NONEXISTENCE OF POSITIVE SUPERSOLUTIONS TO SOME NONLINEAR ELLIPTIC PROBLEMS

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ABSTRACT. In this paper we obtain Liouville type theorems for positive supersolutions of the elliptic problem $-\Delta u + |\nabla u|^q = \lambda f(u)$ in exterior domains of $\mathbb{R}^N$. Here $q > 1$ and the function f can be compared with a power p near zero or infinity. We show that positive supersolutions do not exist in some ranges of the parameters p and q which turn out to be optimal for the model case $f(s) = s^p$. The related problem $-\Delta u - |\nabla u|^q = f(u)$ is also analyzed.

1. INTRODUCTION

Nonlinear Liouville theorems play an important role in the study of some nonlinear partial differential equations of elliptic type. As a typical application, they are used to obtain a priori bounds for solutions to such equations. An example of this is the by now classical result in [17], which in turns relies on the nonexistence theorem of [16]. In the latter it is shown that the problem

$$-\Delta u = u^p \quad \text{in } \mathbb{R}^N$$

does not admit any positive solution provided that $p < \frac{N+2}{N-2}$ ($N \geq 3$), see also [11]. It was later proved in [15] that even positive supersolutions of this equation cannot exist with the more restrictive assumption $p \leq \frac{N}{N-2}$ (see Theorem 8.4 in [27] for a simple proof of this assertion; this restriction on the exponent is optimal).

Subsequently, this result has been generalized to deal with some more general semilinear, quasilinear and fully nonlinear elliptic equations and systems. Without being exhaustive with the huge amount of references concerned with this topic, let us mention the works [2], [4], [5], [6], [7], [8], [9], [11], [12], [19], [20], [25] and [30]. For a more extensive list of references, we refer to the book by Véron [32] and to the survey by Kondratiev, Liskevich and Sobol [21].

A rather general result in this line has been recently obtained by Armstrong and Sirakov [3]. Among other things, they showed that the differential inequality

$$-Qu \geq f(u) \qquad \text{in } \mathbb{R}^N \setminus B_{R_0}$$

where $N \geq 3$ and Q is a fully nonlinear operator does not admit positive (viscosity) solutions provided that f is continuous and positive in $(0, \infty)$ and verifies a condition at zero related with the fundamental solution of Q. Notice that this condition reduces to

$$\liminf_{s \to 0+} \frac{f(s)}{s^{\frac{N}{N-2}}} > 0,$$

when Q is the Laplacian. This approach works due to the homogeneity of the differential operator Q, since it is essential to obtain Hadamard type results (cf. also [11]).

Thus a natural question to ask is whether the previous results are modified when the differential operator is not homogeneous. This is precisely the question we are addressing in the present paper. As a model problem, we will be mainly dealing with

$$-\Delta u + |\nabla u|^q = \lambda f(u) \quad \text{in } \mathbb{R}^N \setminus B_{R_0}, \tag{1.1}$$

where $N \geq 3$, $q > 1$ and $f : (0, \infty) \to \mathbb{R}$ is continuous and positive. This problem has been considered in the reference case $f(u) = u^p$, $p > 0$, by Chipot and Weissler [10] and later by Serrin and Zou [29] and Voirol [33] (cf. the extension to the $p-$Laplacian setting considered in [13], [14]). See also [31] for a survey on the problem. However, the authors of these papers were interested in the existence and nonexistence of *radial solutions* in $\mathbb{R}^N$.

We will be always dealing with positive classical supersolutions to (1.1), that is, functions $u \in C^2(\mathbb{R}^N \setminus B_{R_0})$ verifying (1.1) point-wise. However, we are not making any assumption about asymptotic behavior of the supersolutions near infinity, nor do we assume that they are bounded. In this regard, let us mention that the presence of the gradient term in (1.1) allows the existence of supersolutions which blow up at infinity, that is,

$$\lim_{|x| \to \infty} u(x) = +\infty. \tag{1.2}$$

Hence, to be more precise, we will distinguish in what follows between supersolutions which blow up at infinity and those which do not. Let us remark that in the case where supersolutions are being considered in the whole space $\mathbb{R}^N$, they can not blow up at infinity. This follows because, according to the maximum principle, $\min_{|x|=R} u(x) = \min_{|x| \leq R} u(x)$, and this is a decreasing (hence bounded) function of R.

