On the principal eigenvalue of some nonlocal diffusion problems

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Abstract

In this paper we analyze some properties of the principal eigenvalue $\lambda_1(\Omega)$ of the nonlocal Dirichlet problem $(J*u)(x)-u(x) = -\lambda u(x)$ in Ω with $u(x) = 0$ in $\mathbb{R}^N \setminus \Omega$. Here Ω is a smooth bounded domain of $\mathbb{R}^N$ and the kernel J is assumed to be a C^1 compactly supported, even, nonnegative function with unit integral. Among other properties, we show that $\lambda_1(\Omega)$ is continuous (or even differentiable) with respect to continuous (differentiable) perturbations of the domain Ω. We also provide an explicit formula for the derivative. Finally, we analyze the asymptotic behavior of the decreasing function $\Lambda(\gamma) = \lambda_1(\gamma\Omega)$ when the dilatation parameter $\gamma > 0$ tends to zero or to infinity.

1 Introduction

In the present work we consider the "Dirichlet" eigenvalue problem for a nonlocal operator in a smooth bounded domain Ω:

$$\begin{cases} (J*u)(x) - u(x) = -\lambda u(x), & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.1}$$

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Here $J * u$ stands for the usual convolution,

$$(J * u)(x) = \int_{\mathbb{R}^N} J(x-y)u(y)\,dy,$$

with a the kernel J that is a C^1, compactly supported, nonnegative function with unit integral.

Nonlocal problems related to (1.1) have been recently widely used to model diffusion processes. When $u(x,t)$ is interpreted as the density of a single population at the point x at time t and $J(x-y)$ is the probability of "jumping" from location y to location x, the convolution $(J*u)(x)$ is the rate at which individuals arrive to position x from all other positions, while $-\int_{\mathbb{R}^N} J(y-x)u(x,t)\,dy = -u(x,t)$ is the rate at which they leave position x to reach any other position. If in addition an external source $f(x,u(x,t))$ is present, we obtain the evolution problem

$$\begin{cases} u_t(x,t) = (J*u)(x,t) - u(x,t) + f(x,u(x,t)), & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \mathbb{R}^N \setminus \Omega, t \geq 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

where the "boundary" condition $u = 0$ in $\mathbb{R}^N \setminus \Omega$ means that the habitat Ω is surrounded by a hostile environment (see [23]). Problem (1.2) and its stationary version have been considered recently for several kinds of nonlinearities f. We quote for instance [4], [6], [7], [10], [18], [19], [21], [22], [32] and [33], devoted to travelling front type solutions to the parabolic problem when $\Omega = \mathbb{R}$, and [5], [11], [12], [20], [31], which dealt with the study of problem (1.2) with a logistic type, bistable or power-like nonlinearity. The particular instance of the parabolic problem in $\mathbb{R}^N$ when $f = 0$ is considered in [9], [27], while the "Neumann" boundary condition for the same problem is treated in [1], [16] and [17]. See also [28] for the appearance of convective terms, [2] for a problem with nonlinear nonlocal diffusion and [13], [14], [15] for interesting features in other related nonlocal problems.

We observe that stationary solutions to (1.2) are critical points in $L^2(\Omega)$ of the functional

$$H(u) = \frac{1}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)(u(x)-u(y))^2\,dx\,dy - \int_{\Omega} F(x,u(x))\,dx,$$

where all functions are assumed to vanish outside Ω and F is given by

$$F(x,u) = \int_0^u f(x,s)\,ds.$$

When the kernel J is symmetric, we can expand the first integral in a Taylor series and drop all the terms but for the first one, to obtain the approximate energy

$$\widetilde{H}(u) = \frac{A(J)}{2} \int_\Omega |\nabla u(x)|^2 dx - \int_\Omega F(x, u(x)) dx,$$

where $A(J) = 1/(2N) \int_{\mathbb{R}^N} J(y)|y|^2\, dy$ (see [5]). If we assume that $A(J) = 1$, for simplicity, we have that critical points of $\widetilde{H}$ are weak solutions to the problem

$$\begin{cases} -\Delta u(x) = f(x, u(x)), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

Thus, it is expected that stationary solutions to (1.2) behave in some sense similarly as those of (1.3). This is indeed the case at least for some nonlinearities, even for the parabolic version, see [15], [16], [17], [20]. We also remark in passing that when J is not symmetric, a convection term has to be added in (1.3) to preserve the resemblance between the local and nonlocal problems (see [28]). However, we are restricting in the present work to symmetric kernels.

On the other hand, it is well known that eigenvalue problems are a fundamental tool to deal with problem (1.3). Particularly, when positive solutions are considered, the so-called principal eigenvalue of the problem

$$\begin{cases} -\Delta v(x) = \sigma v(x), & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1.4}$$

plays an important role. The properties of the principal eigenvalue of (1.4) are well-known, and they are frequently used to obtain qualitative information of positive solutions to (1.3).

Our objective in the present work is to study properties of the principal eigenvalue associated to nonlocal problems. Some preliminary properties are already known, as existence, uniqueness and a variational characterization (we collect some of these results with full proofs in Section 2 for the reader's convenience). Here, we are particularly interested in the analysis of the dependence of the principal eigenvalue with respect to the domain. Among the obtained results, two of them seem to be worth stressing. The first one is the continuity and the differentiability of the principal eigenvalue with respect to continuous or differentiable perturbations of the domain. The second one is the precise asymptotic behavior of the principal eigenvalue in scaled domains $\gamma\Omega$ when the parameter γ goes to zero or infinity. In the latter case we find that the eigenvalue behaves essentially as a multiple (that depends on J) of the principal eigenvalue of the local Laplacian.

Next, let us state our main results. We are assuming without further mention that Ω is a bounded C^1 domain and $J \in C^1(\mathbb{R}^N)$ verifies $J > 0$ in B_1 (the unit ball), $J = 0$ in $\mathbb{R}^N \setminus B_1$, $J(-z) = J(z)$, with $\int_{B_1} J(x)dx = 1$.

It is shown in Section 2 that problem (1.1) admits a unique principal eigenvalue, that is, an eigenvalue with an associated positive eigenfunction. This eigenvalue enjoys the usual properties: it is simple and unique, and it can be variationally characterized (see three different characterizations in Theorem 2.1). Let us denote it by $\lambda_1(\Omega)$. We also remark that the associated eigenfunction u_0 verifies $u_0 \in C(\overline{\Omega})$, $u_0 > 0$ in $\overline{\Omega}$, and hence it has a jump discontinuity across $\partial\Omega$, see [8], [9].

