $f(\bar{u}_0) > 0$ and $\mathcal{N} = \{u_1, \ldots, u_m; \bar{u}_0\}$ where each $u_i > 0$ is a zero of f with order k if $F(\bar{u}_0) > 0$ or, alternatively, suppose that $\mathcal{N} = \{0, u_1, \ldots, u_m; \bar{u}_0\}$ and additionally $f(u) \sim -\gamma_0 u^k$ as $u \to 0$ when $F(\bar{u}_0) = 0$ and also $f(0) = 0$. Then* (1) *possesses a family $\{u_\lambda\}$ of positive solutions such that*

(i) *There exist $\varepsilon > 0$, $\lambda_0 > 0$ so that u_λ is the unique positive solution of* (3) *with $\bar{u}_0 < \bar{u}_\lambda \leqslant \bar{u}_0 + \varepsilon$ for $\lambda \geqslant \lambda_0$.*

(ii) *The asymptotic rates as $\lambda \to +\infty$,*

$$\bar{u}_\lambda \sim \begin{cases} \bar{u}_0 + \exp\left\{ -\dfrac{\sqrt{2\lambda}}{4[\beta(2\sqrt{\gamma_0})^{-1} + \sum_{i=1}^m (\sqrt{\gamma_i})^{-1}]} \right\} \\ \bar{u}_0 + \dfrac{1}{f(\bar{u}_0)} \left(\dfrac{8B_k^2 \csc^2(\pi/(k+1))}{(k+1)^2} \right)^{(k+1)/(k-1)} \\ \quad \times \left[\dfrac{B}{2} \left(\dfrac{k+1}{\gamma_0} \right)^{1/(k+1)} + \displaystyle\sum_{i=1}^m \left(\dfrac{k+1}{\gamma_i} \right)^{1/(k+1)} \right]^{2(k+1)/(k-1)} \lambda^{-(k+1)/(k-1)} \end{cases}$$

hold, respectively, for $k = 1$, $k > 1$, being $\beta = 1$ if $F(\bar{u}_0) = 0$ and also $f(0) = 0$, $\beta = 0$ otherwise (B_k is the constant of Theorem 1).

(iii) *Defining*

$$\xi_j = \frac{1}{2} \frac{(\beta/2)\gamma_0^{-1/(k+1)} + \sum_{i=1}^j \gamma_i^{-1/(k+1)}}{(\beta/2)\gamma_0^{-1/(k+1)} + \sum_{i=1}^m \gamma_i^{-1/(k+1)}}, \quad 1 \leqslant j \leqslant m$$

with $\beta = 0$ if $F(\bar{u}_0) > 0$, $\beta = 1$ if $F(\bar{u}_0) = 0$ together with $f(0) = 0$, and setting $\xi_0 = 0$ (respectively, $\xi_{-1} = 0$) if $\beta = 0$ (respectively, $\beta = 1$) then $u_\lambda \to u_{j+1}$ uniformly on compacts of (ξ_j, ξ_{j+1}) as $\lambda \to +\infty$.

(iv) $u'_\lambda(0) \sim \sqrt{(2F(\bar{u}_0))}\lambda^{1/2}$ *when* $\lambda \to +\infty$ *provided* $F(\bar{u}_0) > 0$.

Remark 6. The proof of Theorem 5 is a review of previous ones and therefore omitted. In fact, observe that with the notation in use we now have

$$T(\bar{u}) \sim \begin{cases} -\left(\dfrac{\beta}{2}\sqrt{\left(\dfrac{2}{\gamma_0}\right)} + \displaystyle\sum_1^m \sqrt{\left(\dfrac{2}{\gamma_i}\right)}\right)\log(\bar{u}-\bar{u}_0) & k = 1 \\ \dfrac{\sqrt{2}}{k+1} B_k \csc\left(\dfrac{\pi}{k+1}\right)\left(\dfrac{\beta}{2}\left(\dfrac{k+1}{\gamma_0}\right)^{1/(k+1)} \right. & \\ \qquad \left. + \displaystyle\sum_1^m \left(\dfrac{k+1}{\gamma_i}\right)^{1/(k+1)}\right)(f(\bar{u}_0)(\bar{u}-\bar{u}_0))^{-(k-1)/2(k+1)} & \end{cases}$$

as $\bar{u} \to \bar{u}_0+$, where $\beta = 1$ when $F(\bar{u}_0) = f(0) = 0$, $\beta = 0$ otherwise. On the other hand and for later use we remark that in the present case and according to (16) we easily achieve

$$T'(\bar{u}) \sim -\frac{1}{\sqrt{2}} I_k f(\bar{u}_0)^{-(k-1)/2(k+1)} \sum_{i=1}^m \left(\frac{k+1}{\gamma_i}\right)^{1/(k+1)} (\bar{u}-\bar{u}_0)^{-(3k+1)/2(k+1)}, \quad \bar{u} \to \bar{u}_0+$$