Let us next state our main theorems. We begin by considering positive supersolutions which do not verify (1.2). As for the nonlinear term $f(u)$, aside being positive and continuous in $(0, \infty)$, we will assume that

$$\gamma_1 := \liminf_{s \to 0} \frac{f(s)}{s^p} > 0, \tag{1.3}$$

for some $p > 0$. The first nonexistence result is concerned with a case which could be termed as "subcritical".

Theorem 1. *Assume $f : (0, \infty) \to \mathbb{R}$ is a continuous positive function verifying* (1.3). *If $1 < q < \frac{N}{N-1}$ and $p < \frac{q}{2-q}$ or $q \geq \frac{N}{N-1}$ and $p < \frac{N}{N-2}$, then there are no classical positive supersolutions to* (1.1) *which do not blow up at infinity.*

Let us consider now the first "critical" case, that in which $p = \frac{q}{2-q}$, but with $1 < q < \frac{N}{N-1}$ (the problem can be termed critical because of the scale invariance in the reference situation $f(u) = u^p$, $p > 0$: if u is a supersolution then $\mu^{\frac{2-q}{q-1}} u(\mu x)$ also is for every $\mu > 0$). To deal with it we need to assume in addition that the function f is nondecreasing.

It turns out that in this case the presence of the parameter λ in (1.1) is important, because the existence or nonexistence of positive supersolutions depends on the size of λ. Actually, observe that a supersolution to (1.1) for some value of $\tilde{\lambda}$ is also a supersolution for every value $\lambda < \tilde{\lambda}$. Thus we can define

$$\lambda^* = \inf \left\{ \begin{array}{c} \lambda > 0: \text{ there are no positive supersolutions} \\ \text{for (1.1) which do not blow up at infinity} \end{array} \right\} \tag{1.4}$$

and we obtain that the set of values of λ for which a positive supersolution to (1.1) does not exist is precisely one of the intervals (λ^*, ∞) or $[\lambda^*, \infty)$, provided that $\lambda^* < \infty$. Our next result shows that this is indeed the situation and gives some bounds for the value λ^*.

Theorem 2. *Assume $f : (0, \infty) \to \mathbb{R}$ is a continuous, nondecreasing positive function verifying* (1.3) *and $1 < q < \frac{N}{N-1}$, $p = \frac{q}{2-q}$. Then the value λ^* given in* (1.4) *verifies:*

$$\lambda^* \leq \frac{2-q}{2\gamma_1} \left(\frac{q(2-q)}{2(N - q(N-1))} \right)^{\frac{q}{2-q}}. \tag{1.5}$$

If in addition

$$\gamma_2 := \limsup_{s \to 0} \frac{f(s)}{s^p} < \infty,$$

then

$$\lambda^* \geq \frac{2-q}{2\gamma_2} \left(\frac{q(2-q)}{2(N - q(N-1))} \right)^{\frac{q}{2-q}}$$

and for $\lambda \in (0, \lambda^*)$ there exist positive supersolutions to (1.1) *which do not blow up at infinity.*

Now we consider the second critical case, that in which $q > \frac{N}{N-1}$ and $p = \frac{N}{N-2}$. In this situation, the parameter λ is again of no importance.

Theorem 3. *Assume $f : (0, \infty) \to \mathbb{R}$ is a continuous, nondecreasing positive function verifying* (1.3) *and $q > \frac{N}{N-1}$, $p = \frac{N}{N-2}$. Then problem* (1.1) *does not admit positive supersolutions which do not blow up at infinity.*

It is worth remarking that in all cases not covered by the previous theorems, positive supersolutions can be constructed at least for the model case $f(s) = s^p$. In fact, in most cases positive *solutions* exist (see [10], [29], [33]). Therefore our results are seen to be optimal.

