LARGE SOLUTIONS FOR AN ELLIPTIC SYSTEM OF QUASILINEAR EQUATIONS

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ABSTRACT. In this paper we consider the quasilinear elliptic system $\Delta_p u = u^a v^b$, $\Delta_p v = u^c v^e$ in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, with the boundary conditions $u = v = +\infty$ on $\partial\Omega$. The operator Δ_p stands for the p-Laplacian defined by $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, and the exponents verify $a, e > p - 1$, $b, c > 0$ and $(a - p + 1)(e - p + 1) \geq bc$. We analyze positive solutions in both components, providing necessary and sufficient conditions for existence. We also prove uniqueness of positive solutions in the case $(a - p + 1)(e - p + 1) > bc$ and obtain the exact blow-up rate near the boundary of the solution. In the case $(a - p + 1)(e - p + 1) = bc$, infinitely many positive solutions are constructed.

1. INTRODUCTION

This paper is concerned with the study of positive boundary blow-up solutions to a quasilinear elliptic system of competitive type:

$$\left\{ \begin{array}{ll} \Delta_p u = u^a v^b & \text{in } \Omega \\ \Delta_p v = u^c v^e & \text{in } \Omega \\ u = v = +\infty & \text{on } \partial\Omega, \end{array} \right. \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$ and Δ_p stands for the p-Laplacian operator defined by $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$. The exponents a, b, c, e verify $a, e > p-1$, $b, c > 0$. The boundary condition is assumed in the sense $u(x), v(x) \to +\infty$ when $d(x) \to 0+$, where $d(x)$ stands for the distance function $\mathrm{dist}(x, \partial\Omega)$.

Our motivation comes from [25], where problem (1.1) was analyzed in the semilinear case $p = 2$. Actually, three different sets of boundary conditions were considered there:

$$\left\{ \begin{array}{l} u = \lambda \\ v = \mu \end{array} \right. \quad \text{on } \partial\Omega, \tag{F}$$

where $\lambda, \mu > 0$,

$$\left\{ \begin{array}{l} u = +\infty \\ v = \mu \end{array} \right. \quad \text{on } \partial\Omega, \tag{SF}$$

for $\mu > 0$ and

$$\left\{ \begin{array}{l} u = +\infty \\ v = +\infty \end{array} \right. \quad \text{on } \partial\Omega. \tag{I}$$

Under the assumption that $(a-1)(e-1) \geq bc$, necessary and sufficient conditions for existence of positive solutions were found, and uniqueness or multiplicity were also obtained, together with the exact boundary behavior of solutions. In the subsequent paper [20], the same system but with singular weights was considered, where a different proof of uniqueness was the main point.

Our purpose in the present work is to obtain similar results as those in [25] for the quasilinear system (1.1). Thus we will be interested in the so-called "subcritical" and "critical" cases, given by

$$(a-p+1)(e-p+1) > bc$$

or

$$(a-p+1)(e-p+1) = bc$$

respectively. We will focus our attention on nonnegative weak solutions, that is, pairs of functions $(u,v) \in W^{1,p}_{\rm loc}(\Omega)$ verifying (1.1) in the weak sense with $u,v \geq 0$. However, let us observe that, according to standard regularity for the p-Laplacian, weak solutions verify $u,v \in C^{1,\eta}_{\rm loc}(\Omega)$ for some $\eta \in (0,1)$ (cf. [13], [34], [46]), and since $a,e > p-1$, the strong maximum principle in [47] then implies $u,v>0$ in Ω.

We will obtain necessary and sufficient conditions for existence of positive solutions to (1.1) and will also prove that positive solutions are unique in the subcritical case, while infinitely many positive solutions exist in the critical case. In addition we obtain the exact divergence rate of the solutions near $\partial\Omega$. For simplicity, we are only considering the system provided with the boundary conditions (I).

We would like to quote some references in which systems of boundary blow-up solutions related to (1.1) were analyzed. Lotka-Volterra type systems were considered in [14], [15], [37], [33] (competitive type), [9] (predator-prey type) and [27] (cooperative type), while in [24] the objective was a competitive system not of Lotka-Volterra type, which is somehow connected to the supercritical case $(a-1)(e-1) < bc$ in (1.1) with $p=2$.

With regard to boundary blow-up problems with a single equation, we finally quote the papers [1], [2], [3], [4], [5], [6], [7], [10], [18], [19], [23], [26], [30], [31], [35], [36], [38], [39], [44], [48], [49] for semilinear problems, and [11], [16], [17], [21], [22], [28], [29], [40], [41], [42], [43] for problems with the p-Laplacian.

We are stating next our results. We begin with the subcritical case $(a-p+1)(e-p+1) > bc$, which is somehow closer to a single equation.

Theorem 1. *Assume $a,e > p-1$, $b,c>0$ verify $(a-p+1)(e-p+1) > bc$. Then problem* (1.1) *admits a positive solution if and only if $c < a-p+1$, $b < e-p+1$. This solution is moreover unique, and it* *verifies*

$$u(x) \sim Ad(x)^{-\alpha}, \quad v(x) \sim Bd(x)^{-\beta}, \tag{1.2}$$

as $d(x) \to 0+$, *where*

$$\alpha = \frac{p(e-p+1-b)}{(a-p+1)(e-p+1)-bc}\ ,\ \beta = \frac{p(a-p+1-c)}{(a-p+1)(e-p+1)-bc}$$

and

$$A = \left(\frac{((p-1)\alpha^{p-1}(\alpha+1))^{e-p+1}}{((p-1)\beta^{p-1}(\beta+1))^{b}} \right)^{\frac{1}{(a-p+1)(e-p+1)-bc}}$$

$$B = \left(\frac{((p-1)\beta^{p-1}(\beta+1))^{a-p+1}}{((p-1)\alpha^{p-1}(\alpha+1))^{c}} \right)^{\frac{1}{(a-p+1)(e-p+1)-bc}}.$$

Remark 1. The arguments in Section 4 show indeed that we can obtain the exact behavior of the normal derivatives of the solutions:

$$\frac{\partial u}{\partial \nu}(x) \sim \alpha A d(x)^{-\alpha}, \quad \frac{\partial v}{\partial \nu}(x) \sim \beta B d(x)^{-\beta},$$

as $x \to x_0$, where $\nu = \nu(x_0)$, and $x_0 \in \partial\Omega$ is arbitrary. However, we are not pursuing this further.