4.2. *General case. Classification*

In the general case we have some $\bar{u}_0 > 0$ satisfying (5) while the set $\mathcal{N}$ in (6) possesses a group $\{u_1, \ldots, u_m\}$ of zeros with maximum multiplicity k and a remaining class of zeros $\{v_1, \ldots, v_n\}$ with a lower order (it is also possible that $u_0 = 0$ or $\bar{u}_0$ belong to some of such classes). Then, by following word for word the proofs in section 4.1, it can be shown that the existence of a unique family of positive solutions $\{u_\lambda\}$ to (1), $\bar{u}_\lambda \to \bar{u}_0$ as $\lambda \to +\infty$, which exhibits under the present notation, the features of Theorems 4 and 5, provided the class of zeros $\{u_1, \ldots, u_m\}$ with leading multiplicity k matches the requirements of such results. In other words, the zeros with lower multiplicity $\{v_1, \ldots, v_n\}$ do not contribute to the asymptotic properties of the family as $\lambda \to +\infty$.

We can conclude our analysis with a general result. In fact, if $\{u_\lambda\}$ is a general family of solutions $0 \leqslant u_\lambda(x) \leqslant M$, $\lambda \geqslant \lambda_1$, then the maxima sequence $\{\bar{u}_\lambda\}$ has only finitely many accumulation points $0 \leqslant \bar{u}_{01} \leqslant \cdots \leqslant \bar{u}_{0N} \leqslant M$ which satisfy the energy condition (7) (see section 2). Thus, for ε small and λ large every $\bar{u}_\lambda = \max u_\lambda(x)$ satisfy $\bar{u}_{0l} - \varepsilon \leqslant \bar{u}_\lambda \leqslant \bar{u}_{0l} + \varepsilon$ for some $1 \leqslant l \leqslant N$. Since all possible configurations of the sets $\mathcal{N}(\bar{u}_{0l})$ have been treated in sections 3, 4 and due to the uniqueness assertions in each case this means that u_λ fits some of the patterns studied.

Corollary 6. *Every bounded family* $\{u_\lambda\}$ *of positive solutions to* (1) *can be decomposed into a finite number of subfamilies* $\{u_{\lambda_1}\}, \ldots, \{u_{\lambda_N}\}$, $\lambda_i \to +\infty$, $1 \leqslant i \leqslant N$, *each of them corresponding to some of the patterns considered in Theorems* 1–5.

5. Stability analysis

It follows from [12, chapters III–V] that problem (1) defines a local semiflow in $H_0^1(0,1)$ provided the non-linearity f is, say, of class C^1 (compare also with [13]). Moreover, the principle of the stability by linearization can be used under such conditions and the next result holds.

Theorem 7. *If a bounded family* $\{u_\lambda\}$ *of positive solutions to* (1) *develops a 'simple lower flat' pattern* (*Theorem* 1) *then* u_λ *is asymptotically stable for* λ *large. If, on the contrary,* $\{u_\lambda\}$ *gives rise to any of the other patterns* (*Theorems* 2–5) *then* $\{u_\lambda\}$ *becomes unstable for* λ *large.*

Proof. The initial value problem $-u'' = \lambda f(u)$, $0<x<1, u(1/2)=u_0$, $u'(1/2)=0$ admits, for each u_0, a unique local solution $u=u(x,u_0)$ which can be differentiated with respect to u_0. Moreover, $w(x,u_0) := \partial u/\partial u_0(x,u_0)$ defines a local solution of the problem

$$-w'' = \lambda f'_u(u(x,u_0))w, \quad w(1/2)=1, \quad w'(1/2)=0 \tag{17}$$

By fixing $x \neq 1/2$ and differentiating with respect to u_0 in

$$\frac{1}{\sqrt{2}}\int_{u(x,u_0)}^{u_0} \frac{\mathrm{d}s}{\sqrt{(F(\bar{u})-F(s))}} = \sqrt{\lambda}\left(x-\frac{1}{2}\right)$$

we obtain

$$\frac{w(x,u_0)}{\sqrt{(F(u_0)-F(u(x,u_0)))}} = -\frac{1}{2u_0}\int_{u(x,u_0)}^{u_0} \frac{g(u_0)-g(s)}{(F(u_0)-F(s))^{3/2}}\,\mathrm{d}s \tag{18}$$

with $g(u)=uf(u)-2F(u)$. Assume now that $u=u_\lambda(x)$ is any positive solution to (1) with $\bar{u}_\lambda = \max u_\lambda(x) = u_\lambda(1/2)$. By setting $u_0=\bar{u}_\lambda$ and $x=1$ in (18) we arrive at the identity,

$$\frac{w(1,\bar{u}_\lambda)}{\sqrt{2}\sqrt{(F(\bar{u}_\lambda))}} = T'(\bar{u}_\lambda) \tag{19}$$

