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A Local Bifurcation Theorem for Degenerate Elliptic Equations With Radial Symmetry[1]

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In this work we provide local bifurcation results for equations involving the p-Laplacian in balls. We analyze the continua $\mathscr{C}_n$ of radial solutions emanating from $(\lambda_{n,p}, 0)$, $\{\lambda_{n,p}\}$ being the radial eigenvalues of $-\Delta_p$. First, we show that the only nontrivial solutions close to $(\lambda_{n,p}, 0)$ lie on a continuous curve, thus extending the Crandall–Rabinowitz theorem. Second, it is proved that $\mathscr{C}_n \setminus \{(\lambda_{n,p}, 0)\}$ splits into two unbounded connected pieces, characterized by their nodal properties thus sharpening previous results. © 2002 Elsevier Science

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1. INTRODUCTION AND RESULTS

In this paper we study a bifurcation problem for equations of the form

$$\begin{aligned} -\Delta_p u &= \lambda \varphi_p(u) + g(\lambda, x, u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ stands for the p-Laplacian operator, $\varphi_p(z) = |z|^{p-2} z$ for $z \in \mathbb{R}$, $p \geqslant 2$, λ is a real parameter and Ω is a bounded domain in $\mathbb{R}^N$. The function g is C^1, $g(\lambda, x, 0) = 0$ and $\partial g / \partial u = o(|u|^{p-2})$ as $u \sim 0$, uniformly for λ in bounded intervals and $x \in \Omega$. By a solution of (1) it will be understood a pair (λ, u) with $\lambda \in \mathbb{R}$, $u \in W_0^{1,p}(\Omega)$, verifying (1) in the weak sense.

For $p = 2$ ($\Delta_p \equiv \Delta$) problem (1) has been extensively studied. Bifurcation results of local nature are available since the celebrated Crandall–

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Rabinowitz work [4]. Namely, there exists a branch of bifurcated solutions of the form $(\lambda(s), s(\psi_n + y(s)))$, $|s| < \varepsilon$ with $\lambda(0) = \sigma_n$, where σ_n, ψ_n are any simple eigenvalue and an associated eigenfunction of the operator $-\Delta$ in Ω. Moreover, every nontrivial solution in $\mathbb{R} \times W_0^{1,2}(\Omega)$ near the point $(\sigma_n, 0)$ belongs to that branch. Global results are also well known for the semilinear case: there is a continuum (a closed and connected set) of solutions $\mathscr{C}$ containing this branch, which is unbounded and does not possess other trivial solutions aside $(\lambda, u) = (\sigma_n, 0)$ (cf. [17] where it is merely assumed that σ_n has odd algebraic multiplicity).

However, the situation for $p \neq 2$ is quite different. For $N = 1$, the bifurcation properties of (1) in the case $g(u)/u^{p-1}$ decreasing were studied in [13]. If $N \geqslant 2$ and $\Omega \subset \mathbb{R}^N$ is a general bounded domain the only kind of result available at the moment is the existence of a continuum of solutions $\mathscr{C} \subset \mathbb{R} \times W_0^{1,p}(\Omega)$ emanating from the point $(\lambda_{1,p}, 0)$, where $\lambda_{1,p}$ is the first eigenvalue of $-\Delta_p$ in Ω (see [5, 6] for pioneering results and [16] for an extension). On the other hand, if Ω is a ball B of $\mathbb{R}^N$ and only radially symmetric solutions are considered it was shown in [5] the existence of a continuum $\mathscr{C}_n$ of solutions to (1) bifurcated at every radial eigenvalue $\lambda_{n,p}$ of $-\Delta_p$ (see [8] for the analysis of more general radial problems). However, no information on the multiplicity of solutions and local nature of the $\mathscr{C}_n$'s near $(\lambda_{n,p}, 0)$ is given while the characterization by nodal properties of such continua is not optimal. Also in the context of radial solutions a local bifurcation result for (1), with a slight perturbation of $-\Delta_p$, was treated in [7] where a more precise description of the bifurcated solutions under the form $\lambda = \lambda_{1,p} + s\mu(s)$, $u = s\phi_1 + s^2 v(s, \cdot)$, $0 < s \leqslant \varepsilon$, was obtained, ϕ_1 being the first eigenfunction of $-\Delta_p$ in B. However, no *uniqueness* results for this type of solutions near the bifurcation point $(\lambda_{1,p}, 0)$ were shown, in contrast with the semilinear case.

Our objective in this paper is to produce a Crandall–Rabinowitz theorem for problem (1) at every radial eigenvalue $\lambda_{n,p}$ and radial solutions. Namely, to achieve the existence, uniqueness and regularity of local branches. Thus, our work improves the results of [7] in several respects. First we obtain parametrized branches of solutions which emanate from the trivial solution for every radial eigenvalue $\lambda_{n,p}$ and not merely from the first one $\lambda_{1,p}$ providing the uniqueness of the bifurcated nontrivial solutions, while we feel that our approach is more natural. In addition, our local result enables us to provide a more detailed description of the global structure of the continua $\mathscr{C}_n$ of nontrivial solutions branched at every $\lambda = \lambda_{n,p}$. Thus, we are substantially sharpening the results in Section 4 of [5] (particularly, Theorem 4.1).

On the other hand, it should be remarked that to obtain such a local bifurcation statement beyond the radial case, i.e. for a general domain

$\Omega \subset \mathbb{R}^N$, still seems unreachable. In fact, the linearization of (1) at a solution u_0 degenerates at the critical points of u_0 whose number and distribution in Ω are "a priori" unknown. Moreover, in the case of the trivial solution $u_0 = 0$ such linearization plainly has no sense.

In the radial setting problem (1) becomes

$$\begin{aligned} -\Delta_p u &= \lambda \varphi_p(u) + g(\lambda, r, u) \quad && \text{in } B \\ u &= 0 && \text{on } \partial B, \end{aligned} \tag{P}$$

where $r = |x|$. Our main approach will consist in linearizing (P) at any radial eigenfunction ϕ of the operator $-\Delta_p$ instead of $u = 0$. In fact, it is well-known that $-\Delta_p$ exhibits an infinite increasing sequence of radial Dirichlet eigenvalues $0 < \lambda_{1,p} < \lambda_{2,p} < \cdots$, all of them simple with associated eigenfunctions $\phi = \phi_n(r) \in C^1[0, 1]$, $r^{N-1}\varphi_p(\phi_n) \in C^1[0, 1]$ solving (cf. [1, 5]),

$$\begin{aligned} -(r^{N-1}\varphi_p(\phi'))' &= \lambda_{n,p} r^{N-1} \varphi_p(\phi), \qquad 0 < r < 1 \\ \phi'(0) &= \phi(1) = 0. \end{aligned} \tag{2}$$

Moreover, every eigenfunction ϕ_n has exactly $n-1$ simple zeros in the interval $(0, 1)$. As an outstanding property of the linearized eigenvalue problem corresponding to (P) at ϕ_n, we will prove that $\lambda = \lambda_{n,p}$ is again a simple eigenvalue with $v = \phi_n$ as an associated eigenfunction (see Section 3).