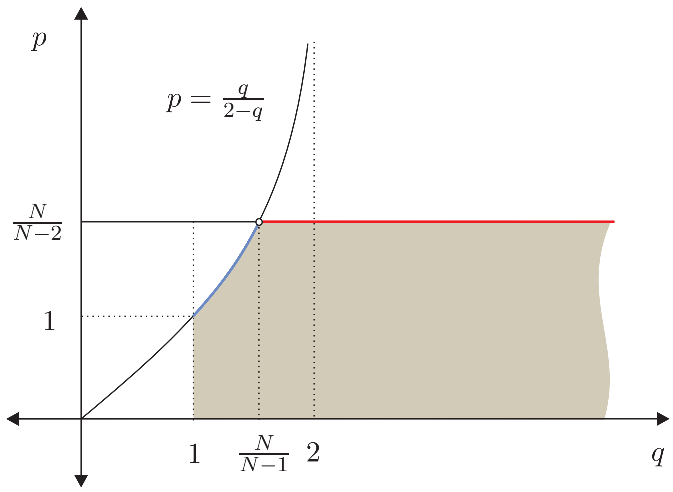

FIGURE 1. Regions of nonexistence of positive supersolutions which do not blow up at infinity in the plane q, p.

Last but not least, we consider positive supersolutions to (1.1) blowing up at infinity. In this case, the behavior of f at zero is of course not important, and the key assumption is:

$$\liminf_{s\to\infty} \frac{f(s)}{s^p} > 0, \tag{1.6}$$

for some $p > 0$. Our result regarding this case is the following:

Theorem 4. *Assume $f : (0,\infty) \to \mathbb{R}$ is a continuous positive function verifying* (1.6) *for some $p > q$. Then there are no positive classical supersolutions to* (1.1) *blowing up at infinity.*

When $\limsup_{s\to\infty} f(s)/s^p < \infty$ for some $p \le q$, it is easy to show that positive supersolutions blowing up at infinity exist: they are just exponential functions of the form $u(x) = e^{\mu|x|}$. Thus Theorem 4 is essentially optimal as well.

When the supersolutions are assumed to be defined throughout $\mathbb{R}^N$ instead of in exterior domains, the results are almost exactly the same as before, except that supersolutions which blow-up at infinity do not exist, and the value λ^* in Theorem 2 cannot be directly estimated from below.

Let us make some comments on our methods of proof. Our method relies in analyzing the function $m(R) = \min_{|x|=R} u(x)$. By means of a device introduced in [11], we obtain an inequality for this function, which, after an iterative procedure, leads us to some suitable upper bounds. At the same time, some lower bounds can be produced, but it is precisely here that the nonhomogeneity of the left-hand side in (1.1) prevents us from obtaining a Hadamard type property, hence we are forced to obtain the properties of $m(R)$ by means of a slightly different (more involved) approach.

In the subcritical cases, the upper bound and the lower bound are in contradiction, hence the proof of nonexistence. However, in both critical cases there is no contradiction whatsoever so that a finer analysis is due. Let us explicitly mention that this proof is completely different to that in

[11], which does not seem to be generalized when a gradient term comes into play.

Finally, it is important to notice that the essential feature in all proofs is the comparison principle. In the framework of classical solutions which we have chosen, the comparison is furnished for instance by Theorem 10.1 in [18]. But the reader will notice that almost all results have an extension to some more general nonlinear elliptic problems like

$$-Q(u) + |\nabla u|^q = \lambda f(u) \qquad \text{in } \mathbb{R}^N \setminus B_{R_0},$$

where Q is a fully nonlinear operator invariant under rotation and supersolutions are considered in the viscosity sense (cf. [3] when the gradient term is absent).

Likewise, some similar problems can be treated with much the same techniques. As an example, we briefly consider the related problem

$$-\Delta u - |\nabla u|^q = f(u) \quad \text{in } \mathbb{R}^N \setminus B_{R_0}, \tag{1.7}$$

which is slightly simpler than (1.1), since its supersolutions are superharmonic functions.

We finally mention that our results have immediate implications on the analysis of the growth of positive solutions to equations of the type $-\Delta u = f(x, u, \nabla u)$ near a singularity, in the spirit of [26]. In particular, Theorem 6.1 there can be extended to deal with nonlinearities with a critical growth in the gradient (see also the comments right after Theorem 6.1 in [26]).