We now turn our attention to the critical case $(a-p+1)(e-p+1) = bc$. In contrast with the subcritical case, the solutions are not unique, and existence is much easier to obtain.

Theorem 2. *Assume* $a, e > p-1$, $b, c > 0$ *with* $(a-p+1)(e-p+1) = bc$. *Then problem* (1.1) *admits a positive solution if and only if* $a-p+1 = c$, $e-p+1 = b$. *In that case, if* (u, v) *is a positive solution, then* $(\lambda^{\frac{b}{b+c}} u, \lambda^{-\frac{c}{b+c}} v)$ *is also a solution for every* $\lambda > 0$. *Hence, there are infinitely many positive solutions.*

In view of the multiplicity result in Theorem 2, a natural question to ask is: are all solutions to (1.1) of the form $(\lambda^{\frac{b}{b+c}} u, \lambda^{-\frac{c}{b+c}} v)$ for a fixed (u, v)? We expect the answer to be affirmative, but this is a difficult task even for the semilinear case $p = 2$, where it remains open. However, this turns out to be true if the domain is a ball B and solutions are radially symmetric.

Theorem 3. *Assume* Ω *is a ball* B *and* $a, e > p-1$, $c = a-p+1$, $b = e-p+1$. *If* (u, v) *is a radial positive solution to* (1.1), *then there exists* $\lambda > 0$ *such that* $u = \lambda^{\frac{b}{b+c}} U$, $v = \lambda^{-\frac{c}{b+c}} U$, *where* U *is the unique positive solution to*

$$\begin{cases} \Delta_p U = U^{b+c+p-1} & \text{in } B \\ U = +\infty & \text{on } \partial B. \end{cases}$$

Thus we also have

$$u(x) \sim \lambda^{\frac{b}{b+c}}((p-1)\omega^{p-1}(\omega+1))^{\frac{1}{b+c}}d(x)^{-\omega} \tag{1.3}$$
$$v(x) \sim \lambda^{\frac{-c}{b+c}}((p-1)\omega^{p-1}(\omega+1))^{\frac{1}{b+c}}d(x)^{-\omega}$$

as $d(x) \to 0+$*, where* $\omega = p/(b+c)$*.*

Remark 2. Some slight generalizations of problem (1.1) are possible, and still we can get most of the results in Theorems 1 and 2. For instance, a system of $(p,q)-$Laplacians can be considered instead, that is

$$\begin{cases} \Delta_p u = u^a v^b & \text{in } \Omega \\ \Delta_q v = u^c v^e & \text{in } \Omega \\ u = v = +\infty & \text{on } \partial\Omega, \end{cases}$$

where $p, q > 1$ but $p \neq q$ is possible, and most of the proofs would remain almost unchanged. In these terms, the subcriticality condition is $(a-p+1)(e-q+1) > bc$.

Another possible generalization is to consider the system with positive continuous weights $a(x)$, $b(x)$, which can be even singular on $\partial\Omega$:

$$\begin{cases} \Delta_p u = a(x) u^a v^b & \text{in } \Omega \\ \Delta_p v = b(x) u^c v^e & \text{in } \Omega \\ u = v = +\infty & \text{on } \partial\Omega. \end{cases}$$

At least for the subcritical case almost everything rests unchanged for this system.

Let us comment on the proofs of our results. We remark that most of the proofs are an adaptation of the corresponding ones in [25], especially those concerning existence and boundary behavior of solutions. In particular, the iterative method used there to obtain rough estimates for solutions is still valid with the obvious modifications. There is however an exception: the proof of Lemma 5 in Section 2, which is needed for the iterative method to work, has to be drastically changed.

There are some other important differences, for instance regarding uniqueness results. The proof of uniqueness in Theorem 1 follows a similar idea as in [25], but it is technically different, mainly due to the lack of regularity of solutions. Another uniqueness theorem, which deals with problem (1.1) in a half-space, needs a completely different proof, because of the nonlinearity of the p-Laplacian. We believe these two results are completely new.

The paper is organized as follows: in Section 2 we consider some preliminaries on a single equation which are instrumental in our proofs, and in Section 3 the two uniqueness results for problems related to (1.1) will be proved. Sections 4 and 5 are dedicated to cover the subcritical and critical cases, respectively. Finally, some results related to the method of sub and supersolutions are collected in an Appendix.

2. Preliminaries

In this section we will establish some preliminary properties of positive solutions to a scalar equation related to the system (1.1). For $q > p-1$ and $\gamma \geq 0$, we consider the problem

$$\left\{ \begin{array}{ll} \Delta_p u = d^{-\gamma} u^q & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{array} \right. \tag{2.1}$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$. This problem has been recently considered in [21], where all issues concerning existence, uniqueness and asymptotic behavior near the boundary of positive solutions were obtained. We collect these results for later use and refer the reader to [21] for a proof.

Lemma 4. *Let $q > p-1$ and $\gamma \in [0, p)$. Then problem (2.1) admits a unique positive solution denoted by $U_{q,\gamma}$. Moreover:*

$$U_{q,\gamma}(x) \sim ((p-1)\alpha^{p-1}(\alpha+1))^{\frac{1}{q-p+1}} d(x)^{-\alpha}$$

as $d(x) \to 0$, where $\alpha = (p-\gamma)/(q-p+1)$.

As an immediate consequence of Theorem 4, it makes sense to define

$$A_{q,\gamma} = \sup_{\Omega} d(x)^{\alpha} U_{q,\gamma}(x), \quad B_{q,\gamma} = \inf_{\Omega} d(x)^{\alpha} U_{q,\gamma}(x), \tag{2.2}$$

and both quantities are finite and positive. For the purposes of estimates of solutions we need a further property of $A_{q,\gamma}$ and $B_{q,\gamma}$. We mention in passing that a similar property was obtained in [25] for $p = 2$, but the proof there is of no use here since solutions are not C^2. Thus we provide a different proof.

Lemma 5. *The quantities $A_{q,\gamma}$ and $B_{q,\gamma}$ are bounded and bounded away from zero when γ is bounded away from p. Moreover,*

$$\lim_{\gamma \to p-} A_{q,\gamma} = \lim_{\gamma \to p-} B_{q,\gamma} = 0.$$

Proof. We are proving that there exist positive constants C_1 and C_2, independent of γ such that

$$C_1 \alpha^{\frac{p-1}{q-p+1}} \leq B_{q,\gamma} \leq A_{q,\gamma} \leq C_2 \alpha^{\frac{p-1}{q-p+1}}, \tag{2.3}$$

and the theorem will follow, since α is bounded and bounded away from zero when γ is bounded away from p, while $\alpha \to 0$ as $\gamma \to p-$.