In order to state our main results we introduce the spaces: $\mathscr{E} = C[0, 1]$, $\mathscr{F} = \{u \in C^1[0, 1] : u'(0) = u(1) = 0\}$, and $\mathscr{Y}_n = \{u \in \mathscr{F} : \int_0^1 r^{N-1} |\phi_n(r)|^{p-2} \phi_n(r) u(r)\, dr = 0\}$ endowed with their natural norms. Then we have

THEOREM 1. *Suppose g is C^1, $g(\lambda, r, 0) = 0$ and $\partial g/\partial u = o(|u|^{p-2})$ as $u \to 0$, uniformly for λ in bounded intervals and $r \in [0, 1]$. Then for every n there exist $\varepsilon = \varepsilon(n) > 0$ and continuous mappings $\lambda_n \colon (-\varepsilon, \varepsilon) \to \mathbb{R}$, $y_n \colon (-\varepsilon, \varepsilon) \to \mathscr{Y}_n$ such that $\lambda_n(0) = \lambda_{n,p}$, $y_n(0) = 0$ and every radial solution $(\lambda, u) \neq (\lambda_{n,p}, 0)$ of (P) in a neighbourhood of $(\lambda_{n,p}, 0)$ in $\mathbb{R} \times W_0^{1,p}(B) \cap L^\infty(B)$ has the form $(\lambda_n(s), s(\phi_n + y_n(s)))$.*

Remarks 1. (a) Notice that since the mapping $y_n \colon (-\varepsilon, \varepsilon) \to \mathscr{Y}_n$ is continuous, it follows that for $s \sim 0$, the solutions u have the same nodal behaviour as ϕ_n.

(b) For a large class of perturbation terms g, with a suitable power type behaviour near $u = 0$, the regularity extent of the bifurcated branch can be improved to reach differentiability. This permits determining the bifurcation direction in such kind of problems. See Remark 4 in Section 4.

(c) If g satisfies the growth condition $g(\lambda, r, u) \operatorname{sgn} u \leqslant C(\lambda)(1 + |u|^{p-1})$, $u \in \mathbb{R}$, $r \in [0, 1]$, $C = C(\lambda) > 0$, then every weak solution $u \in W_0^{1,p}(B)$ to (P) belongs to $L^\infty(B)$ (cf. Lemma 10.8 and the observation in

[12, p. 277]). Therefore the uniqueness assertion in Theorem 1 holds for radial $(\lambda, u) \in \mathbb{R} \times W_0^{1,p}(B)$.

To state our next result we are introducing the open sets $\mathscr{S}_n^{\pm} \in C^1[0,1]$ given by $\mathscr{S}_n^{\pm} = \{u \in \mathscr{F} : u$ exactly exhibits $n-1$ simple zeros in $0<r<1$, $\mp u'(1) > 0\}$. The radial eigenfunctions ϕ_n will be always chosen so that $\phi_n \in \mathscr{S}_n^+$. We are also using the notation $C_0^1(B) = \{u \in C^1(\overline{B}) : u_{\partial B} = 0\}$. Observe that the local bifurcation in Theorem 1 actually occurs in $\mathbb{R} \times C_0^1(B)$.

THEOREM 2. *Under the assumptions of Theorem* 1 *let* $\mathscr{C}_n \subset \mathbb{R} \times C_0^1(B)$ *be the continuum of radial solutions* (λ, u) *to (P) bifurcated from* $(\lambda_{n,p}, 0)$, *i.e., the connected component of* $(\lambda_{n,p}, 0)$ *in the closure of the set of nontrivial solutions,*

$$\mathscr{S} = \{(\lambda, u) \in \mathbb{R} \times C_0^1(B) : u \text{ solves } (P), u \neq 0\}.$$

Then, for every $n \in \mathbb{N}$ *the following properties hold,*

(i) $\mathscr{C}_n = \mathscr{C}_n^+ \cup \{(\lambda_{n,p}, 0)\} \cup \mathscr{C}_n^-$, *where* $\mathscr{C}_n^{\pm}$ *are connected,* $\mathscr{C}_n^{\pm} \subset \mathbb{R} \times \mathscr{S}_n^{\pm}$, *and therefore* $\mathscr{C}_n^- \cap \mathscr{C}_n^+ = \varnothing$.

(ii) *Both connected pieces* $\mathscr{C}_n^+$ *and* $\mathscr{C}_n^-$ *are unbounded and do not contain trivial solutions* $(\lambda, 0)$.

Remarks 2. (a) The results in Section 4 of [5] are not so fine as to discriminate the existence of the two *different* pieces $\mathscr{C}_n^{\pm}$ in $\mathscr{C}_n \setminus \{(\lambda_{1,p}, 0)\}$. Here, we are in particular providing the existence of two unbounded branches $\mathscr{C}_1^+$ and $\mathscr{C}_1^-$ of, respectively, positive and negative radial solutions to (P).

(b) It can be shown that all possible weak nontrivial—not necessarily radial—solutions $(\lambda, u) \in \mathbb{R} \times W_0^{1,p}(B)$ to (P) close enough to $(\lambda_{1,p}, 0)$ in the L^∞ norm keep one sign. However, and except for some few exceptions (cf. [2, 14]) the celebrated symmetry result in [11] fails, in general, for the p-Laplacian environment. Thus, in principle, such possible non radial solutions could escape from our local uniqueness result.

The paper is organized as follows. Section 2 introduces some preliminary properties concerning the linearization of the p-Laplacian in balls of $\mathbb{R}^N$. These results seem to be completely new, and in addition are valid for every $p > 1$. Some properties of the linearized eigenvalue problem corresponding to (2) are shown later in Section 3. Finally, Section 4 is devoted to the proofs of Theorems 1 and 2. In addition, results on the bifurcation direction are obtained which permit showing the nonuniqueness of positive solutions to a kind of problems.

2. PRELIMINARIES

In this section we will establish some basic facts concerning the linearization of the p-Laplacian under radial symmetry. It should be remarked that the results in the present section are valid in the whole range $p>1$.

If B denotes the unit ball in $\mathbb{R}^N$ then it is well known that for every $f \in C(\overline{B})$ there exists a unique weak solution to the equation

$$\begin{aligned} -\Delta_p u &= f \qquad \text{in } B \\ u &= 0 \qquad \text{on } \partial B, \end{aligned}$$

which belongs to $C^{1,\alpha}(\overline{B})$ for some $0<\alpha<1$ (cf. [15, 18]). If further f is radially symmetric, $f=f(r)$, $r=|x|$, then so is u. It can be shown in that case that, modulo redefinition outside a null set, $u=u(r) \in C^1[0,1]$, $r^{N-1}\varphi_p(u') \in C^1[0,1]$ and solves the problem,

$$\begin{aligned} -(r^{N-1}\varphi_p(u'))' &= r^{N-1} f(r), \qquad 0<r<1 \\ u'(0) &= u(1) = 0 \end{aligned} \qquad \left({}' = \frac{d}{dr}\right).$$

Moreover, u can be represented by a solution operator $K\colon \mathscr{E} \to \mathscr{F}$ ($\mathscr{E}$, $\mathscr{F}$ as introduced in Section 1) acting on f as,

$$Kf(r) = \int_r^1 \varphi_{p'}\left(\int_0^s \left(\frac{\rho}{s}\right)^{N-1} f(\rho)\, d\rho\right) ds,$$

where $p' = p/(p-1)$ is the Hölder conjugate of $p>1$. For later use denote for $g \in C^1[0,1]$, $Z_g := \{r \in [0,1] : g'(r) = 0\}$. Some basic properties of K are next introduced without proof (cf. [9, Theorem 2.1]).