The paper is organized as follows: in Section 2 we deal with some preliminary properties of the supersolutions, in particular of the function $m(R) = \inf_{|x|=R} u(x)$ and Section 3 is devoted to the proof of Theorems 1, 2 and 3. In Section 4 we consider supersolutions to (1.1) which blow up at infinity, while the final Section 5 contains nonexistence theorems for problem (1.7) which can be achieved with similar methods.

2. Some preliminaries

In this section we perform a preliminary analysis of supersolutions of (1.1). Thus if u is a supersolution of (1.1), we mainly deal with monotonicity properties and upper estimates for the function:

$$m(R) = \min_{|x|=R} u(x), \tag{2.1}$$

for $R > R_0$. Observe that, thanks to the strong maximum principle, and enlarging R_0 if necessary, we can always assume that $u > 0$ in $\mathbb{R}^N \setminus B_{R_0}$, so that $m(R) > 0$ if $R > R_0$.

When our problem is analyzed in $\mathbb{R}^N$, this minimum coincides with the minimum in the ball of radius R by the maximum principle, hence $m(R)$ is a decreasing function. In the case at hand, however, this need not be so. Let us begin by clearing this point.

Lemma 5. *Let $u \in C^2(\mathbb{R}^N \setminus B_{R_0})$ be a positive function verifying $-\Delta u + |\nabla u|^q \geq 0$ in $\mathbb{R}^N \setminus B_{R_0}$ with $q > 1$ and $m(R)$ be given by (2.1). Then there exists $R_1 > R_0$ such that $m(R)$ is monotone for $R > R_1$.*

Proof. For $R_2 > R_1 > R_0$, consider the annulus $A(R_1, R_2) = \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$. Thanks to the maximum principle applied to u in this annulus we obtain

$$\min_{A(R_1,R_2)} u = \min\{m(R_1), m(R_2)\}.$$

Since the annulus decreases with R_1 and increases with R_2, we obtain $\min_{A(R_1,R_2)} u$ increases with R_1 and decreases with R_2. Therefore the function $\min\{m(R_1), m(R_2)\}$ is increasing in R_1 and decreasing in R_2 whenever $R_2 > R_1 > R_0$.

It follows from this property that $m(R)$ cannot have a strict local minimum. Indeed, if we had a strict local minimum at some R then we could choose $\delta_1 > 0$ so that $m(R) < m(R - \delta_1)$, $m(R) < m(R + \delta_1)$. As we have seen, the function

$$h(\delta) = \min\{m(R - \delta), m(R + \delta)\}$$

is decreasing in $0 \le \delta \le \delta_1$, contradicting the fact that $h(\delta_1) > h(0)$.

Since local minima cannot exist, we obtain one of the three alternatives: either $m(R)$ is increasing, or it is decreasing, or it is increasing in (R_0, R_1) for some $R_1 > R_0$ and then decreasing in (R_1, ∞). Whatever the case, $m(R)$ is monotone for $R > R_1$. This concludes the proof. □

To proceed further, we will assume that f is positive in $(0, \infty)$, and that it has a prescribed behavior near zero. More precisely, we will impose the hypothesis

$$\liminf_{s \to 0} \frac{f(s)}{s^p} > 0 \tag{2.2}$$

for some $p > 0$. The proof of the next result is based on a device introduced in [11] and refined in [12].

Lemma 6. *Let $q > 1$ and $u \in C^2(\mathbb{R}^N \setminus B_{R_0})$ be a positive supersolution to (1.1) in $\mathbb{R}^N \setminus B_{R_0}$ and $m(R)$ be given by (2.1). Then*

(a) *If $p > 0$ and $m(R)$ is bounded, then $m(R)$ is decreasing for large R and it converges to zero as $R \to \infty$. Moreover, if f verifies (2.2), then there exists $C > 0$ such that*

$$m(2R)^p \le C\left(\frac{m(R)}{R^2} + \frac{m(R)^q}{R^q}\right) \tag{2.3}$$

for $R > R_0$.