On the other hand, the eigenvalue problem associated to the linearization of (1) around such solution $u_\lambda(x)$ is

$$\begin{aligned} -v'' &= \lambda f'_u(u_\lambda(x))v + \theta v, \quad 0<x<1 \\ v(0) &= v(1) = 0 \end{aligned} \tag{20}$$

Classical Sturm–Liouville theory asserts that the eigenvalues to (20) consist of an increasing sequence $\theta_n \to +\infty$ of simple eigenvalues, the first one of which, θ_1, is the unique associated with a one sign eigenfunction in $0<x<1$. Observe that $w=\partial u/\partial u_0(x,u_0)$ with $u_0=\bar{u}_\lambda$ solves the equation in (20) when $\theta=0$. Since $u_\lambda(x)$ is symmetric with respect to $x=1/2$, it is clear that $\theta=0$ is an eigenvalue to (20) if and only if $w(1,\bar{u}_\lambda)=0$. In fact, such condition is equivalent to $w(0,\bar{u}_\lambda)=0$, while $w(\cdot,\bar{u}_\lambda)$ is non-trivial due to $w(1/2,\bar{u}_\lambda)=1$.

As a main first conclusion and in virtue of (19), all possible bounded families $\{u_\lambda\}$ of positive solutions to (1), which have been classified in Theorems 1–5, are non-degenerate for λ large, in the sense that the linearized problem around them are not critical, i.e. invertible (compare with the more restrictive analysis in [16]). In fact $\lim_{\lambda\to+\infty} T'(u_\lambda) = +\infty$ in the case of simple lower flat patterns (Theorem 1), and so $\lim w(1,\bar{u}_\lambda) = +\infty$ in this case, while $\lim_{\lambda\to+\infty} T'(\bar{u}_\lambda) = \lim_{\lambda\to+\infty} w(1,\bar{u}_\lambda) = -\infty$ in all remaining classes of patterns (Theorems 2–5).

On the other hand, the asymptotic stability condition for $u_\lambda(x)$, namely $\theta_1 > 0$, is equivalent to the fact $w(x,u_\lambda) > 0$ for $1/2 < x \leqslant 1$. This is a consequence of the result on separation of zeros for second-order equations. Hence, (19) implies that every positive solution in any family different from the simple lower flat pattern becomes unstable for λ large. In fact, since $T'(\bar{u}_\lambda) \to -\infty$ as $\lambda\to+\infty$ in all those cases then $w(1,\bar{u}_\lambda) < 0$ when λ is large and $\theta_1 < 0$ for such solutions.

Let us turn to show that solutions $u_\lambda(x)$ in a simple lower flat pattern are asymptotically stable for λ large. As mentioned, it suffices with proving that $w(x,\bar{u}_\lambda) > 0$ for $1/2 < x \leqslant 1$. Assume, on the contrary, that $w(\bar{x}_\lambda,\bar{u}_\lambda) = 0$ for some $1/2 < \bar{x}_\lambda \leqslant 1$ when $\lambda\to+\infty$. The case $\bar{x}_\lambda \to 1/2$ is easily discarded for, otherwise, suppose that $\bar{x}_\lambda$ is the first zero of w in $x > 1/2$. Observing that $f_u'(u_\lambda(\cdot)) < 0$ in $1/2 \leqslant x \leqslant 1-\delta$ for some $\delta > 0$ small, the fact that w solves (17) leads to $w > 1$ for $|x-1/2| < \varepsilon$ ($\varepsilon > 0$ and small). This contradicts $\bar{x}_\lambda \to 1/2$.

Assume alternatively that $\bar{x}_\lambda \to \bar{x}$ (pick a subsequence if necessary) with $1/2 < \bar{x} \leqslant 1$. From (18) we obtain that,

$$\int_{u(\bar{x}_\lambda,u_0)}^{\bar{u}_\lambda} \frac{g(\bar{u}_\lambda)-g(s)}{(F(\bar{u}_\lambda)-F(s))^{3/2}}\,\mathrm{d}s = 0$$

However that is certainly impossible since such integral becomes negative when λ is great enough. This completes the proof. □

Remarks 7. (a) Under the conditions of Theorem 1, the existence of simple lower flat patterns u_λ for the general n-dimensional version of (1), $u_t = \Delta u + \lambda f(u)$, $x\in\Omega$ with $u_{|\partial\Omega} = 0$, $\Omega\subset\mathbb{R}^n$ a bounded smooth domain, was shown in [20]. In addition, the asymptotic stability and a precise estimate of the normal derivative $\partial u_\lambda/\partial\nu|_{\partial\Omega}$ as the one in (iv) of Theorem 1, are a by-product of the results therein (cf. also [4] for earlier results).