LEMMA 3. *The operator K is continuous from $\mathscr{E}$ into $\mathscr{F}$. Moreover, if there exists $r_0 \in Z_u$ with $u = Kf$ and $f(r_0) \neq 0$, then*

$$\lim_{r \to r_0} \frac{u'(r)}{\varphi_{p'}(r-r_0)} = -C\varphi_{p'}(f(r_0)),$$

where $C=1$ if $r_0>0$ and $C=\varphi_{p'}(1/N)$ for $r_0=0$.

Remark 3. The previous lemma shows that if $f \in C[0, 1]$, $u = Kf$ and $f(r_0) \neq 0$ for every $r_0 \in Z_u$, then $1/|(Kf)'|^{p-2} \in L^q(0, 1)$ for every $1 \leqslant q < \hat{p}$, $\hat{p}$ being given by

$$\hat{p} := \begin{cases} (p-1)/(p-2), & p > 2 \\ +\infty, & 1 < p \leqslant 2. \end{cases}$$

Actually, a finer information is available.

LEMMA 4. *Let $f \in \mathscr{E}$, $u = Kf$ and assume that $f(r_0) \neq 0$ for some $r_0 \in Z_u$. If $g \to f$ in $\mathscr{E}$ then*

$$\frac{1}{|(Kg)'|^{p-2}} \to \frac{1}{|(Kf)'|^{p-2}} \qquad in \quad L^q(r_0 - \eta, r_0 + \eta),$$

where $1 \leqslant q < \hat{p}$ and $\eta > 0$ is such that $(r_0 - \eta, r_0 + \eta) \cap Z_u = \{r_0\}$, provided $r_0 > 0$, while if $r_0 = 0$ the same conclusion holds in $L^q(0, \eta)$ where $\eta > 0$ is chosen so that $[0, \eta) \cap Z_f = \{0\}$.

Proof. We will only consider the case $p > 2$ since the assertion of the theorem when $1 < p \leqslant 2$ is a consequence of Lemma 3. We will also assume that $r_0 > 0$. Notice that we can take $\delta > 0$ and a neighbourhood $\mathscr{U}$ of f such that, for $g \in \mathscr{U}$ both conditions $Z_{Kg} \cap (r_0 - \delta, r_0 + \delta) \neq \emptyset$ and $g \neq 0$ in $Z_{Kg} \cap (r_0 - \delta, r_0 + \delta)$ hold. Fix $g \in \mathscr{U}$ and let $\bar{r}_0 \in Z_{Kg} \cap (r_0 - \delta, r_0 + \delta)$. If we denote $u = Kf$, $v = Kg$, then in virtue of Lemma 3, the functions $u'(r)/\varphi_p(r - r_0)$, $v'(r)/\varphi_p(r - \bar{r}_0)$ (conveniently defined for $r = r_0$ and $r = \bar{r}_0$, respectively) are continuous in $(r_0 - \delta, r_0 + \delta)$. Moreover

$$\begin{aligned}
&\left| \varphi_p \left(\frac{v'(r)}{\varphi_p'(r - \bar{r}_0)} \right) - \varphi_p \left(\frac{u'(r)}{\varphi_p'(r - r_0)} \right) \right| \\
&\quad \leqslant |g - f|_\infty + \left| \frac{1}{r - \bar{r}_0} \int_{\bar{r}_0}^{r} \left(\frac{\rho}{r} \right)^{N-1} f(\rho)\, d\rho - \frac{1}{r - r_0} \int_{r_0}^{r} \left(\frac{\rho}{r} \right)^{N-1} f(\rho)\, d\rho \right| \\
&\quad = |g - f|_\infty + \left| \left(\frac{\xi(r)}{r} \right)^{N-1} f(\xi(r)) - \left(\frac{\eta(r)}{r} \right)^{N-1} f(\eta(r)) \right| \\
&\quad \leqslant |g - f|_\infty + \left| \left(\frac{\xi(r)}{r} \right)^{N-1} - \left(\frac{\eta(r)}{r} \right)^{N-1} \right| |f(\xi(r))| \\
&\qquad + \left| \frac{\eta(r)}{r} \right|^{N-1} |f(\xi(r)) - f(\eta(r))|,
\end{aligned}$$

where the function ξ is comprised between r and r_0, η takes its values between $\bar{r}_0$ and r. Thus,

$$\left|\varphi_p\left(\frac{v'}{\varphi'_p(r-\bar{r}_0)}\right)-\varphi_p\left(\frac{u'}{\varphi'_p(r-r_0)}\right)\right|$$

$$\leqslant |g-f|_\infty+\frac{M}{(r_0-\delta)^{N-1}}|\xi^{N-1}-\eta^{N-1}|+\frac{1}{(r_0-\delta)^{N-1}}|f(\xi)-f(\eta)|,$$

at every $r\in(r_0-\delta, r_0+\delta)$, where $M=\sup|f|$. Since f is uniformly continuous it follows that for $\varepsilon>0$ fixed, and diminishing δ if necessary the previous quantities are less than ε if $r\in(r_0-\delta/2, r_0+\delta/2)$ (notice that $\xi(r)$, $\eta(r)\in(r_0-\delta, r_0+\delta)$). Thus for every $\varepsilon>0$ there exist $\delta>0$ and a neighbourhood $\mathcal{U}$ of f such that

$$\left|\frac{v'(r)}{\varphi_{p'}(r-\bar{r}_0)}-\frac{u'(r)}{\varphi_{p'}(r-r_0)}\right|<\varepsilon$$

for $|r-r_0|<\delta/2$ and $g\in\mathcal{U}$. Moreover, since $v'\to u'$ uniformly in $\delta/2\leqslant |r-r_0|\leqslant\eta$, we have that for g in a neighbourhood $\mathcal{U}$ of f such inequality also holds in the whole interval $|r-r_0|\leqslant\eta$. In particular, this gives that $\varphi_{p'}(r-\bar{r}_0)/v'(r)\to\varphi_{p'}(r-r_0)/u'(r)$ as $g\to f$, uniformly in $|r-r_0|\leqslant\eta$.