As we have mentioned, we are interested in the dependence of the first eigenvalue on the domain Ω. A first consequence of the variational characterization is the strict monotonicity of $\lambda_1(\Omega)$:

Theorem 1.1 *The principal eigenvalue of problem* (1.1) *in* Ω*,* $\lambda_1(\Omega)$*, is decreasing with respect to the domain, that is, if* $\Omega_1 \subsetneq \Omega_2$*, then* $\lambda_1(\Omega_1) > \lambda_1(\Omega_2)$*.*

Next, we analyze perturbations Ω_δ of a fixed domain Ω, where δ is a small parameter, and consider the issues of continuity and differentiability of $\lambda_1(\Omega_\delta)$ with respect to δ. We assume that the perturbed domain verifies $\Omega_\delta = \Psi(\delta, \Omega)$, where $\Psi : (-\varepsilon, \varepsilon) \times \overline{\Omega} \to \mathbb{R}^N$ takes the form

$$\Psi(\delta, x) = x + \Phi(\delta, x), \tag{1.5}$$

with $\Phi(0, \cdot) = 0$. The continuity of $\lambda_1(\Omega_\delta)$ is a more or less simple consequence of the continuity of Φ with respect to δ. We denote by $D\Phi$ the differential of Φ with respect to x.

Theorem 1.2 *Let* $\lambda_1(\Omega_\delta)$ *be the principal eigenvalue of* (1.1) *in* Ω_δ*, and assume* $\Omega_\delta = \Psi(\delta, \Omega)$*, where* Ψ *has the form* (1.5) *with* $\Phi, D\Phi \in C((-\varepsilon, \varepsilon) \times \overline{\Omega})$ *for some* $\varepsilon > 0$ *and* $\Phi(0, \cdot) = 0$*. Then,* $\lambda_1(\Omega_\delta) \to \lambda_1(\Omega)$ *as* $\delta \to 0$*.*

We now consider the question of differentiability of $\lambda_1(\Omega_\delta)$. We assume the function Ψ in (1.5) is differentiable and prove that $\lambda_1(\Omega_\delta)$ is differentiable at $\delta = 0$, providing in addition an explicit formula for the derivative (see [30] for the analogous formula for the Laplacian and [25] for the p-Laplacian).

Theorem 1.3 *Let* $\lambda(\delta) = \lambda_1(\Omega_\delta)$ *be the principal eigenvalue of problem* (1.1) *in* Ω_δ*, and assume* $\Omega_\delta = \Psi(\delta, \Omega)$*, where* Ψ *is of the form* (1.5) *with* $\Phi \in C^1((-\varepsilon, \varepsilon) \times \overline{\Omega})$ *for some* $\varepsilon > 0$ *and* $\Phi(0, \cdot) = 0$*. Then* $\lambda(\delta)$ *is differentiable* *with respect to* δ *at* $\delta = 0$*, and*

$$\lambda'(0) = -(1 - \lambda_1(\Omega)) \int_{\partial\Omega} u_0^2(x) \left\langle \frac{\partial \Phi}{\partial \delta}(0, x), \nu(x) \right\rangle dS(x), \tag{1.6}$$

where u_0 *is the positive eigenfunction associated to* $\lambda_1(\Omega)$ *normalized with* $|u_0|_{L^2(\Omega)} = 1$ *and* $\nu(x)$ *is the outward unit normal to* $\partial\Omega$.

Note that the eigenfunction u_0 is strictly positive on $\partial\Omega$ in spite of the boundary condition in (1.1), see [8], [9]. Thus, the integral in (1.6) is not necessarily zero.

An important example of perturbation of a domain is provided when Ω is enlarged in the direction of the unit normal an amount δ. To make this precise, assume $\partial\Omega$ splits into m connected components, and select k of these components $\Gamma_1, \dots, \Gamma_k$. Set

$$\Omega_\delta = \Omega \bigcup_{i=1}^{k} \{x \in \mathbb{R}^N : \ \mathrm{dist}(x, \Gamma_i) < \delta\}. \tag{1.7}$$

According to Theorem 3.1 in [30], we have $\Omega_\delta = \Psi(\delta, \Omega)$, where $\Psi(\delta, x) = x + \delta\widetilde{\Phi}(x)$. Moreover, the derivative with respect to δ, $\widetilde{\Phi} = \frac{\partial \Phi}{\partial \delta}(0, \cdot)$. verifies $\widetilde{\Phi} = \nu$ on the components Γ_i while $\widetilde{\Phi} = 0$ on the remaining components of the boundary. Hence, we obtain that $\lambda_1(\Omega_\delta)$ decreases linearly as δ goes to zero.

Corollary 1.1 *Let* Ω *be a bounded* C^1 *domain of* $\mathbb{R}^N$*, and assume* Ω_δ *is the perturbation of* Ω *given by* (1.7)*. Then* $\lambda(\delta) = \lambda_1(\Omega_\delta)$ *is differentiable with respect to* δ *at* $\delta = 0$*, and*

$$\lambda'(0) = -(1 - \lambda_1(\Omega)) \sum_{i=1}^{k} \int_{\Gamma_i} u_0^2(x)\, dS(x) < 0,$$

where u_0 *is the positive eigenfunction of* $\lambda_1(\Omega)$ *normalized with* $|u_0|_{L^2(\Omega)} = 1$.

Having established the smoothness and monotonicity properties of $\lambda_1(\Omega)$, we come to the analysis of its asymptotic behavior both for small and large domains Ω. In this context $\Omega_n \to \mathbb{R}^N$ means that the sequence of sets Ω_n contains balls B_{R_n} (centered at a fixed point) with radii $R_n \to +\infty$. Our first result in this direction is the following:

Theorem 1.4 *For the principal eigenvalue* $\lambda_1(\Omega)$ *we have* $\lambda_1(\Omega) \to 1$ *when* $|\Omega| \to 0$ *and* $\lambda_1(\Omega_n) \to 0$ *when* $\Omega_n \to \mathbb{R}^N$.

To make more precise the information given by Theorem 1.4, we fix a C^1 bounded domain Ω and consider dilatations of it, $\Omega_\gamma = \gamma\Omega$, where $\gamma > 0$ is the dilatation parameter. As a consequence of the previous theorems, we have that $\lambda_1(\Omega_\gamma)$ is a decreasing function of γ and $\lambda_1(\Omega_\gamma) \to 1$ when $\gamma \to 0$, $\lambda_1(\Omega_\gamma) \to 0$ as $\gamma \to +\infty$. Our last theorem describes precisely the asymptotic behavior of $\lambda_1(\Omega_\gamma)$ both when $\gamma \to 0$ and when $\gamma \to \infty$.