(b) After the proof of non-degeneracy for λ large it follows from the implicit function theorem that all families $\{u_{\lambda_1}\},\ldots,\{u_{\lambda_N}\}$ obtained in Corollary 6 are indeed C^1 curves for λ_i large, $1\leqslant i\leqslant N$.

6. Degenerate cases. Non-linear diffusion

6.1. *Degenerate zeros*

To complete our analysis of bounded families of positive solutions of (1) we are next considering continuous non-linearities f, C^1 except at their zeros. Thus, the zeros

$u_1 < \cdots < u_m$ in $\mathcal{N}$ (and so $F(u_i) = F(\bar{u}_0)$) will be assumed to have orders $0 < k_i < 1$. For simplicity, we are only dealing with the case $F(\bar{u}_0) > 0$ and $f(\bar{u}_0) > 0$. Also for the sake of clarity only *non-oscillating* positive solutions u to (1) will be considered. A positive solution u to (1) will be called non-oscillating if u is not decreasing in $0 \leqslant x \leqslant r_1$, r_1 the first x where u achieves its maximum $\bar{u} = \sup u$, u is not increasing in $r_2 \leqslant x \leqslant 1$, r_2 the last x where $u(x) = \bar{u}$ while $u(x) = \bar{u}$ for each $r_1 \leqslant x \leqslant r_a$ (see Remark 8(c) for an insight on the remaining kind of positive solutions to (1)).

In the present framework, positive solutions u_λ satisfying $\bar{u}_\lambda \to \bar{u}_0$ are exactly described along the following lines. Let us first introduce the numbers

$$T_{i-1} = \frac{1}{\sqrt{2}} \int_{u_{i-1}}^{u_i} \frac{\mathrm{d}s}{\sqrt{(F(\bar{u}_0) - F(s))}}, \quad 1 \leqslant i \leqslant m+1$$

where $u_0 := 0$, $u_{m+1} := \bar{u}_0$, together with the functions $u = U_i(x)$,

$$\frac{1}{\sqrt{2}} \int_{u_i}^{u} \frac{\mathrm{d}s}{\sqrt{(F(\bar{u}_0) - F(s))}} = x, \quad 0 \leqslant x \leqslant T_i$$

$0 \leqslant i \leqslant m$. In addition set $U_i = 0$ outside the interval $[0, T_i]$. The restrictions $0 < k_i < 1$ ensure the convergence of the integrals.

If λ is so large as $2 \sum_{i=0}^{m} T_i < \sqrt{\lambda}$ then it is possible to find non-negative numbers d_i, d_i' such that

$$\sum_{i=0}^{m-1} (d_i + d_i') = \sqrt{\lambda} - 2 \sum_{i=0}^{m} T_i. \tag{21}$$

Defining now the points $\xi_0, \ldots, \xi_m, \xi_0', \ldots, \xi_m'$ as,

$$\begin{aligned} \xi_0 &= 0 \\ \xi_{i+1} &= \xi_i + T_i + d_i, \quad i = 0, 1, \ldots, m-1 \\ \xi_m' &= \xi_m + 2T_m \\ \xi_{i-1}' &= \xi_i' + T_{i-1} + d_{i-1}', \quad i = 1, \ldots, m \end{aligned}$$

if $\chi(x)$ is the characteristic function of the unit interval $I = [0, 1]$, $\chi(x) = 1$ if $0 \leqslant x \leqslant 1$, $\chi(x) = 0$ elsewhere, we can introduce the function $u = u(\xi, d_i, d_i')$,

$$\begin{aligned} u(\xi) = &\sum_{i=0}^{m} \{U_i(\xi - \xi_i) + U_i(\xi_i' - \xi)\} \\ &+ \sum_{i=0}^{m-1} u_{i+1} \{\chi(d_i^{-1}(\xi - \xi_i - T_i)) + \chi(d_i'^{-1}(\xi - \xi_{i+1}'))\} \end{aligned} \tag{22}$$

(Fig. 3). These elements easily lead to the proof of the next result.