Now let us prove that $1/|r-\bar{r}_0|^\alpha$ converges to $1/|r-r_0|^\alpha$ in $L^q(r_0-\eta, r_0+\eta)$ as $\bar{r}_0\to r_0$, where $\alpha=1/\hat{p}$ and $1\leqslant q<\hat{p}$. Indeed, consider the integral

$$I=\int_{|r-r_0|\leqslant\eta}\left|\frac{1}{|r-\bar{r}_0|^\alpha}-\frac{1}{|r-r_0|^\alpha}\right|^q dr$$

and perform the change of variables $r-r_0=s(r_0-\bar{r}_0)$. Then,

$$I=|r_0-\bar{r}_0|^{1-\alpha q}\left\{\int_{|s|\leqslant R_0}+\int_{R_0\leqslant|s|\leqslant\frac{\eta}{|r_0-\bar{r}_0|}}\right\}\left|\frac{1}{|s+1|^\alpha}-\frac{1}{|s|^\alpha}\right|^q ds.$$

Let us estimate the second of these integrals. Notice that for fixed $R_0>1$ we have

$$\left|\frac{1}{|s+1|^\alpha}-\frac{1}{|s|^\alpha}\right|^q\leqslant\frac{C}{|s|^{(\alpha+1)q}}$$

for $|s|\geqslant R_0$ and some $C>0$. Thus

$$\int_{R_0\leqslant|s|\leqslant\eta/|r_0-\bar{r}_0|}\left|\frac{1}{|s+1|^\alpha}-\frac{1}{|s|^\alpha}\right|^q ds\leqslant\frac{2C}{1-(\alpha+1)q}\left(\frac{\eta}{|r_0-\bar{r}_0|}\right)^{1-(\alpha+1)q}.$$

This yields

$$I \leqslant |r_0 - \bar{r}_0|^{1-\alpha q} \int_{|s| \leqslant R_0} \left| \frac{1}{|s+1|^\alpha} - \frac{1}{|s|^\alpha} \right|^q ds + \frac{2C}{1-(\alpha+1)\,q} |r_0 - \bar{r}_0|^q \, \eta^{1-(\alpha+1)\,q},$$

and then $I \to 0$ as $\bar{r}_0 \to r_0$. To conclude the proof notice that the uniform convergence of $|r-\bar{r}_0|^\alpha / |v'(r)|^{p-2}$ to $|r-r_0|^\alpha / |u'(r)|^{p-2}$ in $|r-r_0| \leqslant \eta$ implies that $1/|v'(r)|^{p-2} \to 1/|u'(r)|^{p-2}$ in $L^q(r_0-\eta, r_0+\eta)$. ∎

In what follows we will consider the operator K with values in the space $\mathscr{G} := \{u \in W^{1,q}(0,1) : u(1) = 0\}$ for some $1 \leqslant q < \hat{p}$. Placing some conditions on $f \in \mathscr{E}$ we will be able to prove that the operator K is differentiable when considered from $\mathscr{E}$ to $\mathscr{G}$. It turns out that a convenient condition is $f(r_0) \neq 0$ for every $r_0 \in Z_u$, $u = Kf$.

THEOREM 5. *Let $f \in \mathscr{E}$, $u = Kf$ with $f(r_0) \neq 0$ for every $r_0 \in Z_u$. Then K: $\mathscr{E} \to \mathscr{G}$ is differentiable at f. Moreover, for every $g \in \mathscr{E}$,*

$$DK(f)\, g(r) = \frac{1}{p-1} \int_r^1 \frac{1}{|u'(s)|^{p-2}} \left(\int_0^s \left(\frac{\rho}{s} \right)^{N-1} g(\rho)\, d\rho \right) ds.$$

Proof. First, observe that the right hand member $v := Lg$ of the equality has full sense. In fact, Remark 3 implies that $1/|u'|^{p-2} \in L^1(0,1)$. Thus,

$$|(Lg)'(r)|^q \leqslant \frac{1}{p-1} \frac{1}{|u'(r)|^{(p-2)\,q}} |g|_\infty^q,$$

which gives $|Lg|_{\mathscr{G}} \leqslant C|g|_\infty$, proving that L: $\mathscr{E} \to \mathscr{G}$ is a continuous linear operator. To see that L is indeed the Fréchet derivative of K at f, fix $g \in \mathscr{E}$. Then,

$$|(K(f+g) - Kf - Lg)'| \leqslant |\varphi_{p'}(\mathscr{I}(f+g)) - \varphi_{p'}(\mathscr{I}(f)) - \frac{1}{p-1} \frac{1}{|u'(r)|^{p-2}} \mathscr{I}(g)|,$$

where $\mathscr{I}(h)$, $h \in \mathscr{E}$, is a short version of $\int_0^r (\rho/r)^{N-1} h(\rho)\, d\rho$. The mean value theorem can be applied to the difference of the first two integrals to obtain

$$|(K(f+g) - Kf - Lg)'(r)| \leqslant \frac{1}{p-1} |g|_\infty \left| |\xi_g(r)|^{p'-2} - \frac{1}{|u'(r)|^{p-2}} \right|,$$

where $\xi_g = \xi_g(r)$ is an intermediate function between $\mathscr{I}(f+g)$ and $\mathscr{I}(f)$. This yields

$$|K(f+g) - Kf - Lg|_{\mathscr{G}} \leqslant \frac{1}{p-1} |g|_\infty \left(\int_0^1 \left| |\xi_g(s)|^{p'-2} - \frac{1}{|u'(s)|^{p-2}} \right|^q ds \right)^{1/q}.$$

But since $|\xi_g|^{p'-2}$ is comprised between the functions $1/|u'|^{p-2}$ and $1/|(K(f+g))'|^{p-2}$, Lemma 4 gives that the last integral tends to zero as $|g|_\infty \to 0$. Thus L is the Fréchet derivative of K at f and the theorem is proved. ▮

COROLLARY 6. *Let $f \in \mathscr{E}$ such that $f(r_0) \neq 0$ for every $r_0 \in Z_u$ where $u = Kf$. Then $K: \mathscr{U} \subset \mathscr{E} \to \mathscr{G}$ is C^1, being $\mathscr{U}$ a certain neighbourhood of f.*

Proof. As already remarked, a neighbourhood $\mathscr{U}$ of f can be chosen so that every $h \in \mathscr{U}$ verifies $h \neq 0$ in Z_{Kh}. Thus K is differentiable on h. Moreover

$$|(DK(f)\, g - DK(h)\, g)'\,(r)| \leqslant \frac{1}{p-1}\, |g|_\infty \left| \frac{1}{|u'(r)|^{p-2}} - \frac{1}{|v'(r)|^{p-2}} \right|,$$

which leads to

$$|DK(f)\, g - DK(h)\, g|_{\mathscr{G}} \leqslant \frac{1}{p-1}\, |g|_\infty \left(\int_0^1 \left| \frac{1}{|u'(r)|^{p-2}} - \frac{1}{|v'(r)|^{p-2}} \right|^q dr \right)^{1/q},$$

and Lemma 4 gives the desired conclusion. ▮

3. EIGENVALUE PROBLEMS

In this section we are investigating an eigenvalue problem associated to the operator DK. We recall that $\{\lambda_{n,p}\}$ and $\{\phi_n\}$ stand for the set of radial eigenvalues and eigenfunctions, respectively, of $-\Delta_p$ in B. Thus they solve the eigenvalue problem (2) (Section 1).