Theorem 1.5 *Let Ω be a smooth bounded domain of $\mathbb{R}^N$, and for $\gamma > 0$ denote $\Omega_\gamma = \gamma\Omega$. Then*

$$\lambda_1(\Omega_\gamma) \sim 1 - J(0)|\Omega|\gamma^N \quad \text{as } \gamma \to 0+ . \tag{1.8}$$

If in addition J is radially symmetric and radially decreasing, then

$$\lambda_1(\Omega_\gamma) \sim A(J)\sigma_1(\Omega)\gamma^{-2} \quad \text{as } \gamma \to +\infty, \tag{1.9}$$

where $\sigma_1(\Omega)$ is the principal eigenvalue of the Laplacian in Ω with Dirichlet boundary conditions,

$$\begin{cases} -\Delta v(x) = \sigma_1(\Omega) v(x), & x \in \Omega, \\ \quad v(x) = 0, & x \in \partial\Omega \end{cases} \tag{1.10}$$

and the constant $A(J)$ is given by

$$A(J) = \frac{1}{2N} \int_{\mathbb{R}^N} J(z)|z|^2 \, dz.$$

Roughly speaking, when conveniently scaled to a large domain, our nonlocal problem resembles a local one. Indeed, for the first eigenvalue of the Laplacian it is well known that $\sigma_1(\Omega_\gamma) = \sigma_1(\Omega)\gamma^{-2}$, therefore the asymptotic behavior as $\gamma \to \infty$ for both problems coincide (up to a factor that depends on J, $A(J)$). This resemblance has been already observed for related problems in previous works, for instance in [2], [15] and [17]. Notice that the vanishing rate of $1 - \lambda_1(\Omega_\gamma)$ at $\gamma = 0$ and of $\lambda_1(\Omega_\gamma)$ at $\gamma = +\infty$ is different, which is in contrast with the already mentioned scaling invariance of the Laplacian. This phenomenon is caused by the lack of homogeneity of the convolution term $J * u$. Hence, there is a strong difference between the behaviour of the first eigenvalue for local diffusion and for nonlocal diffusion when the domain is small (case $\gamma \sim 0$) but there is no big difference for large domains (case $\gamma \sim \infty$).

The rest of the paper is organized as follows: in Section 2 we consider the issues of existence, simplicity and monotonicity of the principal eigenvalue. Section 3 is devoted to prove the differentiability with respect to differentiable perturbations of the domain, while in Section 4 the asymptotic behavior of the principal eigenvalue in big and small domains is analyzed.

2 Preliminaries

In this section we consider some preliminary facts related with the principal eigenvalue of (1.1). First notice that, since the eigenfunctions u of (1.1) verify $u = 0$ in $\mathbb{R}^N \setminus \Omega$, the integral in the convolution term can indeed be considered only in Ω. Thus we define the operator

$$L_0 u(x) = \int_\Omega J(x-y)u(y)\,dy.$$

Although the integral makes sense when $u \in L^1(\Omega)$, we are considering L_0 as an operator defined in $L^2(\Omega)$ with values in $L^2(\Omega)$. This operator $L_0 : L^2(\Omega) \to L^2(\Omega)$ is self-adjoint and compact.

Now observe that λ is an eigenvalue of (1.1) if and only if $\mu = 1-\lambda$ is an eigenvalue of L_0 in $L^2(\Omega)$. Since L_0 is compact and selfadjoint, the classical theory of compact operators in Hilbert spaces apply. However, we are interested only in the existence of a principal eigenvalue, that is, an eigenvalue associated to a nonnegative eigenfunction. Notice that an eigenfunction $u \in L^2(\Omega)$ (or even in $L^1(\Omega)$) automatically verifies that $u \in C(\overline{\Omega})$, and thus thanks to the strong maximum principle nonnegative eigenfunctions are strictly positive in $\overline{\Omega}$ (see Theorem 7 in [24]).

We summarize in the next result the essential properties of the principal eigenvalue (see [20] and [26] for a proof of existence).

Theorem 2.1 *Problem* (1.1) *admits an eigenvalue* $\lambda_1(\Omega)$ *associated to a positive eigenfunction* $\phi \in C(\overline{\Omega})$. *Moreover, it is simple and unique, and it verifies* $0 < \lambda_1(\Omega) < 1$. *Furthermore,* $\lambda_1(\Omega)$ *can be variationally characterized as*

$$\lambda_1(\Omega) = 1 - \left(\sup_{\substack{u\in L^2(\Omega)\\ u\neq 0}} \frac{\displaystyle\int_\Omega \left(\int_\Omega J(x-y)u(y)\,dy \right)^2 dx}{\displaystyle\int_\Omega u^2(x)\,dx} \right)^{1/2} \qquad (2.1)$$

or

$$\lambda_1(\Omega) = 1 - \sup_{\substack{u \in L^2(\Omega) \\ u \neq 0}} \frac{\displaystyle\int_\Omega \int_\Omega J(x-y)u(x)u(y)\,dy\,dx}{\displaystyle\int_\Omega u^2(x)\,dx} \tag{2.2}$$

or

$$\lambda_1(\Omega) = \frac{1}{2} \inf_{\substack{u \in L^2(\Omega) \\ u \neq 0}} \frac{\displaystyle\int_\Omega \int_\Omega J(x-y)(u(x)-u(y))^2\,dy\,dx}{\displaystyle\int_\Omega u^2(x)\,dx}. \tag{2.3}$$

Proof. Motivated by the fact that the positive cone of $L^2(\Omega)$ has empty interior and since eigenfunctions are continuous in $\overline{\Omega}$, we consider in this proof the operator L_0 defined in $C(\overline{\Omega})$ instead of $L^2(\Omega)$. Note that L_0 is a positive operator: $L_0 u \geq 0$ for every $u \geq 0$. Moreover, L_0 is strongly positive in the sense that for every nonnegative $u \in C(\overline{\Omega})$, there exists n such that $L_0^n u > 0$ in $\overline{\Omega}$. Indeed, if $u(x_0) > 0$, then it follows that $L_0 u > 0$ in $B_1(x_0)$, and after finitely many steps we arrive at $L_0^n u > 0$ in $\overline{\Omega}$. According to Theorem 6.3 in [29], this property is enough to obtain that the spectral radius $\mathrm{spr}_{C(\overline{\Omega})}(L_0)$ of L_0 is an eigenvalue associated to a positive eigenfunction, and it is the unique eigenvalue of L_0 with this property.