(b) *If $m(R)$ is unbounded, then $m(R)$ is increasing for large R and it diverges to infinity as $R \to \infty$.*

Proof. Choose a cut-off function $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi = 0$ in $(0, 1) \cup (4, \infty)$, $0 \le \varphi \le 1$ and $\varphi = 1$ in $[\frac{3}{2}, \frac{7}{2}]$. For $R > R_0$ consider the function

$$v(x) = u(x) - m(2R)\varphi\left(\frac{|x|}{R}\right).$$

Observe that there exists a point x_R with $|x_R| = 2R$ and $u(x_R) = m(2R)$, so that $v(x_R) = 0$. Since $v > 0$ in $B_R \setminus B_{R_0}$ and in $\mathbb{R}^N \setminus B_{4R}$, we conclude that

v achieves a nonpositive minimum at some point y_R with $R \le |y_R| \le 4R$. This implies $\Delta v(y_R) \ge 0$, $\nabla v(y_R) = 0$, so that

$$-\Delta u(y_R) + |\nabla u(y_R)|^q \le -\frac{m(2R)}{R^2}\Delta\varphi\left(\frac{|y_R|}{R}\right) + \frac{m(2R)^q}{R^q}\left|\nabla\varphi\left(\frac{|y_R|}{R}\right)\right|^q$$
$$\le C\left(\frac{m(2R)}{R^2} + \frac{m(2R)^q}{R^q}\right),$$

where C is a positive constant which only depends on φ (from now on, we will use the letter C to denote positive constants, not necessarily the same everywhere), and in particular since u is a supersolution of (1.1), we obtain

$$\lambda f(u(y_R)) \le C\left(\frac{m(2R)}{R^2} + \frac{m(2R)^q}{R^q}\right). \tag{2.4}$$

Notice that we also have $u(y_R) \le m(2R)$, since the minimum of v is nonpositive and $\varphi \le 1$.

Now we show part (a). Since $m(R)$ is bounded, we deduce from (2.4) that $f(u(y_R)) \to 0$ as $R \to \infty$, and since $u(y_R) \le C$, then it follows that $u(y_R) \to 0$. Hence Lemma 5 implies that $m(R)$ is decreasing for large R and $\lim_{R\to\infty} m(R) = 0$. From (2.2) we then achieve $f(u(y_R)) \ge Cu(y_R)^p \ge Cm(4R)^p$. Hence

$$m(4R)^p \le C\left(\frac{m(2R)}{R^2} + \frac{m(2R)^q}{R^q}\right),$$

and part (a) follows.

Showing part (b) is straightforward. Indeed, since $m(R)$ is unbounded, then $m(R)$ has to be increasing for $R > R_0$, again by Lemma 5, so that $\lim_{R\to\infty} m(R) = \infty$. □

The last two results in the section are key points, since they allow to extract upper bounds for the function $m(R)$ by means of the inequality (2.3). This upper bounds are essential to achieve the nonexistence of supersolutions to (1.1).

Lemma 7. *Let $q > 1$, $p > 0$ and $h(R)$ be a positive decreasing function defined for $R > R_0$ and verifying*

$$h(2R)^p \le C\left(\frac{h(R)}{R^2} + \frac{h(R)^q}{R^q}\right) \tag{2.5}$$

for $R > R_0$ and some positive constant $C > 0$. Then:

(a) *If $1 < q < 2$ and $0 < p \le 1$, then for every $\theta > 0$ there exists a constant C such that*

$$h(R) \le CR^{-\theta}$$

for every $R > R_0$.

(b) *If $1 < q < 2$ and $1 < p < \frac{q}{2-q}$, then for every $\theta \in (0, \frac{2}{p-1})$ there exists a constant C such that*

$$h(R) \le CR^{-\theta}$$

for every $R > R_0$.

(c) *If* $q \geq 2$ *and* $0 < p \leq 1$*, then for every* $\theta > 0$ *there exists a constant* C *such that*
$$h(R) \leq CR^{-\theta}$$
for every $R > R_0$.