Notice that $\phi_n = K(\lambda_{n,p} \varphi_p(\phi_n))$, and thus $Z_n := Z_{K(\lambda_{n,p}\varphi_p(\phi_n))} = \{r \in [0, 1] : \phi'_n(r) = 0\}$. Since the zeros of ϕ_n are all simple, it turns out that $\phi_n \neq 0$ in Z_n, and this yields, in virtue of Theorem 5, that K is differentiable at $\lambda_{n,p} \varphi_p(\phi_n)$. This allows us to consider the linearized eigenvalue problem,

$$\begin{gathered} -(r^{N-1}\, |\phi'_n|^{p-2}\, v')' = \lambda r^{N-1}\, |\phi_n|^{p-2}\, v, \qquad 0 < r < 1 \\ v'(0) = v(1) = 0. \end{gathered} \tag{3}$$

Our objective is to show that (3) admits at least an exceptional eigenvalue. Namely,

THEOREM 7. *The eigenvalue problem* (3) *has $\lambda_{n,p}$ and ϕ_n as eigenvalue and eigenfunction, respectively. Moreover, every other eigenfunction associated to $\lambda_{n,p}$ is a constant multiple of ϕ_n.*

Proof. It is clear that $\lambda=\lambda_{n,p}$ is an eigenvalue to (3) with ϕ_n as a corresponding eigenfunction. As for the simplicity of $\lambda_{n,p}$, the crucial point is that the Cauchy problem

$$\begin{aligned} -(r^{N-1}\,|\phi_n'|^{p-2}\,v')' &= \lambda_{n,p} r^{N-1}\,|\phi_n|^{p-2}\,v, \qquad 0<r<1 \\ v(r_0)=v_0, &\qquad v'(r_0)=v_0', \end{aligned} \tag{4}$$

has a unique solution for $0\leqslant r_0<1$ and $v_0, v_0'\in\mathbb{R}$, in an interval of the form $r_0\leqslant r\leqslant r_0+\delta$ for $\delta>0$ small enough. Notice that the general theory of ordinary differential equations (see [3]) is not directly applicable if $r_0=0$ or $\phi_n'(r_0)=0$, because the equation in (4) is singular there, or when $1<p<2$ and $\phi_n(r_0)=0$. Thus we will achieve existence and uniqueness of solutions in $r_0\leqslant r\leqslant r_0+\delta$ by proving that the operator,

$$\begin{aligned} Tv(r) = v_0 &+ v_0'\int_{r_0}^{r}\left(\frac{r_0}{s}\right)^{N-1}\frac{|\phi_n'(r_0)|^{p-2}}{|\phi_n'(s)|^{p-2}}\,ds \\ &-\int_{r_0}^{r}\frac{1}{|\phi_n'(s)|^{p-2}}\int_{r_0}^{s}\left(\frac{\rho}{s}\right)^{N-1}\lambda_{n,p}\,|\phi_n(\rho)|^{p-2}\,v(\rho)\,d\rho\,ds, \end{aligned}$$

whose fixed points coincide with the solutions to (4), is contractive in $C[r_0, r_0+\delta]$ for δ small enough. For the sake of brevity we will only consider the case $\phi_n'(r_0)=0$, $v_0'=0$, the other cases being handled similarly. According to Lemma 3,

$$\frac{(s-r_0)^{\frac{p-2}{p-1}}}{|\phi_n'(s)|^{p-2}}\leqslant C, \qquad r_0<s\leqslant r_0+\delta,$$

for some $\delta>0$ small and $C>0$. We have for every $v, w\in C[r_0, r_0+\delta]$,

$$\begin{aligned} |Tv(r)-Tw(r)| &\leqslant C\int_{r_0}^{r}\frac{1}{(s-r_0)^{\frac{p-2}{p-1}}}\int_{r_0}^{s}\lambda_{n,p}M\,|v-w|_\infty\,d\rho\,ds \\ &= \frac{\lambda_{n,p}CM}{p'}\,\delta^{p'}\,|v-w|_\infty, \end{aligned}$$

where $M := \sup_{r_0\leqslant s\leqslant r_0+\delta}|\phi_n(s)|^{p-2}$ (notice that $\phi_n\neq 0$ in a neighbourhood of r_0), and the contractivity of T for small δ follows. By Banach's contraction principle (4) has a unique solution in $r_0\leqslant r\leqslant r_0+\delta$.

Now let v be an eigenfunction associated to $\lambda_{n,p}$. Set $v_0=v(0)$ and observe that both v and $v_0/\phi_n(0)\,\phi_n$ are solutions to (4) with $r_0=0$. Hence $v=v_0/\phi_n(0)\,\phi_n$ in $0\leqslant r\leqslant\delta$. An standard continuation argument based upon the uniqueness of problem (4) leads to $v=v_0/\phi_n(0)\,\phi_n$ in $0\leqslant r\leqslant 1$. This concludes the proof. ∎

4. BIFURCATION RESULTS

Our first goal will be to study local bifurcation from the branch of trivial solutions $\{(\lambda, 0)\}_{\lambda \in \mathbb{R}}$ of the problem

$$\begin{aligned} -\Delta_p u &= \lambda \varphi_p(u) + g(\lambda, r, u) && \text{in } B \\ u &= 0, && \text{on } \partial B, \end{aligned} \tag{P}$$

where $\lambda > 0$ and g is a C^1 function satisfying $g(\lambda, r, 0) = 0$, $\partial g/\partial u = o(|u|^{p-2})$ as $u \to 0$ uniformly in bounded intervals of λ and $r \in [0, 1]$. For differentiability reasons, we will assume $p \geqslant 2$ throughout.

As in [4], we are looking for small solutions of the form $u = s(\phi_n + y)$, $s \sim 0$, ϕ_n is a radial eigenfunction (normalized as $\int_0^1 r^{N-1} |\phi_n(r)|^p \, dr = 1$) and y varying in some appropriate subspace of $\mathscr{F}$. However, and due to the lack of differentiability of K near $u = 0$, we must proceed in a completely different way than in [4] to show the uniqueness of this type of solutions.

Proof of Theorem 1. For the sake of brevity we will drop the subindex n in ϕ_n fixing also $\lambda_0 = \lambda_{n, p}$.

In $\mathscr{G}$ consider the closed subspace $\tilde{Y} = \{y \in \mathscr{G} : \int_0^1 r^{N-1} |\phi(r)|^{p-2} \phi(r) y(r) \, dr = 0\}$ (notice that $\tilde{Y} \cap \mathscr{F} = \mathscr{Y}$), and for $\lambda, s \in \mathbb{R}$ and $y \in \tilde{Y}$ define

$$H(s, \lambda, y) := \begin{cases} \phi + y - K\left(\lambda \varphi_p(\phi + y) + \dfrac{1}{\varphi_p(s)} G(s, \lambda, y)\right), & s \neq 0 \\ \phi + y - K(\lambda \varphi_p(\phi + y)), & s = 0, \end{cases}$$

where $G(s, \lambda, \cdot) \colon \mathbb{R} \times \mathbb{R} \times \mathscr{E} \to \mathscr{E}$ is the Nemytskii operator associated to $g(\lambda, r, s\phi + sy)$. By our hypothesis on g, and since $p \geqslant 2$, the mapping $(s, \lambda, y) \to \lambda \varphi_p(\phi + y) + G(s, \lambda, y)/\varphi_p(s)$ is continuous from $\mathbb{R} \times \mathbb{R} \times \mathscr{E}$ to $\mathscr{E}$ while is continuously differentiable with respect to $(\lambda, y) \in \mathbb{R} \times \mathscr{E}$ (but not in general with respect to $s \in \mathbb{R}$). Thus, H is continuous from $\mathbb{R}^2 \times \tilde{Y}$ to $\mathscr{G}$. Moreover, by Corollary 6, H is continuously differentiable with respect to (λ, y) in a neighbourhood of $(s, \lambda, y) = (0, \lambda_0, 0)$ in $\mathbb{R}^2 \times \tilde{Y}$ and values in $\mathscr{G}$. This amount of smoothness will be enough for our objectives. In addition, $H(0, \lambda_0, 0) = \phi - K(\lambda_0 \varphi_p(\phi)) = 0$, and

$$D_{(\lambda, y)} H(0, \lambda_0, 0)(\hat{\lambda}, \hat{y}) = \hat{y} - DK(\lambda_0 \varphi_p(\phi))(\lambda_0 (p-1) |\phi|^{p-2} \hat{y}) - \frac{1}{p-1} \frac{\hat{\lambda}}{\lambda_0} \phi.$$