To proceed further, we consider again L_0 defined in $L^2(\Omega)$. Since L_0 is self-adjoint, it follows that $\mathrm{spr}_{L^2(\Omega)}(L_0) = \|L_0\|$ (where $\|L_0\|$ denotes the operator norm in $L^2(\Omega)$), and there exists and eigenvalue $\lambda \in \mathbb{R}$ of L_0 such that $|\lambda| = \|L_0\|$ (cf. for instance [3]). We deduce then that $\mathrm{spr}_{C(\overline{\Omega})}(L_0) = \|L_0\|$, and thus $\|L_0\|$ is an eigenvalue associated to a positive eigenfunction, and it is the unique eigenvalue of L_0 with this property.

Hence, the principal eigenvalue of problem (1.1) is given by $\lambda_1(\Omega) = 1 - \|L_0\|$. This immediately implies $\lambda_1(\Omega) < 1$. To prove that $\lambda_1(\Omega) > 0$, we use the maximum principle. Indeed, assume that $\lambda_1(\Omega) \leq 0$ and let ϕ be an associated positive eigenfunction. Then,

$$J * \phi - \phi = -\lambda_1(\Omega)\phi \geq 0.$$

The maximum principle implies $\phi \leq 0$, which is impossible. Thus $\lambda_1(\Omega) > 0$.

The variational characterizations (2.1) and (2.2) are can be obtained at once since

$$\|L_0\|^2 = \sup_{\substack{u\in L^2(\Omega)\\ u\neq 0}} \frac{|L_0 u|^2_{L^2(\Omega)}}{|u|^2_{L^2(\Omega)}} = \sup_{\substack{u\in L^2(\Omega)\\ u\neq 0}} \frac{\int_\Omega \left(\int_\Omega J(x-y)u(y)\,dy\right)^2 dx}{\int_\Omega u^2(x)\,dx},$$

and

$$\|L_0\| = \sup_{\substack{u\in L^2(\Omega)\\ u\neq 0}} \frac{|\langle L_0 u, u\rangle|}{|u|^2_{L^2(\Omega)}} = \sup_{\substack{u\in L^2(\Omega)\\ u\neq 0}} \frac{\int_\Omega\int_\Omega J(x-y)u(x)u(y)\,dy\,dx}{\int_\Omega u^2(x)\,dx},$$

since L_0 is self-adjoint. Finally, by expanding the square in the numerator and applying Fubini's theorem, it is easily seen that

$$\frac{1}{2}\frac{\int_\Omega\int_\Omega J(x-y)(u(x)-u(y))^2\,dy\,dx}{\int_\Omega u^2(x)\,dx} = 1 - \frac{\int_\Omega\int_\Omega J(x-y)u(x)u(y)\,dy\,dx}{\int_\Omega u^2(x)\,dx},$$

since J is even and $u_0 = 0$ in $\mathbb{R}^N \setminus \Omega$. Thus (2.3) follows. □

As an immediate consequence of the variational characterizations (2.1) and (2.2), we have an estimate for $\lambda_1(\Omega)$, which will be useful when dealing with the asymptotic behavior of $\lambda_1(\Omega)$ in large and small domains in Section 4.

Corollary 2.1 *For the principal eigenvalue $\lambda_1(\Omega)$ we have the estimates:*

$$\left(\frac{1}{|\Omega|}\int_\Omega A^2(x)\,dx\right)^{1/2} \leq 1-\lambda_1(\Omega) \leq \sup_{y\in\Omega}\left(\int_\Omega A(x)J(x-y)\,dx\right), \tag{2.4}$$

where $A(x) = \int_\Omega J(x-y)\,dy.$

Proof. Taking $u \equiv 1$ as test function in (2.1), we obtain

$$1-\lambda_1(\Omega) \geq \left(\frac{1}{|\Omega|}\int_\Omega A^2(x)\,dx\right)^{1/2}. \tag{2.5}$$

On the other hand, thanks to Cauchy-Schwartz inequality, we have:

$$\int_\Omega \left(\int_\Omega J(x-y)u(y)\,dy \right)^2 dx \leq \int_\Omega A(x) \left(\int_\Omega J(x-y)u^2(y)\,dy \right) dx.$$

Using Fubini's theorem, we get

$$\begin{aligned} \int_\Omega \left(\int_\Omega J(x-y)u(y)\,dy \right)^2 dx &\leq \int_\Omega u^2(y) \left(\int_\Omega A(x)J(x-y)dx \right) dy \\ &\leq \sup_{y\in\Omega} \left(\int_\Omega A(x)J(x-y)dx \right) \int_\Omega u^2(y)\,dy, \end{aligned}$$

and this implies that

$$1 - \lambda_1(\Omega_0) \leq \sup_{y\in\Omega} \left(\int_\Omega A(x)J(x-y)dx \right). \tag{2.6}$$

Finally, (2.4) follows from (2.5) and (2.6). This concludes the proof of the corollary. □

Remark 2.1 (a) We observe that since $\lambda_1(\Omega) = 1 - \|L_0\| > 0$, it follows that the norm of the operator $\|L_0\|$ (considered in $L^2(\Omega)$) verifies $\|L_0\| < 1$.

(b) The estimates (2.4) obtained in Corollary 2.1 are not sharp: if the domain Ω contains a ball of radius 2, say, then the right-hand side in (2.4) equals one, so the estimate is useless. However, these estimates will be enough to deal with with small domains.

We end this section analyzing the monotonicity of the principal eigenvalue $\lambda_1(\Omega)$ with respect to the domain.

Proof.[Proof of Theorem 1.1] We notice that $L^2(\Omega_1) \subset L^2(\Omega_2)$, provided we extend all functions of the first space by zero outside Ω_1. Hence we have, thanks to the characterization (2.1), that $\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2)$. To show that the inequality is strict, we notice that if $\lambda_1(\Omega_1) = \lambda_1(\Omega_2)$, then we obtain an associated eigenfunction which is positive in $\overline{\Omega}_1$, but zero in $\Omega_2 \setminus \overline{\Omega}_1$, which contradicts the strong maximum principle. □

3 Continuity and differentiability of the principal eigenvalue

In this section we prove that the principal eigenvalue of a domain varies smoothly with respect to smooth perturbations. To his end, we always assume that $\Omega_\delta = \Psi(\delta, \Omega)$ is a perturbation of Ω such that the function Ψ is of the form (1.5), where $\Phi(0, \cdot) = 0$. First, we show that $\lambda_1(\Omega_\delta)$ varies continuously with δ.