Fix $x \in \Omega$. We introduce the function $v(y) = d(x)^{\alpha} u(x + d(x)y)$ for $y \in B_1(0)$, where we are denoting $U_{q,\gamma}$ by u to simplify the notation. Then it is easily seen that

$$\Delta_p v = d(x)^{\gamma} d(x + d(x)y)^{-\gamma} v^q \geq 2^{-\gamma} v^q$$

in $B_1(0)$, since $d(x + d(x)y) \leq 2d(x)$ if $y \in B_1(0)$. We are next constructing a supersolution to the same equation which blows up on the boundary of $B_1(0)$. We claim that

$$w(y) = C\alpha^{\frac{p-1}{q-p+1}} (1 - |y|^{p'})^{-\alpha}$$

is a supersolution for large enough C, where $p' = p/(p-1)$. Indeed, a calculation shows that

$$\Delta_p w = C^{p-1}\alpha^{\frac{(p-1)^2}{q-p+1}}(\alpha p')^{p-1}(1-|y|^{p'})^{-\alpha q}(p(\alpha+1)|y|^{p'} + N(1-|y|^{p'})),$$

and thus w will be a supersolution provided that

$$C^{p-1}\alpha^{\frac{(p-1)^2}{q-p+1}}(\alpha p')^{p-1}(p(\alpha+1)|y|^{p'} + N(1-|y|^{p'})) \le 2^{-\gamma}C^q\alpha^{\frac{q(p-1)}{q-p+1}},$$

which is equivalent to

$$2^\gamma(p')^{p-1}(p(\alpha+1)|y|^{p'} + N(1-|y|^{p'})) \le C^{q-p+1}. \tag{2.4}$$

But (2.4) is certainly true if C is large enough (independent of γ) since $\alpha \le p/(q-p+1)$ and $2^\gamma \le 2^p$. Thus w is a supersolution and we obtain by comparison that $v \le w$ in $B_1(0)$. Setting $y = 0$ we arrive at

$$d(x)^\alpha u(x) \le C_2\alpha^{\frac{p-1}{q-p+1}}, \tag{2.5}$$

where C_2 does not depend on γ. This shows the rightmost inequality in (2.3).

To show the other inequality, we let $u = v^{-\beta}$, for some $\beta > 0$ to be chosen. Then v verifies:

$$-\Delta_p v + (\beta+1)(p-1)\frac{|\nabla v|^p}{v} = \frac{1}{\beta^{p-1}}d^{-\gamma}v^{(\beta+1)(p-1)-\beta q} \quad \text{in } \Omega,$$

with $v = 0$ on $\partial\Omega$, while (2.5) gives

$$v \ge C^{-\frac{1}{\beta}}\alpha^{-\frac{p-1}{(q-p+1)\beta}}d^{\frac{\alpha}{\beta}}.$$

If β is chosen to verify $\beta > (p-1)/(q-p+1)$, then

$$-\Delta_p v \le \frac{1}{\beta^{p-1}}\left(C\alpha^{-\frac{p-1}{q-p+1}}\right)^{\frac{p-1}{\beta}-(q-p+1)} d^{-p+\frac{\alpha}{\beta}(p-1)}. \tag{2.6}$$

We now restrict β further to verify $1 < p - \frac{\alpha}{\beta}(p-1) < p$, that is, $\beta > \alpha$. It suffices with setting

$$\beta = \frac{p}{p-1}\alpha$$

to verify both restrictions. Then (2.6) reads

$$-\Delta_p v \le CC^{\frac{(p-1)^2}{p\alpha}}\alpha^{-\frac{(p-1)^3}{(q-p+1)p\alpha}}d^{-\frac{2p-1}{p}}.$$

Now let ϕ be the unique solution to $-\Delta_p\phi = d^{\frac{2p-1}{p}}$ in Ω with $\phi = 0$ on $\partial\Omega$. According to Theorem 2 in [21], ϕ exists, is unique and verifies $\phi \le Cd^{\frac{p-1}{p}}$ for a positive constant C (notice that $1 < \frac{2p-1}{p} < p$). The comparison principle then implies

$$v \le C^{\frac{(p-1)}{p\alpha}}\alpha^{-\frac{(p-1)^2}{(q-p+1)p\alpha}}d^{\frac{p-1}{p}},$$

which in turn gives for u:

$$u \ge C\alpha^{\frac{p-1}{q-p+1}}d^{-\alpha}.$$

This shows the lower inequality in (2.3), and completes the proof of the lemma. □

To conclude this section we state a comparison theorem which will be of use when dealing with solutions of the system (1.1). Its proof follows thanks to a scaling and uniqueness of solutions to (2.1), and it will be omitted.

Lemma 6. *Let $u \in C^{1,\eta}_{\rm loc}(\Omega)$ for some $\eta \in (0,1)$ verify $\Delta_p u \geq C d^{-\gamma} u^q$ in Ω with $u = +\infty$ on $\partial\Omega$. Then $u \leq C^{-\frac{1}{q-p+1}} U_{q,\gamma}$ in Ω. Similarly, if $\Delta_p u \leq C d^{-\gamma} u^q$ in Ω and $u = +\infty$ on $\partial\Omega$, then $u \geq C^{-\frac{1}{q-p+1}} U_{q,\gamma}$ in Ω.*

3. Two uniqueness results

In this section we obtain two new uniqueness results for system (1.1) in two different situations. The first one is concerned with the system with finite boundary conditions which, at the best of our knowledge, has not been considered before.

Theorem 7. *Let (u_1, v_1), (u_2, v_2) be positive weak solutions to the system*

$$\begin{cases} \Delta_p u = u^a v^b & \text{in } \Omega, \\ \Delta_p v = u^c v^e & \text{in } \Omega, \\ u = f(x), \ v = g(x) & \text{on } \partial\Omega, \end{cases}$$

with $f > 0$, $g > 0$ on $\partial\Omega$, and assume $a > p-1$, $e > p-1$ and $(a-p+1)(e-p+1) > bc$. Then $u_1 = u_2$, $v_1 = v_2$.