Let us see that $D_{(\lambda, y)}H(0, \lambda_0, 0)\colon \mathbb{R}\times\tilde{Y}\to\mathcal{G}$ is an isomorphism. We will prove first that it is injective. For this aim take $(\hat{\lambda}, \hat{y})\in\mathbb{R}\times\tilde{Y}$ such that $D_{(\lambda, y)}H(0, \lambda_0, 0)(\hat{\lambda}, \hat{y})=0$. This implies that

$$-\left(r^{N-1}\,|\phi'|^{p-2}\left(\hat{y}'-\frac{1}{p-1}\frac{\hat{\lambda}}{\lambda_0}\phi'\right)\right)'=r^{N-1}\lambda_0\,|\phi|^{p-2}\,\hat{y},\qquad 0<r<1 \tag{5}$$
$$\hat{y}'(0)=\hat{y}(1)=0.$$

Multiplying Eq. (5) by ϕ and integrating by parts leads to $\hat{\lambda}\int_0^1 r^{N-1}\,|\phi'(r)|^p\,dr=0$, from which $\hat{\lambda}=0$ follows. But then $\hat{y}$ verifies

$$-(r^{N-1}\,|\phi'|^{p-2}\,\hat{y}')'=\lambda_0 r^{N-1}\,|\phi|^{p-2}\,\hat{y},\qquad 0<r<1$$
$$\hat{y}'(0)=\hat{y}(1)=0,$$

that is, $\hat{y}$ is an eigenfunction of the eigenvalue problem (3) considered in Section 3. By Theorem 7, $\hat{y}=c\phi$ for some $c\in\mathbb{R}$, and since $\hat{y}\in\tilde{Y}$, $c=0$, i.e., $\hat{y}=0$ follows.

To prove that $D_{(\lambda, y)}H(0, \lambda_0, 0)$ is surjective, let us first observe that $\mathcal{G}=\operatorname{span}\{\phi\}\oplus\tilde{Y}\approx\mathbb{R}\times\tilde{Y}$. On the other hand, the operator $\mathcal{L}\hat{y}=DK(\lambda_0\varphi_p(\phi))(\lambda_0(p-1)\,|\phi|^{p-2}\,\hat{y})$ leaves $\tilde{Y}$ invariant. To check this, it suffices with multiplying by ϕ both sides of the equation satisfied by $z=DK(\lambda_0\varphi_p(\phi))(\lambda_0(p-1)\,|\phi|^{p-2}\,\hat{y})$ and integrating by parts to arrive at $z\in\tilde{Y}$. Finally, observe that as an operator $D_{(\lambda, y)}H(0, \lambda_0, 0)\colon\mathbb{R}\times\tilde{Y}\to\mathbb{R}\times\tilde{Y}$ it can be written as,

$$D_{(\lambda, y)}H(0, \lambda_0, 0)(\hat{\lambda}, \hat{y})$$
$$=(\hat{\lambda}, \hat{y})-\left(\frac{1}{p-1}\frac{\hat{\lambda}}{\lambda_0}, DK(\lambda_0\varphi_p(\phi))(\lambda_0(p-1)\,|\phi|^{p-2}\,\hat{y})\right).$$

Thus, it defines a compact perturbation of the identity. Therefore, surjectivity is already implied by its nonsingular character.

The implicit function theorem (see the special version in the Appendix to [4]) can then be applied to conclude the existence of $\varepsilon, \delta>0$ and continuous functions $\lambda\colon(-\varepsilon, \varepsilon)\to\mathbb{R}$, $y\colon(-\varepsilon, \varepsilon)\to\tilde{Y}$ such that every solution of $H(s, \lambda, y)=0$ with $|\lambda-\lambda_0|<\delta$, $|s|<\delta$ and $\|y\|<\delta$ is of the form $\lambda=\lambda(s)$, $y=y(s)$. In particular, this provides us with a branch of nontrivial radial solutions $(\lambda(s), s(\phi+y(s)))$ to (P) emanating from $(\lambda, u)=(\lambda_0, 0)$.

To show the uniqueness assertion first observe that radial solutions $u\in W_0^{1,p}(B)\cap L^\infty(B)$ to (P) verify $u\in\mathcal{F}\subset\mathcal{G}$. Thus, any such solution can be written as $u=s\phi+w$, where $w\in\tilde{Y}$ and $s=\int_0^1 r^{N-1}\,|\phi(r)|^{p-2}\,\phi(r)\,u(r)\,dr$.

Suppose now that uniqueness does not hold. Then, there exist sequences $\tilde{\lambda}_m \to \lambda_0$ and $u_m \to 0$ such that $u_m = s_m \phi + w_m$ and $w_m \neq s_m\, y(s_m)$. Let $\tilde{u}_m = u_m / |u_m|_\infty$. Then

$$\tilde{u}_m = K\left(\tilde{\lambda}_m \varphi_p(\tilde{u}_m) + \frac{G(\tilde{\lambda}_m, u_m)}{|u_m|_\infty^{p-1}} \right).$$

Since $|\tilde{u}_m|_\infty = 1$ and $K\colon \mathscr{E} \to \mathscr{G}$ is a completely continuous operator, it follows that there exists a subsequence, denoted again by $\tilde{u}_m$ such that $\tilde{u}_m \to u_0$ in $\mathscr{G}$, with $|u_0|_\infty = 1$. Notice that u_0 verifies $u_0 = K(\lambda_0 \varphi_p(u_0))$ and thus the simplicity of λ_0 as an eigenvalue of $-\Delta_p$ gives $u_0 = c\phi$ for some $c \in \mathbb{R}$, $c \neq 0$. By using the previous integral for s in terms of u we have the convergence

$$\frac{s_m}{|u_m|_\infty} = \int_0^1 r^{N-1}\, |\phi(r)|^{p-2}\, \phi(r)\, \tilde{u}_m\, dr \to c.$$

But then $w_m / s_m = (|u_m|_\infty / s_m)\, \tilde{u}_m - \phi \to 0$. The uniqueness assertion given by the implicit function theorem then leads to the equality $w_m = s_m\, y(s_m)$, against the assumption.