Proof.[Proof of Theorem 1.2] We first notice that for small δ we can always assume $\Omega_1 \subset \Omega_\delta \subset \Omega_2$ for some smooth domains Ω_1 and Ω_2 not depending on δ. Thanks to Theorem 1.1 this implies

$$0 < \lambda_1(\Omega_2) < \lambda_1(\Omega_\delta) < \lambda_1(\Omega_1) < 1. \tag{3.1}$$

Now let u_δ be a positive eigenfunction associated to $\lambda_1(\Omega_\delta)$:

$$\int_{\Omega_\delta} J(x-y) u_\delta(x)\, dx = (1 - \lambda_1(\Omega_\delta)) u_\delta(x), \quad x \in \Omega_\delta.$$

We make the change of variables $x = z + \Phi(\delta, z)$, $y = w + \Phi(\delta, w)$ with $x, w \in \Omega$ to obtain

$$\int_\Omega J(z - w + \Phi(\delta, z) - \Phi(\delta, w)) v_\delta(w) \Delta(\delta, w) dw = (1 - \lambda_1(\Omega_\delta)) v_\delta(z), \tag{3.2}$$

for $z \in \Omega$, where $v_\delta(w) = u_\delta(w + \Phi(\delta, w))$ and $\Delta(\delta, w) = \det(I + D\Phi(\delta, w))$. We select v_δ with the normalization $|v_\delta|_{L^2(\Omega)} = 1$. Then, for every sequence $\delta_n \to 0$, we have a subsequence – still denoted by δ_n – such that $v_{\delta_n} \rightharpoonup v$ weakly in $L^2(\Omega)$. Since

$$J(z - w + \Phi(\delta_n, z) - \Phi(\delta_n, w))\ \Delta(\delta_n, w) \to J(z - w)$$

uniformly in $z, w \in \Omega$, we obtain thanks to weak convergence

$$\begin{aligned} &\int_\Omega J(z - w + \Phi(\delta_n, z) - \Phi(\delta_n, w)) v_{\delta_n}(w)\ \Delta(\delta, w)\, dw \\ &\qquad \to \int_\Omega J(z - w) v(w)\, dw \end{aligned} \tag{3.3}$$

for almost every $w \in \Omega$. Using the dominated convergence theorem, we also have the convergence in (3.3) in $L^2(\Omega)$.

On the other hand, since $\lambda_1(\Omega_{\delta_n})$ is bounded, we may pass to a further subsequence to have $\lambda_1(\Omega_{\delta_n}) \to \mu$, where $0 < \mu < 1$, thanks to (3.1). Then, setting $\delta = \delta_n$ in (3.2) and passing to the limit we have that the convergence of v_{δ_n} to v_0 is strong in $L^2(\Omega)$. Therefore, $|v_0|_{L^2(\Omega)} = 1$. By (3.2) we finally have

$$\int_\Omega J(x-y)v_0(y)\,dy = (1-\mu)v_0(x), \quad x \in \Omega,$$

with $v_0 \geq 0$, $v_0 \not\equiv 0$. According to Theorem 2.1, we obtain that $\mu = \lambda_1(\Omega)$, that is, $\lambda_1(\Omega_{\delta_n}) \to \lambda_1(\Omega)$. Since δ_n was arbitrary, this shows that $\lambda_1(\Omega_\delta) \to \lambda_1(\Omega)$ as $\delta \to 0$, as we wanted to prove. □

Next, we prove differentiability of $\lambda_1(\Omega_\delta)$ under the additional hypotheses that Φ is C^1 in both variables. Our proof is based on estimates of the incremental quotients for $\lambda_1(\Omega_\delta)$, inspired by [25].

Proof.[Proof of Theorem 1.3] We use the variational characterization (2.2) to estimate the incremental quotients of $\lambda_1(\Omega_\delta)$. For simplicity, let us write $\mu(\delta) = 1 - \lambda_1(\Omega_\delta)$. If we denote

$$H_\delta(u) = \frac{\displaystyle\int_{\Omega_\delta}\int_{\Omega_\delta} J(x-y)u(x)u(y)\,dx\,dy}{\displaystyle\int_{\Omega_\delta} u^2(x)\,dx},$$

we have, thanks to (2.2), that

$$\frac{\mu(\delta)-\mu(0)}{\delta} \geq \frac{H_\delta(u_0)-\mu(0)}{\delta} \tag{3.4}$$

for $\delta > 0$ (recall that $u_0 = 0$ outside Ω). Now, we perform the change of variables $x = z + \Phi(\delta, z)$, $y = w + \Phi(\delta, w)$ in the integrals in H_δ and we obtain

$$\begin{aligned} H_\delta(u_0) &= \frac{\displaystyle\int_{\Omega_\delta}\int_{\Omega_\delta} J(x-y)u_0(x)u_0(y)\,dx\,dy}{\displaystyle\int_{\Omega_\delta} u_0^2(x)\,dx} \\ &= \int_\Omega\int_\Omega J(z-w+\Phi(\delta,z)-\Phi(\delta,w))u_0(z+\Phi(\delta,z))u_0(w+\Phi(\delta,w))\times \\ &\qquad \times \Delta(z)\Delta(w)\,dzdw \Big/ \int_\Omega u_0^2(z+\Phi(\delta,z))\Delta(z)\,dz \end{aligned} \tag{3.5}$$

where $\Delta(z) = \det(I + D\Phi(\delta, z))$ and D stands for differentiation with respect to the second variable. By our regularity assumptions we have that

$$
\begin{aligned}
&J(z-w+\Phi(\delta,z)-\Phi(\delta,w)) \times \\
&\qquad \times u_0(z+\Phi(\delta,z))\; u_0(w+\Phi(\delta,w))\; \Delta(z)\Delta(w) \\
&\qquad = J(z-w)u_0(z)u_0(w) + K(z,w)\delta + o(\delta),
\end{aligned}
\tag{3.6}
$$

where

$$
\begin{aligned}
K(z,w) &= \langle \nabla J(z-w), \Phi'(0,z)-\Phi'(0,w)\rangle\, u_0(z)u_0(w) \\
&+ J(z-w)u_0(w)\,\langle \nabla u_0(z), \Phi'(0,z)\rangle + J(z-w)u_0(z)\,\langle \nabla u_0(w), \Phi'(0,w)\rangle \\
&+ J(z-w)u_0(z)u_0(w)\ \mathrm{div}(\Phi'(0,z)) + J(z-w)u_0(z)u_0(w)\ \mathrm{div}(\Phi'(0,w)),
\end{aligned}
$$

and $'$ stands for differentiation with respect to δ. Integrating (3.6) with respect to z and w in Ω, we get,

$$
\begin{aligned}
&\int_\Omega\int_\Omega J(z-w+\Phi(\delta,z)-\Phi(\delta,w))u_0(z+\Phi(\delta,z))u_0(w+\Phi(\delta,w))\times \\
&\qquad \Delta(z)\Delta(w)\, dz\, dw \\
&\qquad = \int_\Omega\int_\Omega J(z-w)u_0(z)u_0(w)\, dz\, dw + \delta \int_\Omega\int_\Omega K(z,w)\, dz\, dw + o(\delta) \\
&\qquad = \mu(0) + \delta \int_\Omega\int_\Omega K(z,w)\, dz\, dw + o(\delta).
\end{aligned}
\tag{3.7}
$$