Theorem 8. *Let* $0 < u_1 < \cdots < u_m$ *be zeros with orders* $0 < k_i < 1$ *of* f *such that* $F(u_i) = F(\bar{u}_0) > 0$ *and* $f(\bar{u}_0) > 0$. *The following assertions hold true*:

(i) *If* $\{u_\lambda\}$ *is any family of non-oscillating positive solutions to* (1) *satisfying* $\bar{u}_\lambda = \max u_\lambda \to \bar{u}_0$ *as* $\lambda \to +\infty$ *then for* $\lambda > \lambda_0 := 4(\sum_{i=0}^{m} T_i)^2$ *there exist non-negative*

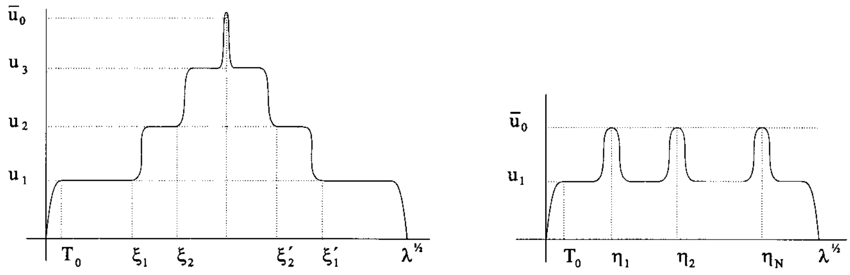

Fig. 3. Degenerate zeros: non-oscillating and oscillating solutions

functions $d_i = d_i(\lambda), d_i' = d_i'(\lambda)$ *satisfying* (21) *such that*

$$u_\lambda(x) = u(\sqrt{\lambda}x, d_i(\lambda), d_i'(\lambda)), \quad 0 \leqslant x \leqslant 1 \tag{23}$$

In particular, $\bar{u}_\lambda = \bar{u}_0$ *for* $\lambda > \lambda_0$.

(ii) *Reciprocally, given any family of non-negative numbers* $d_i = d_i(\lambda), d_i' = d_i'(\lambda)$ *which satisfy* (21) *then* (23) *defines for* $\lambda > \lambda_0$ *a family of non-oscillating positive solutions* $\{u_\lambda\}$ *to* (1) *such that* $\bar{u}_\lambda = \bar{u}_0$.

Remark 8. (a) Item (ii) asserts that in the present case, infinitely many families of solutions $\{u_\lambda\}$ can be generated from the same set of zeros $u_1 < \cdots < u_m$. On the other hand, the development of flat cores now occurs in a trivial way. In fact $u_\lambda = u_i$ in the pair of intervals $[\xi_i/\sqrt{\lambda} - d_{i-1}/\sqrt{\lambda}, \xi_i/\sqrt{\lambda}] \cup [\xi_i'/\sqrt{\lambda}, \xi_i'/\sqrt{\lambda} + d_{i-1}'/\sqrt{\lambda}]$. The limit position of such intervals as $\lambda \to +\infty$ can be computed for (21) ensures the existence of subfamilies such that $d_i(\lambda)/\sqrt{\lambda} \to d_i^*, d_i'(\lambda)/\sqrt{\lambda} \to (d_i')^*$. This means that,

$$\frac{\xi_i}{\sqrt{\lambda}} \to d_0^* + \cdots + d_{i-1}^*, \quad \frac{\xi_i'}{\sqrt{\lambda}} \to d_0^* + \cdots + d_{m-1}^* + (d_{m-1}')^* + \cdots + (d_{i-1}')^*$$

as $\lambda \to +\infty$. Note that $d_0^* + \cdots + d_{m-1}^* + (d_1')^* + \cdots + (d_{m-1}')^* = 1$. Reciprocally, families of solutions $\{u_\lambda\}$ with arbitrary pre-fixed limit positions of such intervals can be constructed by a suitable choice of convergent sequences $d_i(\lambda)/\sqrt{\lambda}$ and $d_i'(\lambda)/\sqrt{\lambda}$ in (21).

(b) As explained in sections (4.2), a group of zeros $u_1 < \cdots < u_m$ with multiplicities lower than unity can coexist in $\mathcal{N}$ with $v_1 < \cdots < v_n$ zeros with orders at least one (thus fitting into some of the patterns studied in sections 3 and 4). In this case, the second group clearly leads the asymptotic pattern of the infinitely many families $\{u_\lambda\}$ associated with the whole group $\{u_1, \ldots, u_m, v_1, \ldots, v_n\}$.

(c) With non-oscillating solutions aside, a great amount of families of positive solutions $\{u_\lambda\}$, satisfying $\bar{u}_\lambda = \bar{u}_0$ can be constructed from a fixed set $0 < u_1 < \cdots < u_m$ of degenerate zeros of f in the precise conditions of this section.