To conclude the proof of Theorem 1 it only remains to show that $y\colon (-\varepsilon, \varepsilon) \to \mathscr{Y}$ is continuous (observe that $y(s) \in \mathscr{Y}$ for every $s \in (-\varepsilon, \varepsilon)$). In fact, assume that $s_m \to s_0 \in (-\varepsilon, \varepsilon)$. Then $\phi + y(s_m) \to \phi + y(s_0)$ in $\mathscr{G}$ and so in $\mathscr{E}$. The continuity of $K\colon \mathscr{E} \to \mathscr{F}$ and the own equation $\phi + y(s_m) = K(\lambda_0(s_m)\, \varphi_p(\phi + y(s_m)) + G(\lambda_0(s_m), s_m(\phi + y(s_m))) / \varphi_p(s_m))$ provides that $\phi + y(s_m) \to \phi + y(s_0)$ in $\mathscr{F}$. ∎

Remark 4 (*Bifurcation direction*). If g is slightly more smooth or has a suitable structure it is possible, as in the case $p = 2$ to compute the local bifurcation direction of each branch $\mathscr{C}_n$. For the purposes of Remark 5, let us only consider an special case of perturbation terms g in (P). Namely,

$$g(u) = \varphi_p(u)\, g_1(\varphi_{\gamma+1}(u)),$$

with $\varphi_{\gamma+1}(u) = |u|^{\gamma-1}\, u$ and where $\gamma > 0$, $g_1 \in C^1(\mathbb{R})$, $g_1(0) = 0$, $g_1'(0) = a$. It is now very convenient to search for solutions (λ, u) close to $(\lambda_{n,p}, 0)$ in the two-scale form,

$$u = s(\phi_n + y(\cdot, \sigma)) \qquad \lambda = \lambda(\sigma), \tag{6}$$

where $\sigma = |s|^{\theta-1}\, s$, θ to be determined. We will be searching for C^1 mappings $y(\cdot, \sigma) \in \mathscr{Y}_n$, $\lambda(\sigma) \sim \lambda_{n,p}$, $|\sigma|$ small. It is precisely the value of $\lambda'(0) = \frac{d\lambda}{d\sigma}(0)$ which will give, when non zero, the bifurcation direction. Proceeding as in the proof of Theorem 1, the expression (6) for the solutions leads to the equation,

$$(\phi_n+y)-K(\lambda\varphi_p(\phi_n+y)+\varphi_p(\phi_n+y)\,G_1(|s|^{\gamma-1}\,s\varphi_{\gamma+1}(\phi_n+y)))=0. \quad (7)$$

Setting $\sigma=|s|^{\gamma-1}\,s$ (i.e., $\theta=\gamma$) permits observing (7) as an equation which is C^1 in (σ, λ, y) and that defines, now by means of the standard implicit function theorem, $y=y(\cdot\,,\sigma)\in\mathscr{Y}_n$, $\lambda=\lambda(\sigma)$ as C^1 mappings near $\sigma=0$. In addition, the same argument as in Theorem 1 ensures that every radial solution $(\lambda, u)\in\mathbb{R}\times(W_0^{1,p}(B)\cap L^\infty(B))$ to (P) close to $(\lambda_{n,p}, 0)$ can be expressed as (6) for certain $\sigma\sim 0$.

To find $\lambda'(0)$ it suffices with differentiating (7) at $\sigma=0$ to arrive at

$$y_1-DK(\lambda_{n,p}\varphi_p(\phi_n))[(p-1)\,\lambda_{n,p}\,|\phi_n|^{p-2}\,y_1+\lambda'(0)\,\varphi_p(\phi_n)+a\varphi_{p+\gamma+1}(\phi_n)]=0.$$

This means that

$$\begin{aligned}-(p-1)(r^{N-1}\,|\phi_n|^{p-2}\,y_1)=r^{N-1}\{&(p-1)\,\lambda_{n,p}\,|\phi_n|^{p-2}\,y_1+\lambda'(0)\,\varphi_p(\phi_n)\\ &+a\varphi_{p+\gamma+1}(\phi_n)\}\\ y_1'(0)=y_1(1)=0,&\end{aligned}$$

with $0<r<1$. Multiplying the equation by ϕ_n and integrating by parts give the desired value for $\lambda'(0)$,

$$\lambda'(0)=-\int_0^1 a\,|\phi_n|^{p+\gamma+1}\,r^{N-1}\,dr.$$

Notice that we have produced the expression $\lambda=\lambda_{n,p}+\lambda'(0)\,|s|^{\gamma-1}\,s+o(|s|^\gamma)$ as $s\to 0$. Thus, $\lambda'(0)\neq 0$ certainly permits determining the bifurcation direction.

Let us proceed now to the proof of Theorem 2.

Proof of Theorem 2. Theorem 1 ensures that the connected component of $(\lambda_{n,p}, 0)$ in $\bar{\mathscr{S}}$ is nontrivial, i. e. $\mathscr{C}_n\neq\{(\lambda_{n,p}, 0)\}$ for every $n\in\mathbb{N}$. Moreover, a standard argument in topological degree permits showing that either $\mathscr{C}_n$ is unbounded or $\mathscr{C}_n$ meets some point $(\bar{\lambda}, 0)$, $\bar{\lambda}\neq\lambda_{n,p}$ (see [17, 5]).

First consider the case $n=1$ and designate by $\mathscr{C}^+_{1,\varepsilon}$(respectively $\mathscr{C}^-_{1,\varepsilon}$) $=\{(\lambda, u)=(\lambda_1(s), s(\phi_1+y_1(\cdot\,,s))) : 0<s<\varepsilon$ (respectively, $-\varepsilon<s<0)\}$. Since both $\mathscr{C}^\pm_{1,\varepsilon}$ are connected, the connected components $\mathscr{C}^\pm_1$ in $\mathscr{C}_1\setminus\{(\lambda_{1,p}, 0)\}$ containing $\mathscr{C}^\pm_{1,\varepsilon}$ will exist.

It should be remarked that thanks to the local uniqueness

$$\begin{aligned}&\mathscr{C}_1\cap\{|\lambda-\lambda_{1,p}|<\varepsilon_1, |u|_{C_0^1(B)}<\varepsilon_1\}\\ &\quad=(\mathscr{C}^+_{1,\varepsilon}\cup\{(\lambda_{1,p}, 0)\}\cup\mathscr{C}^-_{1,\varepsilon})\cap\{|\lambda-\lambda_{1,p}|<\varepsilon_1, |u|_{C_0^1(B)}<\varepsilon_1\}\end{aligned}$$

for $\varepsilon_1 > 0$ small. This means that $\mathscr{C}_1 \setminus \{(\lambda_{1,p}, 0)\}$ has at most $\mathscr{C}_1^+$ and $\mathscr{C}_1^-$ as connected components, but it could be also possible that $\mathscr{C}_1^+ = \mathscr{C}_1^-$.

However, we are next proving that $\mathscr{C}_1^\pm \subset \mathbb{R} \times \mathscr{S}_1^\pm$. This will imply that $\mathscr{C}_1^+ \cap \mathscr{C}_1^- = \varnothing$ (so $\mathscr{C}_1 \setminus \{(\lambda_{1,p}, 0)\}$ has exactly two components) while $(\lambda_{1,p}, 0)$ is the unique trivial solution in $\mathscr{C}_1$. Therefore, $\mathscr{C}_1$ will be unbounded.