Taking into account that J is even – and hence ∇J is odd – and using Fubini's theorem we have that

$$
\begin{aligned}
\int_\Omega\int_\Omega K(z,w)\, dz\, dw &= 2\int_\Omega\int_\Omega \langle \nabla J(z-w), \Phi'(0,z)\rangle\, u_0(z)u_0(w)\, dz\, dw \\
&+ 2\int_\Omega\int_\Omega J(z-w)u_0(w)\,\langle \nabla u_0(z), \Phi'(0,z)\rangle\ dz\, dw \\
&+ 2\int_\Omega\int_\Omega J(z-w)u_0(z)u_0(w)\ \mathrm{div}(\Phi'(0,z))\, dz\, dw.
\end{aligned}
$$

Integrating by parts in the last integral, we arrive at

$$
\int_\Omega\int_\Omega K(z,w)\, dz\, dw = 2\int_\Omega\int_{\partial\Omega} J(z-w)u_0(z)u_0(w)\ \langle \Phi'(0,z), \nu(z)\rangle\ dS(z)\, dw.
$$

Noticing that u_0 is an eigenfunction, this expression can be further transformed into:

$$\int_{\Omega}\int_{\Omega} K(z,w)\,dz\,dw = 2\mu(0)\int_{\partial\Omega} u_0^2(z)\ \langle \Phi'(0,z),\nu(z)\rangle\ dS(z)\,dw.$$

Hence, from (3.7) we obtain

$$\begin{aligned}\int_{\Omega}\int_{\Omega} & J(z-w+\Phi(\delta,z)-\Phi(\delta,w))u_0(z+\Phi(\delta,z))u_0(w+\Phi(\delta,w))\times \\ & \times\, \Delta(z)\Delta(w)\,dz\,dw \qquad (3.8)\\ & = \mu(0)+2\mu(0)\delta\int_{\partial\Omega} u_0^2(z)\ \langle \Phi'(0,z),\nu(z)\rangle\ dS(z)+o(\delta).\end{aligned}$$

On the other hand, with a similar procedure, we obtain:

$$\int_{\Omega} u_0^2(z+\Phi(\delta,z))\Delta(z)dz = 1+\delta\int_{\partial\Omega} u_0^2(z)\,\langle \Phi'(0,z),\nu(z)\rangle\,dS(z)+o(\delta). \qquad (3.9)$$

Taking into account (3.8) and (3.9), we obtain from (3.5):

$$H_\delta(u_0) = \mu(0)+\mu(0)\delta\int_{\partial\Omega} u_0^2(z)\,\langle \Phi'(0,z),\nu(z)\rangle\,dS(z)+o(\delta).$$

Hence (3.4) gives:

$$\frac{\mu(\delta)-\mu(0)}{\delta} \geq \mu(0)\int_{\partial\Omega} u_0^2(z)\,\langle \Phi'(0,z),\nu(z)\rangle\,dS(z)+o(1),$$

and thus

$$\liminf_{\delta\to 0+}\frac{\mu(\delta)-\mu(0)}{\delta} \geq \mu(0)\int_{\partial\Omega} u_0^2(z)\,\langle \Phi'(0,z),\nu(z)\rangle\,dS(z).$$

The remaining limits, $\limsup_{\delta\to 0+}$, $\liminf_{\delta\to 0-}$ and $\limsup_{\delta\to 0-}$ of the incremental quotients $\frac{\mu(\delta)-\mu(0)}{\delta}$ can be proved with similar calculations (we only remark that for the upper estimate the continuity of u_δ is needed), and therefore we finally conclude that

$$\lim_{\delta\to 0}\frac{\mu(\delta)-\mu(0)}{\delta} = \mu(0)\int_{\partial\Omega} u_0^2(z)\,\langle \Phi'(0,z),\nu(z)\rangle\,dS(z).$$

This proves (1.6), and concludes the proof of the theorem. □

4 Asymptotic behavior in large and small domains

In this last section we determine the behavior of the principal eigenvalue $\lambda_1(\Omega)$ when the domain Ω goes to zero or to infinity. We first prove the preliminary result in this direction contained in Theorem 1.4.

Proof.[Proof of Theorem 1.4] We make use of Corollary 2.1. First, notice that if $|\Omega| \to 0$, the integral in the second inequality in (2.4) goes to zero, and thus $\lambda_1(\Omega) \to 1$.

To prove that $\lambda_1(\Omega_n) \to 0$ when $\Omega_n \to \mathbb{R}^N$, we first show that $\lambda_1(B_R) \to 0$ when $R \to \infty$, where B_R is the ball centered at the origin with radius R. According to (2.4) we have

$$\lambda_1(B_R) \le 1 - \left(\frac{1}{|B_R|} \int_{B_R} \left(\int_{B_R} J(x-y)\, dy \right)^2 dx \right)^{1/2},$$

hence we need to prove

$$\frac{1}{|B_R|} \int_{B_R} \left(\int_{B_R} J(x-y)\, dy \right)^2 dx \to 1 \tag{4.1}$$

as $R \to \infty$. We set in the inner integral $y = x - z$, and then $x = Rw$, and arrive at

$$\frac{1}{|B_R|} \int_{B_R} \left(\int_{B_R} J(x-y)\, dy \right)^2 dx = \frac{1}{|B_1|} \int_{B_1} \left(\int_{|z-Rw|<R} J(z)\, dz \right)^2 dw.$$

Now observe that for fixed w with $|w| < 1$ it holds

$$\int_{|z-Rw|<R} J(z)\, dz \to \int_{\mathbb{R}^N} J(z)\, dz = 1,$$

as $R \to \infty$, and (4.1) follows thanks to the dominated convergence theorem.