To prove Theorem 7 we need a technical result. Recall that by standard calculus if u, v are C^2 functions in a domain Ω with $u \leq v$ and $x_0 \in \Omega$ is such that $u(x_0) = v(x_0)$ then $\Delta u(x_0) \leq \Delta v(x_0)$. This fact, very useful for equations with the Laplacian, is not at all straightforward when dealing with the p-Laplacian, $p \neq 2$, mainly due to the fact that solutions are not in general C^2. Even if we assume that $\nabla u(x_0) \neq 0$, this seems to be true only if $p > 2$. However, a slightly weaker result will suffice for our purposes.

Lemma 8. *Let $f, g \in C(\overline{\Omega})$ and $u, v \in C^{1,\eta}(\overline{\Omega})$ be weak solutions to $\Delta_p u = f$, $\Delta_p v = g$ in Ω with $u \leq v$ and $u = v$ at some point of Ω. Assume moreover that $u < v$ on $\partial\Omega$. Then there exists $x_0 \in \Omega$ such that $u(x_0) = v(x_0)$ and $f(x_0) \leq g(x_0)$.*

Proof. Let $A = \{x \in \Omega : \ u(x) = v(x)\}$. By our assumptions, A is nonempty and it is strictly contained in Ω. Assume for a contradiction that $f > g$ in A. Then we can choose an open neighborhood $\mathcal{U}$ of A such that $f > g$ in $\mathcal{U}$ and $u < v$ on $\partial\mathcal{U}$. Then for small $\varepsilon > 0$ we have $u + \varepsilon \leq v$ on $\partial\mathcal{U}$ together with $\Delta_p(u+\varepsilon) = f > g = \Delta_p v$ in $\mathcal{U}$. The comparison principle implies $u + \varepsilon \leq v$ in $\mathcal{U}$, which is clearly a contradiction since $A \subset \mathcal{U}$. Thus $f > g$ is not possible in A, and there exists $x_0 \in A$ with $f(x_0) \le g(x_0)$. This concludes the proof. □

Remark 3. Lemma 8 is also useful for obtaining an alternative proof of uniqueness of positive solutions to the problem

$$\left\{ \begin{array}{ll} -\Delta_p u = f(x,u) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{array} \right. \tag{3.1}$$

when f is a continuous nonlinearity with $f(x,u)/u^{p-1}$ decreasing in u for fixed x, $\liminf_{u\to 0+} f(x,u)/u^{p-1} \ge c > -\infty$ and $g > 0$. Indeed, if u, v are positive solutions, then $k = \sup u/v$ is finite. If we assume $k > 1$, Lemma 8 can be applied to give the existence of a point with $f(x_0, kv(x_0)) \ge k^{p-1} f(x_0, v(x_0))$, which is incompatible with the monotonicity of $f(x,u)/u^{p-1}$. Thus $k \le 1$ and $u \le v$. The reversed argument gives $u = v$. With a little more effort, the case $g = 0$ can also be covered (see [12] for existence and uniqueness of positive solutions to (3.1) in this case).

Proof of Theorem 7. Since $v_1, u_2 > 0$ in $\overline{\Omega}$, we can select a large k so that

$$u_1 \le k u_2, \quad v_1 \ge k^{-\frac{c}{e-p+1}} v_2 \quad \text{in } \Omega. \tag{3.2}$$

Choose the least k with this property, and assume $k > 1$. Then one of the two inequalities in (3.2) is not strict. Assume it is the second one. We can apply Lemma 8 to obtain a point $x_0 \in \Omega$ with $v_1(x_0) = k^{-\frac{c}{e-p+1}} v_2(x_0)$ and

$$u_1(x_0)^c v_1(x_0)^e \ge k^{-\frac{c(p-1)}{e-p+1}} u_2(x_0)^c v_2(x_0)^e,$$

which implies $u_1(x_0) = k u_2(x_0)$. That is, we can always assume that the first inequality in (3.2) is not the optimal one.

Thus we may apply Lemma 8 once more to get a point $x_0 \in \Omega$ with $u_1(x_0) = k u_2(x_0)$ and

$$u_1(x_0)^a v_1(x_0)^b \le k^{p-1} u_2(x_0)^a v_2(x_0)^b,$$

which gives $v_1(x_0) \le k^{-\frac{a-p+1}{b}} v_2(x_0)$. But then the second inequality in (3.2) gives

$$k^{\frac{(a-p+1)(e-p+1)-bc}{b(e-p+1)}} \le 1,$$

which is impossible since $(a-p+1)(e-p+1) > bc$ and $k > 1$. This contradiction shows $k \le 1$, that is $u_1 \le u_2$, $v_1 \ge v_2$, and a similar argument proves the reversed inequalities. This concludes the proof. □

Our second uniqueness result is concerned with problem (1.1) in a half-space $D := \{x \in \mathbb{R}^N : \ x_1 > 0\}$, where for a point x in $\mathbb{R}^N$ we write $x = (x_1, x')$, with $x' \in \mathbb{R}^{N-1}$. This problem is obtained when analyzing the boundary behavior of positive solutions to (1.1). We remark again that the proof of the analog statement for $p=2$ obtained in [25] is not applicable here, since linearity was important there.

Theorem 9. *Assume $a>p-1$, $e>p-1$ and $(a-d+1)(e-p+1)>bc$, and let (u,v) be a positive weak solution to*

$$\begin{cases} \Delta_p u = u^a v^b & \text{in } D,\\ \Delta_p v = u^c v^e & \text{in } D,\\ u=v=+\infty & \text{on } \partial D, \end{cases}$$

verifying

$$C_1 x_1^{-\alpha} \le u \le C_2 x_1^{-\alpha},\ C_1 x_1^{-\beta} \le v \le C_2 x_1^{-\beta} \quad \text{in } D, \tag{3.3}$$

for some positive constants C_1, C_2, where

$$\alpha = \frac{p(e-p+1-b)}{(a-p+1)(e-p+1)-bc}\ ,\ \beta = \frac{p(a-p+1-c)}{(a-p+1)(e-p+1)-bc}\ .$$

Then

$$u(x) = A x_1^{-\alpha},\ v(x) = B x_1^{-\beta}$$

where

$$A = \left(\frac{((p-1)\alpha^{p-1}(\alpha+1))^{e-p+1}}{((p-1)\beta^{p-1}(\beta+1))^b}\right)^{\frac{1}{(a-p+1)(e-p+1)-bc}}$$

$$B = \left(\frac{((p-1)\beta^{p-1}(\beta+1))^{a-p+1}}{((p-1)\alpha^{p-1}(\alpha+1))^c}\right)^{\frac{1}{(a-p+1)(e-p+1)-bc}}.$$

Proof. We are going to show that $u \le A x_1^{-\alpha}$, $v \ge B x_1^{-\beta}$, since the complementary inequality can be proved in a completely similar way. Thanks to (3.3) we have

$$u(x) \le K x_1^{-\alpha},\ v \ge K^{-\frac{c}{e-p+1}} x_1^{-\beta}$$

for large enough K. We may take $K = \max\{\lambda,\mu\}$, where the quantities

$$\lambda = \sup_D (x_1^\alpha u(x)),\quad \mu = (\inf_D (x_1^\beta v(x)))^{-\frac{e-p+1}{c}}$$

are finite and positive. Assume that $\lambda \ge \mu$, the other case being treated similarly.