To give an account of the possibilities just consider the simplest case where $\mathcal{N} = \{u_1, \bar{u}_0\}, 0 < u_1 < \bar{u}_0$ with u_1 a degenerate zero and $f(\bar{u}_0) > 0$. To achieve a family u_λ of positive solutions with exactly N oscillations in the interval $I=[0,1]$ first assume $2(T_0+NT_1)<\sqrt{\lambda}$, the values being T_0, T_1 and the functions U_0, U_1 those defined at the beginning of the section. Next, set arbitrary points $\eta_1,\dots,\eta_N$:

$$\eta_1 = T_0 + d_0 + T_1, \quad \eta_k = \eta_{k-1} + d_{k-1} + 2T_1, \quad 2 \leqslant k \leqslant N$$

the numbers $d_0,\dots,d_N$ being known once the η_k are given, while the identity

$$2T_0 + 2NT_1 + d_0 + \cdots + d_N = \sqrt{\lambda}$$

holds. If χ stands again for the characteristic function of the interval $I=[0,1]$ then the family of functions:

$$u(\xi, d_i, d_i') = U_0(\xi) + U_0(\sqrt{\lambda}-\xi) + \sum_{k=1}^{N}\{U_1(\xi-\eta_k+T_1)+U_1(T_1+\eta_k-\xi)\}$$
$$+ \sum_{k=1}^{N} u_1\chi(d_{k-1}^{-1}(\xi-(\eta_k-T_1-d_{k-1}))) + \chi(d_N^{-1}(\xi-(\eta_N+T_1)))$$

$\xi=\sqrt{(\lambda)}x$, defines, for an arbitrary choice of the functions $d_i=d_i(\lambda)$ satisfying (24), a family $\{u_\lambda\}$ of positive solutions to (1) such that every u_λ exactly undergoes N oscillations in the interval I. Finally, observe that the number of possible configurations and the complexity of oscillating patterns considerably increases with the number m of degenerate zeros in $\mathcal{N}$ (Fig. 3).

6.2. *The p-Laplacian operator*

Let us explain how all the results described remain true for the one-dimensional p-Laplacian version of problem (1). Namely

$$\begin{aligned} &-(|u'|^{p-2}u')' = \lambda f(u), \quad 0<x<1 \\ &u(0)=u(1)=0 \end{aligned} \tag{1a}$$

$p>1$. As we precise later, in general (1a) fails to have classical solutions. Thus, solutions to (1a) are understood in the weak sense, that is $u \in W_0^{1,p}(0,1)$ and

$$\int_0^1 |u'|^{p-2}u'v = \lambda\int_0^1 f(u)v$$

for every $v \in W_0^{1,p}(0,1)$. It can be shown that such solutions verify, possibly after the redefinition in a zero set, $u \in C^1[0,1]$, $|u'|^{p-2}u' \in C^1[0,1]$ and (1a) is pointwise satisfied (see [6] for details). This implies that u is also C^2 except at critical points.

On the other hand, direct computations show that the combination $|u'|^p + p'\lambda F(u)$ keeps constant on solutions to (1a). Therefore, if $u \in W_0^{1,p}(0,1)$ is a positive solution to (1a) with maximum $\bar{u}=\max u(x)$, $f(\bar{u})\neq 0$ (see the item (d) below) then u can be represented as

$$\frac{1}{(p')^{1/p}}\int_{u(x)}^{\bar{u}} \frac{\mathrm{d}s}{(F(\bar{u})-F(s))^{1/p}} = \lambda^{1/p}\left(x-\frac{1}{2}\right)$$

with $p' = p/(p-1)$. By using $T(\bar{u}) = (p')^{-1/p}\int_0^{\bar{u}} (F(\bar{u}) - F(s))^{-1/p}\,\mathrm{d}s$ the features in section 2 hold (now $\bar{u}$ solves the equation $T(\bar{u}) = \lambda^{1/p}/2$).

It should also be pointed out that the multiplicity $k = p-1$ for zeros of f plays now the rôle of $k=1$ in (1) (i.e. $p=2$). In fact, the integral in $T(\bar{u})$ converges near $\bar{u}$ if $f(\bar{u}) = 0$ only when $0 < k < p-1$. Thus, only such kind of zeros can be the critical values of positive solutions to (1) (see (d) below).

Finally, just a few remarks on the smoothness of weak solutions to (1a). If u is a positive solution with maximum $\bar{u}$ and $f(\bar{u}) \neq 0$ (and so $f(\bar{u}) > 0$) then the representation

$$u = \bar{u} - \frac{f(\bar{u})^{p'-1}}{p'}\left|x - \frac{1}{2}\right|^{p'} + o\left(\left|x - \frac{1}{2}\right|^{p'}\right) \quad \text{as } \left|x - \frac{1}{2}\right| \to 0+$$