To show that $\mathscr{C}_1^+ \subset \mathbb{R} \times \mathscr{S}_1^+$ let us use a connectedness argument and set $C_1^+ := \{(\lambda, u) \in \mathscr{C}_1^+ : u \in \mathscr{S}_1^+\}$. It is clear that for $\varepsilon > 0$ small, $\mathscr{C}_{1,\varepsilon}^+ \subset C_1^+$ so $C_1^+ \neq \varnothing$ while C_1^+ is open in $\mathscr{C}_1^+$. On the other hand, let $\{(u_n, \lambda_n)\} \subset C_1^+$ such that $(\lambda_n, u_n) \to (\bar{\lambda}, \bar{u})$ in $\mathscr{C}_1^+$. Then $(\bar{\lambda}, \bar{u}) \neq (\lambda_{1,p}, 0)$ ($(\lambda_{1,p}, 0) \notin \mathscr{C}_1^+$) and so $\bar{u} \neq 0$ since a well-known compactness argument shows that $\bar{u} = 0$ is only compatible with $\bar{\lambda} = \lambda_{1,p}$. Thus $\bar{u} \geqslant 0$ with $\bar{u} \not\equiv 0$ in B. The strong maximum principle (cf. [19]) implies then that $u \in \mathscr{S}_1$. Therefore, $C_1^+ = \mathscr{C}_1^+$ and $\mathscr{C}_1^+ \subset \mathbb{R} \times \mathscr{S}_1^+$. Arguing in the same way leads to $\mathscr{C}_1^- \subset \mathbb{R} \times \mathscr{S}_1^-$.

Let us show now that both $\mathscr{C}_1^\pm$ are unbounded. For this aim, we introduce the auxiliary problem,

$$\begin{aligned} -\Delta_p u &= \lambda \varphi_p(u) + \tilde{g}(\lambda, r, u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{$(P)'$}$$

$\tilde{g}(\lambda, r, u) = g(\lambda, r, u)$ if $u \geqslant 0$, $\tilde{g}(\lambda, r, u) = -g(\lambda, r, -u)$ otherwise. Our previous argument shows that an unbounded continuum $\tilde{\mathscr{C}}_1 \subset \mathbb{R} \times C_0^1(B)$ bifurcates from $(\lambda_{1,p}, 0)$ and can be split into

$$\tilde{\mathscr{C}}_1 = \tilde{\mathscr{C}}_1^+ \cup \{(\lambda_{1,p}, 0)\} \cup \tilde{\mathscr{C}}_1^-,$$

with $\tilde{\mathscr{C}}_1^\pm$ connected, $\tilde{\mathscr{C}}_1^\pm \subset \mathbb{R} \times \mathscr{S}_1^\pm$. Since, by local uniqueness and symmetry, $\tilde{\mathscr{C}}_1^- = -\tilde{\mathscr{C}}_1^+$ it follows that both $\tilde{\mathscr{C}}_1^\pm$ are unbounded. However, it is clear that $\tilde{\mathscr{C}}_1^+ \subset \mathscr{C}_1^+$. Therefore $\mathscr{C}_1^+$ must be unbounded. A symmetric argument shows that $\mathscr{C}_1^-$ is also unbounded.

Finally, the same ideas give the structure of $\mathscr{C}_n$ for arbitrary n. In fact the maximum principle step should be replaced by the nodal properties of the eigenfunctions ϕ_n and the functions in $\mathscr{S}_n^\pm$. This concludes the proof. ∎

Remark 5 *(Multiplicity of solutions).* We are next involved in discussing the nonuniqueness of *positive* solutions to the class of perturbed logistic problems,

$$\begin{aligned} -\Delta_p u &= \lambda u^{p-1} - u^q + f(u), && x \in B \\ u &= 0, && x \in \partial B, \end{aligned} \tag{8}$$

where $q > p-1 \geqslant 1$, $f \in C^1(\mathbb{R})$, while $f = o(|u|^{p-1})$ as $u \to 0$, $f = o(u^q)$ when $u \to +\infty$. It has been recently shown in [9] that (8) exhibits a *unique* positive solution $u = u_\lambda \in W_0^{1,p}(B) \cap L^\infty(B)$ for each $\lambda \geqslant \bar{\lambda}$ and some large enough $\bar{\lambda} \geqslant \lambda_{1,p}$. This solution is in addition found to be radially symmetric (recall Remark 2(b)).

We are next showing that this is the best possible result by means of constructing a suitable perturbation term f for which (8) exhibit two positive radial solutions for $\lambda \sim \lambda_{1,p}$.

We first observe that no positive solutions to (8) are possible for $\lambda < \lambda^*$, with $\lambda^* := \inf_{u \in \mathbb{R}^+} \{u^{q-p+1} - f(u)/u^{p-1}\}$ (notice that $\lambda^* \leqslant 0$). On the other hand, uniform "a priori" L^∞ bounds can be easily produced for all positive solutions u to (8) with λ varying in bounded intervals of $\lambda \geqslant \lambda^*$. In fact, every positive weak solution $u \in W_0^{1,p}(B)$ to (8) satisfies $0 \leqslant u \leqslant u_0(\lambda)$ in B, where $u_0(\lambda)$ is the maximum positive zero to the right-side member in (8) (cf. [9, Theorem 4.1, 10] for more general results). This means that the subcontinuum of positive radial solutions $\mathscr{C}_1^+$ in Theorem 2, connects $(\lambda_{1,p}, 0)$ to the smooth branch $\{(\lambda, u_\lambda)\}_{\lambda \geqslant \bar{\lambda}}$. In other words,

$$\{\lambda \in \mathbb{R} : \exists u \in C_0^1(B) \text{ a positive radial solution to (8) with } (\lambda, u) \in \mathscr{C}_1^+\}$$
$$\subset [\lambda^*, +\infty).$$

On the other hand, set $\gamma = q-p+1$ and choose $f(u) = \varphi_p(u)\, f_1(\varphi_{\gamma+1}(u))$ with $f_1 \in C^1(\mathbb{R})$, $f_1(0) = 0$, $f_1'(0) = a_1$ while $f(z) = o(z)$ as $z \to +\infty$. In the layout of Remark 4 the nonlinearity g in (P) takes now the form

$$g(u) = \varphi_p(u)[f_1(\varphi_{\gamma+1}(u)) - \varphi_{\gamma+1}(u)].$$

Thus, $\lambda'(0) < 0$ provided a_1 is taken so that $a_1 > 1$.

Finally, the local uniqueness of positive solutions in $\lambda < \lambda_{1,p}$, $\lambda \sim \lambda_{1,p}$ if $a_1 > 1$, the corresponding nonexistence of small positive solutions for $\lambda > \lambda_{1,p}$, $\lambda \sim \lambda_{1,p}$ together with the connectedness of $\mathscr{C}_1^+$ imply the existence of at least two radial positive solutions u_λ^1, $u_\lambda^2 \in W_0^{1,p}(B) \cap L^\infty(B)$ for $\lambda < \lambda_{1,p}$, $\lambda \sim \lambda_{1,p}$, as desired.

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