We may also assume $K \ge A$, since otherwise there is nothing to prove. According to the definition of K, there exists a sequence $\{x_n\} \subset D$ such that $x_{n,1}^\alpha u(x_n) \to K$, where to simplify the notation $x_{n,1}$ stands for the first component of x_n. Let ξ_n be the projection of x_n onto ∂D and introduce the functions

$$U_n(y) = x_{n,1}^\alpha u(\xi_n + x_{n,1} y),\quad V_n(y) = x_{n,1}^\beta v(\xi_n + x_{n,1} y).$$

It is not hard to see that U_n, V_n verify the equations

$$\begin{cases} \Delta_p U_n = U_n^a V_n^b & \text{in } D,\\ \Delta_p V_n = U_n^c V_n^e & \text{in } D, \end{cases}$$

and the inequalities

$$C_1 y_1^{-\alpha} \le U_n \le K y_1^{-\alpha},\ K^{-\frac{c}{e-p+1}} y_1^{-\beta} \le V_n \le C_2 y_1^{-\beta} \quad \text{in } D. \tag{3.4}$$

Notice that (3.4) gives in particular uniform local bounds for U_n, V_n, so that we can use the standard $C^{1,\eta}$ interior estimates (cf. [13], [34], [46]) to obtain that (up to a subsequence) $U_n \to U$, $V_n \to V$ in $C^1_{\text{loc}}(D)$, where U, V is a solution to

$$\begin{cases} \Delta_p U = U^a V^b & \text{in } D, \\ \Delta_p V = U^c V^e & \text{in } D, \end{cases} \tag{3.5}$$

which in addition verifies $U \le K y_1^{-\alpha}$, $V \ge K^{-\frac{c}{e-p+1}} y_1^{-\beta}$, $U(e_1) = K$, where e_1 is the first vector in the canonical basis of $\mathbb{R}^N$.

On the other hand, it easily seen that if $K \ge A$, then $(\overline{u}, \bar{v})$ given by $\overline{u} = K y_1^{-\alpha}$, $\bar{v} = K^{-\frac{c}{e-p+1}} y_1^{-\beta}$ is a supersolution to (3.5). We choose $M > 0$ such that the function $h(\zeta) = \zeta^a \bar{v}(x)^b - M\zeta$ is decreasing in a neighborhood of K for fixed x in a neighborhood of e_1. Then

$$\begin{aligned} \Delta_p U - MU &= U^a V^b - MU \ge U^a \bar{v}^b - MU \\ &\ge \overline{u}^a \bar{v}^b - M\overline{u} \ge \Delta_p \overline{u} - M\overline{u} \end{aligned}$$

in a neighborhood of e_1. Since $U \le \overline{u}$ and $U(e_1) = K = \overline{u}(e_1)$, the strong comparison principle, Theorem 1.4 in [8] (see also Proposition 3.3.2 in [45]), gives $U \equiv \overline{u}$ in a neighborhood of e_1. Notice that it is important here that $|\nabla \overline{u}(e_1)| = K\alpha > 0$.

We also obtain then that $V \equiv \bar{v}$ in the same neighborhood, and this gives that $(\overline{u}, \bar{v})$ is in fact a solution to (3.5) in a neighborhood of e_1. Thus $K = A$, $K^{-\frac{c}{e-p+1}} = B$, which finally implies $u \le A x_1^{-\alpha}$, $v \ge B x_1^{-\beta}$. The proof is finished. □

4. The subcritical case

We dedicate this section to the proof of Theorem 2: that is, existence, nonexistence, boundary behavior and uniqueness of positive solutions to (1.1) when $a > p-1$, $e > p-1$ and $(a-p+1)(e-p+1) > bc$.

Proof of existence. We employ the method of sub and supersolutions. We look for a subsolution of the form $(\underline{u}, \underline{v}) = (\varepsilon U_{a,\gamma}, \varepsilon^{-\delta} U_{e,\sigma})$, where δ, γ and σ are to be chosen and ε is small enough. It is not hard to see that $(\underline{u}, \underline{v})$ will be a subsolution provided that

$$\begin{aligned} 1 &\ge \varepsilon^{a-p+1-\delta b} d^\gamma U_{e,\sigma}^b \\ 1 &\le \varepsilon^{c-\delta(e-p+1)} d^\sigma U_{a,\gamma}^c \end{aligned} \tag{4.1}$$

in Ω. We may choose δ in such a way that $a-p+1-\delta b > 0$ and $c - \delta(e-p+1) < 0$, that is,

$$\frac{c}{e-p+1} < \delta < \frac{a-p+1}{b}$$

since $(a-p+1)(e-p+1) > bc$. Thus (4.1) will hold for small enough ε provided that $d^\gamma U^b_{e,\sigma}$ and $d^\sigma U^c_{a,\gamma}$ are bounded from above and from below in Ω. According to the boundary behavior of $U_{a,\gamma}$ and $U_{e,\sigma}$ given by Lemma 4, it is enough to have

$$\frac{p-\sigma}{e-p+1} = \frac{\gamma}{b}, \quad \frac{p-\gamma}{a-p+1} = \frac{\sigma}{c},$$

which is also possible thanks to the subcriticality condition $(a-p+1)(e-p+1) > bc$. Thus we have a subsolution as small as desired.

It is shown in a similar way that $(\overline{u},\overline{v}) = (MU_{a,\gamma}, M^{-\delta}U_{e,\sigma})$ is a supersolution for the same choice of δ, γ and σ, provided that M is large enough. Since the sub and supersolution are ordered, it follows from Theorem A.2 in the Appendix that there exists a solution to (1.1). □

To obtain the boundary behavior of solutions we use a blow-up argument, as in previous works (cf. [4], [21], [22] for instance). For this sake we first need some rough estimates of all possible positive solutions. The present proof is modelled on that of the semilinear case contained in [25].