holds in C^1. This means that u is C^2 near $x = 1/2$ only when $1 < p \leqslant 2$. If, on the contrary $f(\bar{u}) = 0$ with $f(\bar{u}) \sim \gamma\varphi_{k+1}(\bar{u} - u), \gamma > 0$, the general setting of the paper implies $0 < k < p-1$. In this case $u = \bar{u}$ in a subinterval $[d_1, d_2]$ ($d_1 = d_2$ in some cases) and the C^1 behaviour near, say, $x = d_1$ is

$$u(x) = \bar{u} - \left(\frac{\gamma}{(\alpha k + 1)\alpha^{p-1}}\right)^{\alpha/p}(d_1 - x)^{\alpha} + o(|x - d_1|^{\alpha}) \quad \text{as } x \to d_1 -$$

where $\alpha = p/(p-1-k)$. This means that u is C^2 near $x = d_1$ only when $p/2 \leqslant k+1$ (recall we require $k+1 < p$). See [8] for precise details.

Let us give now a detailed account of the variations on the results in sections 3 and 4.

(a) When $p \neq 2$, upper and lower flat patterns appear modulus some slight modifications. The asymptotic rates in Theorem 1 (ii) read now as

$$\bar{u}_\lambda \sim \begin{cases} \bar{u}_0 - \exp\left\{-\dfrac{1}{2}\left(\dfrac{\gamma\lambda}{p-1}\right)^{1/p}\right\} \quad \lambda \to +\infty & \text{if } k = p-1 \\ \bar{u}_0 - \left(\dfrac{2\bar{B}_k}{k+1}\left(\dfrac{k+1}{\gamma p'}\right)^{1/p}\right)^{p/(k-p+1)} \lambda^{-1/(k-p+1)} & \text{if } k > p-1 \end{cases}$$

as $\lambda \to +\infty$ where $\tilde{B}_k = B(1/p - 1/(k+1),\ 1 - 1/p)$. As for the upper flat case the corresponding rates in Theorem 2(ii) are

$$\bar{u}_\lambda \sim \begin{cases} \bar{u}_0 + \exp\left\{-\dfrac{1}{2}\left(\dfrac{\gamma\lambda}{p-1}\right)^{1/p}\right\} & \text{if } k = p-1 \\ \bar{u}_0 + \left(\dfrac{2\tilde{B}_k}{k+1}\left(\dfrac{k+1}{\gamma p'}\right)^{1/p}\tilde{C}_k \sin(\pi/p)\right)^{p/(k-p+1)} \lambda^{-1/(k-p+1)} & \text{if } k > p-1 \end{cases}$$

as $\lambda \to +\infty$, being $\tilde{C}_k = \csc(\pi/(k+1)) + \cot(\pi/(k+1)) - \cot(\pi/p)$.

(b) If $p \neq 2$, the estimates in (ii) of Theorem 3 change to

$$\bar{u}_\lambda - \bar{u}_0 \sim \begin{cases} \exp\left\{ -\dfrac{p}{2\beta}\left(\dfrac{\gamma\lambda}{p-1}\right)^{1/p}\right\} \\ \dfrac{1}{f(\bar{u}_0)}\left(\dfrac{2\beta\tilde{B}_k}{(p')^{1/p}(k+1)}\left(\dfrac{k+1}{\gamma}\right)^{1/(k+1)}\sin(\pi/p) \right. \\ \qquad \left. \times \csc(\pi/(k+1))\right)^{p(k+1)/(k-p+1)} \lambda^{-(k+1)/(k-p+1)} \end{cases}$$

respectively, if $k = p-1$, $k > p-1$, as $\lambda \to +\infty$. $\tilde{B}_k$ is defined in (a) and $\beta = 1, 2$ if, respectively, $u_1 = 0$, $u_1 > 0$.

(c) As for multiple flat patterns, the p-versions of the results in Theorem 4 are next detailed. We are simultaneously including the case where $F(\bar{u}_0) = 0$ with $f(0) = 0$ and $f(u) \sim -\gamma_0 u^k$. In that case we will set the coefficient $\beta = 1$, being $\beta = 0$ if $F(\bar{u}_0) > 0$. Thus, the estimates in (ii) hold under the form

$$\bar{u}_\lambda \sim \begin{cases} \bar{u}_0 + \exp\left\{-\dfrac{1}{4}\dfrac{(\lambda/(p-1))^{1/p}}{(\beta/2)\gamma_0^{-1/p} + \sum_{i=1}^m \gamma_i^{-1/p} + (1/2)\gamma_{m+1}^{-1/p}}\right\} \\ \bar{u}_0 + \left(\dfrac{k+1}{\gamma_{m+1}}\right)^{1/\sigma} \tilde{D}_k \left[\dfrac{\beta}{2}\left(\dfrac{\gamma_{m+1}}{\gamma_0}\right)^{-1/(k+1)} + \sum_{i=1}^{m+1}\left(\dfrac{\gamma_{m+1}}{\gamma_i}\right)^{-1/(k+1)} + \tilde{E}_k\right]^{p/\sigma} \lambda^{-1/\sigma} \end{cases}$$

respectively, for $k = p-1$, $k > p-1$, where $\tilde{D}_k$ stands for the constant

$$\tilde{D}_k = \left\{\frac{4^p \tilde{B}_k^p \sin^p(\pi/p)\csc^p(\pi/(k+1))}{p'(k+1)^p}\right\}^{1/\sigma},$$