Finally, let us show that $\lambda_1(\Omega_n) \to 0$ as $\Omega_n \to \mathbb{R}^N$. We can assume $0 \in \Omega_n$ and that there exists balls B_{R_n} such that $B_{R_n} \subset \Omega_n$ with $R_n \to \infty$, and hence $\lambda_1(\Omega_n) < \lambda_1(B_{R_n})$. It follows that

$$\limsup_{n\to\infty} \lambda_1(\Omega_n) \le \lim_{n\to\infty} \lambda_1(B_{R_n}) = 0,$$

which concludes the proof. □

We finally determine the asymptotic behavior of the principal eigenvalue $\lambda_1(\Omega_\gamma)$, when we consider dilatations $\Omega_\gamma = \gamma\Omega$ of a fixed domain Ω. Our next theorem makes more precise the information given by Theorem 1.4.

Proof.[Proof of Theorem 1.5] We first prove (1.8). Let u_γ be an arbitrary positive eigenfunction associated to $\lambda_1(\Omega_\gamma)$. Choose an arbitrary $\varepsilon > 0$. Now, for γ small enough we have

$$J(x-y) \leq J(0) + \varepsilon$$

if $x, y \in \Omega_\gamma$. Then

$$\begin{aligned}(1-\lambda_1(\Omega_\gamma)) \int_{\Omega_\gamma} u_\gamma(x)\, dx &= \int_{\Omega_\gamma}\int_{\Omega_\gamma} J(x-y) u_\gamma(y)\, dy\, dx \\ &\leq (J(0)+\varepsilon) \int_{\Omega_\gamma}\int_{\Omega_\gamma} u_\gamma(y)\, dy\, dx = (J(0)+\varepsilon)|\Omega|\gamma^N \int_{\Omega_\gamma} u_\gamma(y)\, dy.\end{aligned}$$

It follows that

$$\limsup_{\gamma\to 0+} \frac{1-\lambda_1(\Omega_\gamma)}{\gamma^N} \leq J(0)|\Omega|.$$

The reverse inequality for the liminf can be proved in an analogous way. This completes the proof of (1.8).

Let us prove now (1.9), which is much more involved. The first step is to show that $\lambda_1(\Omega_\gamma) \leq C\gamma^{-2}$ for a certain positive constant. Indeed, we will show the more precise estimate,

$$\limsup_{\gamma\to+\infty} \gamma^2 \lambda_1(\Omega_\gamma) \leq \sigma_1(\Omega) A(J). \tag{4.2}$$

Let ϕ be the positive eigenfunction of the Laplacian in Ω, normalized by $\int_\Omega \phi^2(x)\, dx = 1$ and extended by zero outside Ω. Taking as a test function $\phi_\gamma(x) = \phi(x/\gamma)$ in the variational characterization (2.3), we obtain

$$\lambda_1(\Omega_\gamma) \leq \frac{1}{2} \frac{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\left(\phi\left(\frac{x}{\gamma}\right) - \phi\left(\frac{y}{\gamma}\right)\right)^2 dy\, dx}{\displaystyle\int_{\Omega_\gamma} \phi\left(\frac{x}{\gamma}\right)^2 dx}.$$

Setting $x = y + z$ and $y = \gamma w$ in the integrals of the numerator, and $x = \gamma\theta$

in the integral of the denominator, we obtain

$$\begin{aligned}\lambda_1(\Omega_\gamma) &\leq \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(z)\left(\phi\left(w+\frac{z}{\gamma}\right)-\phi(w)\right)^2 dw\, dz\\ &= \frac{1}{2}\int_{B_1}\int_{\mathbb{R}^N} J(z)\left(\phi\left(w+\frac{z}{\gamma}\right)-\phi(w)\right)^2 dw\, dz.\end{aligned}$$

Taking into account that the function ϕ belongs to $W^{1,\infty}(\mathbb{R}^N)$, we have

$$\phi\left(w+\frac{z}{\gamma}\right)-\phi(w)=\frac{1}{\gamma}\int_0^1\left\langle\nabla\phi\left(w+s\frac{z}{\gamma}\right),z\right\rangle ds$$

for every $w\in\mathbb{R}^N$, $z\in B_1$. Hence,

$$\gamma^2\lambda_1(\Omega_\gamma)\leq\frac{1}{2}\int_{B_1}\int_{\mathbb{R}^N} J(z)\left(\int_0^1\left\langle\nabla\phi\left(w+s\frac{z}{\gamma}\right),z\right\rangle ds\right)^2 dw\, dz. \tag{4.3}$$

Thanks to dominated convergence theorem, we can pass to the limit in (4.3) as $\gamma\to+\infty$ to obtain,

$$\limsup_{\gamma\to+\infty}\gamma^2\lambda_1(\Omega_\gamma)\leq\frac{1}{2}\int_{B_1}\int_{\mathbb{R}^N} J(z)\left\langle\nabla\phi(w),z\right\rangle^2 dw\, dz. \tag{4.4}$$

In the last integral, we apply Fubini's theorem to obtain

$$\begin{aligned}\int_{B_1}\int_{\mathbb{R}^N} J(z)\left\langle\nabla\phi(w),z\right\rangle^2 dw\, dz &= \int_{\mathbb{R}^N}\int_{B_1} J(z)\left\langle\nabla\phi(w),z\right\rangle^2 dz\, dw\\ &= \sum_{i,j=1}^N\int_{\mathbb{R}^N}\frac{\partial\phi}{\partial x_i}(w)\frac{\partial\phi}{\partial x_j}(w)\left(\int_{B_1} J(z)z_iz_j\, dz\right) dw.\end{aligned}$$

We notice that the integrals $\int_{B_1} J(z)z_iz_j\, dz$ vanish by symmetry when $i\neq j$, while they are all equal to $2A(J)$ when $i=j$. Thus (4.4) implies (4.2).

Now let φ_γ be a positive eigenfunction associated to $\lambda_1(\Omega_\gamma)$, and set $\psi_\gamma(x)=\varphi_\gamma(\gamma x)$, $x\in\Omega$. We normalize ψ_γ by $\int_\Omega\psi_\gamma^2(x)\, dx=1$. According to the variational characterization (2.3), we have

$$2\lambda_1(\Omega_\gamma)=\int_{\widetilde{\Omega}}\int_{\widetilde{\Omega}} J_\gamma(x-y)(\psi_\gamma(x)-\psi_\gamma(y))^2dxdy,$$

where $J_\gamma(x)=\gamma^N J(\gamma x)$, and $\widetilde{\Omega}$ is a smooth bounded domain such that $\Omega\subset\subset\widetilde{\Omega}$.