Lemma 10. *Let (u,v) be a positive solution to (1.1) with $a,e > p-1$ and $(a-p+1)(e-p+1) > bc$. Then there exist positive constants C_1, C_2 such that*

$$C_1 d(x)^{-\alpha} \le u(x) \le C_2 d(x)^{-\alpha}$$
$$C_1 d(x)^{-\beta} \le v(x) \le C_2 d(x)^{-\beta}$$

where

$$\alpha = \frac{p(e-p+1-b)}{(a-p+1)(e-p+1)-bc}\ , \ \beta = \frac{p(a-p+1-c)}{(a-p+1)(e-p+1)-bc}\ .$$

Proof. Let (u,v) be a positive solution to (1.1). Then if $v_0 = \inf_\Omega v$, we have $\Delta_p u \ge v_0^b u^a$ in Ω, and by Lemma 6

$$u \le v_0^{-\frac{b}{a-p+1}} U_{a,0} \le v_0^{-\frac{b}{a-p+1}} A_{a,0} d^{-\alpha_0} \quad \text{in } \Omega,$$

where $\alpha_0 = p/(a-p+1)$. Set $a_0 = v_0^{-\frac{b}{a-p+1}} A_{a,0}$. Using the second equation in (1.1) we have $\Delta_p v \le a_0^c d^{-\alpha_0 c} v^e$ in Ω, and thanks to Lemma 6 again we obtain

$$v \ge a_0^{-\frac{c}{e-p+1}} U_{e,\alpha_0 c} \ge a_0^{-\frac{c}{e-p+1}} B_{e,\alpha_0 c} d^{-\beta_0},$$

where $\beta_0 = (p-\alpha_0 c)/(e-p+1)$. We can iterate this argument to obtain that

$$u \le a_n d^{-\alpha_n}, \quad v \ge a_n^{-\frac{c}{e-p+1}} B_{e,\alpha_n c} d^{-\beta_n} \quad \in \Omega, \tag{4.2}$$

where

$$\alpha_n = \frac{p-b\beta_{n-1}}{a-p+1}, \quad \beta_n = \frac{p-\alpha_n c}{e-p+1}$$

and

$$a_{n+1} = a_n^{\frac{bc}{(a-p+1)(e-p+1)}} B_{e,\alpha_n c}^{-\frac{b}{a-p+1}} A_{a,\beta_n b}. \tag{4.3}$$

It is easily seen that

$$\alpha_n = \frac{p}{a-p+1}\left(\frac{e-p+1-b}{e-p+1}\right) + \frac{bc}{(a-p+1)(e-p+1)}\alpha_{n-1},$$

and $\alpha_1 < \alpha_0$. Thus $\{\alpha_n\}$ is a decreasing sequence of positive numbers, and it has a limit, which is easily seen to be α. This also entails that $\beta_n \to \beta$.

Observe that $\alpha, \beta > 0$, and this implies that $b\beta_n$ and $c\alpha_n$ are bounded away from p. Thanks to Lemma 5, the quantities $A_{a,b\beta_n}$ and $B_{e,c\alpha_n}$ are bounded and bounded away from zero. Thus thanks to (4.3), there exists $K > 0$ such that $a_{n+1} \le K a_n^\delta$, where $\delta = \frac{bc}{(a-p+1)(e-p+1)} < 1$. This readily gives $a_{n+1} \le K^{1+\delta+\cdots+\delta^n} a_0^{\delta^{n+1}}$, and thus $\limsup_{n\to\infty} a_{n+1} \le K^{\frac{1}{1-\delta}}$.

Passing to the limit in (4.2), we find that there exist positive constants C_1, C_2 such that $u \le C_2 d^{-\alpha}$, $v \ge C_1 d^{-\beta}$ in Ω. A symmetric argument proves the reversed inequalities, and thus the lemma is proved. □

Proof of the boundary behavior (1.2). Let (u, v) be an arbitrary positive solution to (1.1). Fix a point $x_0 \in \partial\Omega$. With no loss of generality we may assume $x_0 = 0$ and $\nu(x_0) = -e_1$, the first vector in the canonical basis of $\mathbb{R}^N$. Take an arbitrary sequence $\{x_n\} \subset \Omega$ with $x_n \to 0$. Denote $d_n = d(x_n)$ and let ξ_n be the projection of x_n onto $\partial\Omega$. Introduce the functions

$$z_n(y) = d_n^\alpha u(\xi_n + d_n y), \quad w_n(y) = d_n^\beta v(\xi_n + d_n y),$$

where $y \in \Omega_n := \{y \in \mathbb{R}^N : \ \xi_n + d_n y \in \Omega\}$. Observe that $\Omega_n \to D$ as $n \to \infty$, while $d(\xi_n + d_n y) \sim d_n y_1$, where $y = (y_1, y')$. It is not hard to see that (z_n, w_n) satisfies the system

$$\begin{cases} \Delta_p z = z^a w^b & \text{in } \Omega_n, \\ \Delta_p w = z^c w^e & \text{in } \Omega_n, \end{cases}$$

and thanks to Lemma 10, also the inequalities

$$C_1 d_n^\alpha d(\xi_n + d_n y)^{-\alpha} \le z(y) \le C_2 d_n^\alpha d(\xi_n + d_n y)^{-\alpha}$$
$$C_1 d_n^\beta d(\xi_n + d_n y)^{-\beta} \le w(y) \le C_2 d_n^\beta d(\xi_n + d_n y)^{-\beta}.$$

Thus it follows that (z_n, w_n) is locally uniformly bounded, and thanks to the $C^{1,\eta}$ interior estimates in [13], [34], [46], we obtain that – passing to a subsequence if necessary – (z_n, w_n) converges in $C^1_{\rm loc}(D)$ to a pair (z, w) which is a weak solution to

$$\begin{cases} \Delta_p z = z^a w^b & \text{in } D, \\ \Delta_p w = z^c w^e & \text{in } D, \end{cases}$$

verifying

$$C_1 y_1^{-\alpha} \le z \le C_2 y_1^{-\alpha}, \quad C_1 y_1^{-\beta} \le w \le C_2 y_1^{-\beta} \quad \text{in } D.$$

According to Theorem 9, $z = A y_1^{-\alpha}$, $w = B y_1^{-\beta}$, and setting $y = e_1$, we obtain that $d_n^\alpha u(x_n) \to A$, $d_n^\beta v(x_n) \to B$. Since the sequence $\{x_n\}$ is arbitrary, the estimate (1.2) is proved. □

Once the boundary behavior of solutions has been elucidated, we proceed to the proof of uniqueness, where Theorem 7 will be used.