$\tilde{E}_k = \sin^2((p-2)/2p + 1/(k+1))(\pi/2)\csc(\pi/p)$ and $\sigma = k+1-p$. The corresponding expressions for the inner layers are $(1 \leqslant j \leqslant m)$

$$\xi_j = \begin{cases} \dfrac{1}{2}\dfrac{(\beta/2)\gamma_0^{-1/p} + \sum_{i=1}^j \gamma_i^{-1/p} + (1/2)\gamma_{m+1}^{-1/p}}{(\beta/2)\gamma_0^{-1/p} + \sum_{i=1}^m \gamma_i^{-1/p} + (1/2)\gamma_{m+1}^{-1/p}}, & k = p-1 \\ \dfrac{1}{2}\dfrac{(\beta/2)\gamma_0^{-1/(k+1)} + \sum_{i=1}^j \gamma_i^{-1/(k+1)}}{(\beta/2)\gamma_0^{-1/(k+1)} + \sum_{i=1}^{m+1} \gamma_i^{-1/(k+1)} + (1/2)\gamma_{m+1}^{-1/(k+1)}\tilde{E}_k}, & k > p-1 \end{cases}$$

Finally, the assumptions of Theorem 5 also lead in the case $p \neq 2$ to the arising of multiple flat patterns and spike as $\lambda \to +\infty$. The expressions for the inner layers remain the same. Namely

$$\xi_j = \frac{1}{2}\frac{(\beta/2)\gamma_0^{-1/(k+1)} + \sum_{i=1}^j \gamma_i^{-1/(k+1)}}{(\beta/2)\gamma_0^{-1/(k+1)} + \sum_{i=1}^m \gamma_i^{-1/(k+1)}}$$

$1 \leqslant j \leqslant m$, $k \geqslant p-1$, while the estimates in (ii) now read

$$\bar{u}_\lambda \sim \begin{cases} \bar{u}_0 + \exp\left\{ -\dfrac{p}{4} \dfrac{(\lambda/(p-1))^{1/p}}{(\beta/2)\gamma_0^{-1/p} + \sum_{i=1}^m \gamma_i^{-1/p}} \right\} \\ \tilde{u}_0 + \dfrac{1}{f(\tilde{u}_0)} \tilde{E}_k \left[(\beta/2) \left(\dfrac{k+1}{\gamma_0} \right)^{-1/(k+1)} + \sum_{i=1}^m \left(\dfrac{k+1}{\gamma_i} \right)^{-1/(k+1)} \right]^{p(k+1)/\sigma} \lambda^{-(k+1)/\sigma} \end{cases}$$

respectively, if $k = p-1$, $k > p-1$.

(d) The scenario of degenerate zeros in section 6.1 corresponds, when $p \neq 2$, to the case where $\mathcal{N}$ consists of positive zeros $u_1 < \cdots < u_m$ with multiplicities $0 < k_i < p-1$. In order to construct the families $u_\lambda(x) = u(\sqrt{\lambda} x, d_i, d_i')$ of Theorem 8 we similarly introduce the numbers T_i and functions U_i as

$$T_{i-1} = \frac{1}{(p')^{1/p}} \int_{u_{i-1}}^{u_i} \frac{\mathrm{d}s}{(F(\bar{u}_0) - F(s))^{1/p}}, \qquad \frac{1}{(p')^{1/p}} \int_{u_i}^{U_i(x)} \frac{\mathrm{d}s}{(F(\bar{u}_0) - F(s))^{1/p}} = x$$

$i = 1, \ldots, m+1$. Assuming that $\lambda > \lambda_0(p) := 2^p (\sum_{i=1}^m T_i)^p$ and choosing d_i, d_i' satisfying (21) with $\lambda^{1/p}$ replacing $\sqrt{\lambda}$, the same set of points ξ_i, ξ_i' as in section 6.1 provides the desired family of solutions $u_\lambda = u(\sqrt{(\lambda)} x, d_i, d_i')$. Of course, its properties are those in Theorem 8 and Remarks 8. It should also be remarked that this kind of solutions (including two sign solutions) was studied in [9] for the special case of logistic-type non-linearities $f(u)$ exhibiting a single zero in the conditions of Theorem 1.

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