Now let $\gamma_n \to +\infty$ be an arbitrary sequence. By passing to a subsequence, we may assume $\psi_n := \psi_{\gamma_n}$ converges weakly in $L^2(\widetilde{\Omega})$ to a function ψ. Since J is radially decreasing and $\lambda_1(\Omega_{\gamma_n}) \le C\gamma_n^{-2}$, thanks to (4.2), we may apply Proposition 3.2 of [2], which implies that $\psi_n \to \psi$ strongly in $L^2(\widetilde{\Omega})$ with $\psi \in H^1(\widetilde{\Omega})$. Since $\psi = 0$ in $\widetilde{\Omega} \setminus \Omega$, we obtain

$$\psi \in H_0^1(\Omega) \quad \text{and} \quad \int_\Omega \psi^2(x)\,dx = 1. \tag{4.5}$$

We claim that ψ is the principal eigenfunction of a multiple of the Laplacian in Ω with Dirichlet boundary conditions, and this will imply $\lim_{n\to\infty} \gamma_n^2 \lambda_1(\Omega_{\gamma_n}) = A(J)\sigma_1(\Omega)$. Indeed, thanks to (4.2), we may assume that $\gamma_n^2\lambda_1(\Omega_{\gamma_n}) \to \lambda_0 \ge 0$. We notice that ψ_n satisfies

$$J_{\gamma_n} * \psi_n - \psi_n = -\lambda_1(\Omega_{\gamma_n})\psi_n. \tag{4.6}$$

Choose an arbitrary function $v \in C_0^\infty(\Omega)$. Multiply (4.6) by v and integrate in Ω to obtain

$$\begin{aligned}\gamma^N \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(\gamma(x-y))\psi_n(y)v(x)\,dy\,dx - \int_{\mathbb{R}^N} \psi_n(x)v(x)\,dx \\ = -\lambda_1(\Omega_{\gamma_n}) \int_{\mathbb{R}^N} \psi_n(x)v(x)\,dx.\end{aligned} \tag{4.7}$$

Note that all the integrals in what follows may be considered in $\mathbb{R}^N$, since v and ψ_n vanish outside Ω. Thanks to Fubini's theorem, the integrals in the left-hand side of (4.7) can be rewritten to have

$$\begin{aligned}\gamma_n^N \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(\gamma_n(x-y))(v(y)-v(x))\psi_n(x)\,dx\,dy \\ = -\lambda_1(\Omega_{\gamma_n}) \int_{\mathbb{R}^N} \psi_n(x)v(x)\,dx,\end{aligned} \tag{4.8}$$

since J has unit integral. Letting $z = -\gamma_n(x-y)$ in the first integral of (4.8), we get

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(z)\left(v\left(x+\frac{z}{\gamma_n}\right) - v(x)\right)\psi_n(x)\,dx\,dz \\ = -\lambda_1(\Omega_{\gamma_n}) \int_{\mathbb{R}^N} \psi_n(x)v(x)\,dx.\end{aligned} \tag{4.9}$$

We now use Taylor expansion up to the second order in v:

$$v\left(x+\frac{z}{\gamma_n}\right)-v(x)=\frac{1}{\gamma_n}\sum_{i=1}^{N}\frac{\partial v}{\partial x_i}(x)z_i+\frac{1}{\gamma_n^2}\sum_{i,j=1}^{N}\int_0^1(1-s)\frac{\partial^2 v}{\partial x_i\partial x_j}\left(x+\frac{sz}{\gamma_n}\right)z_iz_jds,$$

which, when plugged into (4.9), gives

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}J(z)\left(\gamma_n\sum_{i=1}^{N}\frac{\partial v}{\partial x_i}(x)z_i\right.$$
$$\left.+\sum_{i,j=1}^{N}\int_0^1(1-s)\frac{\partial^2 v}{\partial x_i\partial x_j}\left(x+\frac{sz}{\gamma_n}\right)z_iz_jds\right)\psi_n(x)\,dxdz$$
$$=-\gamma_n^2\lambda_1(\Omega_{\gamma_n})\int\psi_n(x)v(x)dx.$$

Next we analyze the integrals involving the first derivatives of v. Notice that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}J(z)\frac{\partial v}{\partial x_i}(x)z_i\psi_n(x)\,dx\,dz=\int_{\mathbb{R}^N}\frac{\partial v}{\partial x_i}(x)\psi_n(x)\left(\int_{\mathbb{R}^N}J(z)z_idz\right)dx=0$$

by the symmetry of J. Hence:

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}J(z)\left(\sum_{i,j=1}^{N}\int_0^1(1-s)\frac{\partial^2 v}{\partial x_i\partial x_j}\left(x+\frac{sz}{\gamma_n}\right)z_iz_j\,ds\right)\psi_n(x)\,dx\,dz\\&\quad=-\gamma_n^2\lambda_1(\Omega_{\gamma_n})\int_{\mathbb{R}^N}\psi_n(x)v(x)\,dx.\end{aligned}\tag{4.10}$$

Now we pass to the limit as $n\to\infty$ in (4.10). Notice that

$$\frac{\partial^2 v}{\partial x_i\partial x_j}\left(x+\frac{sz}{\gamma_n}\right)\to\frac{\partial^2 v}{\partial x_i\partial x_j}(x)$$

uniformly for $x\in\Omega$, $z\in B_1$, and hence the first term in (4.10) converges to

$$\frac{1}{2}\int_{\mathbb{R}^N}\sum_{i,j=1}^{N}\frac{\partial^2 v}{\partial x_i\partial x_j}(x)\psi(x)\left(\int_{\mathbb{R}^N}J(z)z_iz_jdz\right)dx=A(J)\Delta v(x)\psi(x).$$

Thus

$$A(J)\int_{\mathbb{R}^N}\Delta v(x)\psi(x)\,dx=-\lambda_0\int_{\mathbb{R}^N}\psi(x)v(x)\,dx.\tag{4.11}$$

According to (4.5), we may integrate by parts in the integral of the left-hand side in (4.11) to obtain

$$A(J) \int_{\mathbb{R}^N} \nabla v(x) \nabla \psi(x) dx = \lambda_0 \int_{\mathbb{R}^N} \psi(x) v(x)\, dx.$$

Since $v \in C_0^\infty(\Omega)$ is arbitrary, and $\psi \in H_0^1(\Omega)$ with $\psi \not\equiv 0$, we have that ψ is a positive eigenfunction associated to $-\Delta$ in Ω. Thus $\lambda_0 = A(J)\sigma_1(\Omega)$, and since the sequence γ_n was arbitrary, the theorem is proved. □

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