Proof of uniqueness. Let (u_1, v_1) and (u_2, v_2) be positive solutions to (1.1). According to (1.2), we have

$$\lim_{d\to 0} \frac{u_1(x)}{u_2(x)} = \lim_{d\to 0} \frac{v_1(x)}{v_2(x)} = 1.$$

Thus for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} &(1-\varepsilon)u_2 \le u_1 \le (1+\varepsilon)u_2 \\ &(1+\varepsilon)^{-\frac{c}{e-p+1}} v_2 \le v_1 \le (1-\varepsilon)^{-\frac{c}{e-p+1}} v_2 \end{aligned} \tag{4.4}$$

for $d(x) \le \delta$. Now set $\Omega_\delta = \{x \in \Omega : \ d(x) > \delta\}$, and consider the problem

$$\begin{cases} \Delta_p z = z^a w^b & \text{in } \Omega_\delta, \\ \Delta_p w = z^c w^e & \text{in } \Omega_\delta, \\ \ z = u_1, \ w = v_1 & \text{on } \partial\Omega_\delta. \end{cases} \tag{4.5}$$

It is not difficult to see that the pair $((1-\varepsilon)u_2, (1+\varepsilon)^{-\frac{c}{e-p+1}} v_2)$ is a subsolution to (4.5), while $((1+\varepsilon)u_2, (1-\varepsilon)^{-\frac{c}{e-p+1}} v_2)$ is a supersolution. On the other hand, thanks to Theorem 7, problem (4.5) has a unique positive solution, which is precisely (u_1, v_1). Thus we obtain that (4.4) is valid in Ω, and letting ε go to zero we arrive at $u_1 = u_2$, $v_1 = v_2$, which proves uniqueness. □

We finally prove that conditions $c < a - p + 1$, $b < e - p + 1$ are necessary for the existence of solutions to (1.1) under the hypothesis $(a-p+1)(e-p+1) > bc$. We make use again of the iteration procedure introduced in the proof of Lemma 10.

Proof of necessity. We are only showing that $b < e - p + 1$ is necessary, since the other inequality is obtained similarly. Thus assume for a contradiction that there exists a positive solution to (1.1) with $b \ge e - p + 1$. Observe that this implies that $c < a - p + 1$, since $(a - p + 1)(e - p + 1) > bc$.

Let us consider first the case $b > e - p + 1$. We remark that the iterative argument in the proof of Lemma 10 is still valid, although we now have $\alpha_n \to \alpha < 0$. Then we can choose a first n such that $\alpha_{n+1} \leq 0$. Then, thanks to (4.2):

$$\Delta_p u \geq C d^{-b\beta_n} u^a \quad \text{in } \Omega,$$

for a positive constant C. We claim that this implies u to be bounded (notice that we can not directly use Lemmas 6 and 4, since $b\beta_n \geq p$). Indeed, if we fix $x \in \Omega$ and introduce the function $z(y) = d(x)^{\alpha_{n+1}} u(x + d(x)y)$ for $y \in B_1(0)$, we obtain that $\Delta_p z \geq C z^a$ in $B_1(0)$, and then $z \leq V$, the unique positive solution to $\Delta_p V = C V^a$ in $B_1(0)$ with $V = +\infty$ on $\partial B_1(0)$. Then $u(x) = z(0) d(x)^{-\alpha_{n+1}} \leq V(0) d(x)^{-\alpha_{n+1}}$, which implies that u is bounded.

Now suppose $b = e - p + 1$, $c < e - p + 1$. The iterative procedure still holds, with $\alpha_n \to 0$, $\beta_n \to p/(e - p + 1)$. Then $c\alpha_n \to 0$ and $b\beta_n \to p$, so that with the use of Lemma 5 we deduce that $B_{e,c\alpha_n}$ is bounded from below and $A_{a,b\beta_n} \to 0$. Thus, according to (4.3), for every $\varepsilon > 0$ we have $a_{n+1} \leq a_n^\delta \varepsilon$ if n is large enough. It follows from here that $\limsup_{n\to\infty} a_{n+1} \leq \varepsilon^{\frac{1}{1-\delta}}$, and thus $a_{n+1} \to 0$ as $n \to \infty$. Then (4.2) implies $u \equiv 0$. Thus no solution exists in this case, and the proof is concluded. □

5. The critical case

In this section we prove Theorems 2 and 3, which are concerned with the critical case $(a - p + 1)(e - p + 1) = bc$ for (1.1).

Proof of Theorem 2. Let us prove first that $c = a - p + 1$, $b = e - p + 1$ is necessary for existence. Indeed, if we assume $c < a - p + 1$, and thus $b > e - p + 1$, we can use the iterative procedure in Lemma 10, where now

$$\alpha_n = \frac{p}{a - p + 1} \left(\frac{e - p + 1 - b}{e - p + 1} \right) + \alpha_{n-1},$$

and hence $\alpha_n \to +\infty$, which forces $\beta_n \to -\infty$. Thus if we choose the first n so that $b\beta_n \geq p$, we obtain as in Section 4 that u is bounded. The case $c > a - p + 1$ is ruled out similarly, and hence $c = a - p + 1$, $b = e - p + 1$ is necessary for existence.

The proof existence in this case is straightforward, since it is easily seen that (U, U) is a solution to (1.1), where U is the unique solution to

$$\begin{cases} \Delta_p U = U^{b+c+p-1} & \text{in } \Omega \\ U = +\infty & \text{on } \partial\Omega \end{cases}$$

(Lemma 4). Moreover, it is easily seen that if (u, v) is any positive solution to (1.1), then $(\lambda^{\frac{b}{b+c}} u, \lambda^{-\frac{c}{b+c}} v)$ is also a positive solution if $\lambda > 0$. The proof is finished. □

Proof of Theorem 3. Notice that both u and v verify $u(0), v(0) > 0$ with $u'(0) = v'(0) = 0$. Thus if we let $\lambda = u(0)/v(0)$, both functions

u and λv agree on $r = 0$, and their derivatives vanish there. On the other hand, denoting $a(x) = u^c v^b$, we have that u and λv are solutions to the "p-linear" equation $\Delta_p w = a(x) w^{p-1}$, and it follows by uniqueness of the associated Cauchy problem that $u = \lambda v$. Thus we obtain that $\Delta_p v = \lambda^c v^{b+c+p-1}$ in Ω, which implies $v = \lambda^{-\frac{c}{b+c}} U$. Then $u = \lambda v = \lambda^{\frac{b}{b+c}} U$. Finally, the asymptotic behavior (1.3) follows from Lemma 4 with $\gamma = 0$, $q = b + c + p - 1$. This concludes the proof. □

Appendix

We collect in this appendix two results related to the method of sub and supersolutions for our system:

$$\text{(P)} \qquad \begin{cases} \Delta_p u = u^a v^b & \text{in } \Omega \\ \Delta_p v = u^c v^e & \text{in } \Omega, \end{cases}$$

where $a, e > p - 1$ and $b, c > 0$. Since this system is of competitive type a subsolution is a pair $(\underline{u}, \underline{v}) \in (W^{1,p}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega))^2$ such that

$$\begin{cases} \Delta_p \underline{u} \geq \underline{u}^a \underline{v}^b & \text{in } \Omega \\ \Delta_p \underline{v} \leq \underline{u}^c \underline{v}^e & \text{in } \Omega, \end{cases}$$

and a supersolution $(\overline{u}, \bar{v})$ is defined by reversing the inequalities. Since competitive type systems with p-Laplacians are not frequent in the literature, we sketch proofs of the results for completeness.

We begin by considering system (P) with finite boundary conditions $u = f(x)$, $v = g(x)$ on $\partial\Omega$, where $f, g \in C^\eta(\partial\Omega)$ for some $\eta \in (0, 1)$ (this is not the optimal regularity but it will suffice for our purposes).

Theorem A. 1. *Assume $(\underline{u}, \underline{v})$ is a subsolution and $(\overline{u}, \bar{v})$ a supersolution to (P) with $\underline{u} \leq f(x) \leq \overline{u}$, $\underline{v} \geq g(x) \geq \bar{v}$ on $\partial\Omega$ and $\underline{u} \leq \overline{u}$, $\underline{v} \geq \bar{v}$ in Ω. Then problem (P) admits at least a weak solution (u, v) with $\underline{u} \leq u \leq \overline{u}$, $\underline{v} \geq v \geq \bar{v}$ in Ω and $u = f(x)$, $v = g(x)$ on $\partial\Omega$.*

Proof. Denote by u_1 the unique positive solution to the problem

$$\text{(A.1)} \qquad \begin{cases} \Delta_p u = \underline{v}^b u^a & \text{in } \Omega \\ u = f(x) & \text{on } \partial\Omega \end{cases}$$

(cf. Remark 3). Since $\underline{u}$ and $\overline{u}$ are easily seen to be a subsolution and a supersolution, respectively, of (A.1), it follows by uniqueness that $\underline{u} \leq u_1 \leq \overline{u}$ in Ω. We now consider v_1 to be the unique solution to

$$\text{(A.2)} \qquad \begin{cases} \Delta_p v = u_1^c v^e & \text{in } \Omega \\ v = g(x) & \text{on } \partial\Omega. \end{cases}$$

It follows similiarly that $\underline{v} \geq v_1 \geq \bar{v}$. We can continue in this way by defining u_n to be the unique solution to (A.1) with $\underline{v}$ replaced by v_{n-1} and v_n the unique solution to (A.2) with u_1 replaced by u_n. We obtain two sequences $\{u_n\}$, $\{v_n\}$, such that $\underline{u} \leq u_n \leq \overline{u}$, $\underline{v} \geq v_n \geq \bar{v}$, while $\{u_n\}$ is increasing and $\{v_n\}$ is decreasing. It is now standard to pass to the limit thanks to the C^η estimates of [32] and the interior

$C^{1,\eta}$ estimates of [13], [34], [46], and obtain that $u_n \to u$, $v_n \to v$ in $C^\eta(\overline{\Omega}) \cap C^1_{\rm loc}(\Omega)$, where (u,v) is a weak solution to (P). Moreover, $u = f$, $v = g$ on $\partial\Omega$ and $\underline{u} \le u \le \overline{u}$, $\underline{v} \ge v \ge \bar{v}$ in Ω. This finishes the proof. □

Finally, we state and prove a version of the method which is directly applicable to the problem with infinite boundary conditions.

Theorem A. 2. *Assume $(\underline{u}, \underline{v})$ is a subsolution and $(\overline{u}, \bar{v})$ a supersolution to (P) with $\underline{u} = \overline{u} = \underline{v} = \bar{v} = +\infty$ on $\partial\Omega$ and $\underline{u} \le \overline{u}$, $\underline{v} \ge \bar{v}$ in Ω. Then problem (P) admits at least a weak solution (u,v) with $\underline{u} \le u \le \overline{u}$, $\underline{v} \ge v \ge \bar{v}$ in Ω and $u = v = +\infty$ on $\partial\Omega$.*

Proof. Let $\delta > 0$ and consider the problem

$$\text{(A.3)} \qquad \begin{cases} \Delta_p u = u^a v^b & \text{in } \Omega_\delta, \\ \Delta_p v = u^c v^e & \text{in } \Omega_\delta, \\ u = \hat{u}_\delta, \ v = \hat{v}_\delta & \text{on } \partial\Omega_\delta, \end{cases}$$

where $\Omega_\delta = \{x \in \Omega : \ d(x) > \delta\}$ and $\hat{u}_\delta$, $\hat{v}_\delta$ are smooth functions defined on $\partial\Omega_\delta$ with $\underline{u} \le \hat{u}_\delta \le \overline{u}$, $\underline{v} \ge \hat{v}_\delta \ge \bar{v}$ on $\partial\Omega_\delta$. By Theorem A.1, there exists a solution (u_δ, v_δ) of (A.3) verifying $\underline{u} \le u_\delta \le \overline{u}$, $\underline{v} \ge v_\delta \ge \bar{v}$ in Ω_δ. This inequalities give bounds for the solutions (u_δ, v_δ), so that we obtain bounds in $C^{1,\eta}_{\rm loc}(\Omega)$ thanks to the estimates in [13], [34], [46]. Then we obtain that for a sequence $\delta_n \to 0$, $u_{\delta_n} \to u$, $v_{\delta_n} \to v$ in $C^1_{\rm loc}(\Omega)$. Thus u, v is a weak solution to (P) verifying in addition $\underline{u} \le u \le \overline{u}$, $\underline{v} \ge v \ge \bar{v}$ in Ω. In particular, $u = v = +\infty$ on $\partial\Omega$. □

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