(d) *If* $q \geq 2$ *and* $p > 1$*, then for every* $\theta \in (0, \frac{2}{p-1})$ *there exists a constant* C *such that*
$$h(R) \leq CR^{-\theta}$$
for every $R > R_0$.

Proof. We consider first the case $1 < q < 2$ and $0 < p < \frac{q}{2-q}$. Since $h(R)$ is bounded, we have from (2.5) $h(2R) \leq CR^{-\gamma_0}$, where $\gamma_0 = \frac{q}{p}$. Assume that $\gamma_0 \leq \alpha := \frac{2-q}{q-1}$. Then
$$\begin{aligned} h(2R)^p &\leq C(R^{-\gamma_0-2} + R^{-\gamma_0 q - q}) = CR^{-\gamma_0 q - q}(R^{\gamma_0 q + q - \gamma_0 - 2} + 1) \\ &\leq CR^{-\gamma_0 q - q}, \end{aligned}$$
since $\gamma_0 q + q - \gamma_0 - 2 \leq 0$. This yields $h(2R) \leq CR^{-\gamma_1}$, where $\gamma_1 = \frac{\gamma_0 q + q}{p}$. We can iterate this argument: if $h(2R) \leq CR^{-\gamma_{k-1}}$ for some $\gamma_{k-1} \leq \alpha$, then
$$h(2R) \leq CR^{-\gamma_k},$$
where $\gamma_k = \frac{\gamma_{k-1} q + q}{p}$. Now we observe that $\gamma_1 > \gamma_0$, this implying that the sequence $\{\gamma_k\}$ is increasing. If we had $\gamma_k \leq \alpha$ for every k, then we would obtain that γ_k converges to some value $\bar{\gamma} \leq \alpha$. It follows that $\bar{\gamma} = \frac{q}{p-q}$ (this is only possible if $p > q$), and then $\frac{q}{p-q} \leq \frac{2-q}{q-1}$, which implies $p \geq \frac{q}{2-q}$, against the assumption.

Thus there exists a first value of k, called k_0, such that $\gamma_{k_0} > \alpha$. Of course, this holds with $k_0 = 0$ when $\frac{q}{p} > \alpha$. Then

(2.6)
$$\begin{aligned} h(2R)^p &\leq C(R^{-\gamma_{k_0}-2} + R^{-\gamma_{k_0} q - q}) = CR^{-\gamma_{k_0}-2}(1 + R^{\gamma_{k_0}+2-\gamma_{k_0} q - q}) \\ &\leq CR^{-\gamma_{k_0}-2}, \end{aligned}$$
and we obtain $h(2R) \leq CR^{-\tau_0}$, where $\tau_0 = \frac{\gamma_{k_0}+2}{p}$. We can iterate similarly as before to obtain an increasing sequence $\{\tau_j\}$ given by $\tau_j = \frac{\tau_{j-1}+2}{p}$ and $h(R) \leq CR^{-\tau_j}$ for every j. Since $\tau_{j+1} - \tau_j = (\tau_j - \tau_{j-1})/p$, we deduce $\tau_{j+1} = \tau_1 + (\tau_1 - \tau_0)\sum_{k=1}^{j} p^{-k}$. Therefore, when $0 < p \leq 1$, $\tau_j \to \infty$, which shows part (a). When $1 < p < \frac{q}{2-q}$,
$$\tau_j \to \tau_\infty = \tau_1 + (\tau_1 - \tau_0)\frac{1}{p-1} = \frac{2}{p-1},$$
which proves part (b). Observe that we need only a finite number of iterations to reach each value of θ in the statement, and therefore we do not need to keep track of the different constants appearing in each step.

Let us prove parts (c) and (d), so we assume $q \geq 2$ and $p > 0$. We notice that $\gamma_0 = \frac{2}{p} < \frac{2}{p-1}$ and we can argue using (2.6) to construct an increasing sequence γ_k given by $\gamma_k = \frac{\gamma_{k-1}+2}{p}$. We arrive as in parts (a) and (b) at $\gamma_k \to \infty$ when $0 < p \leq 1$, while $\gamma_k \to \frac{2}{p-1}$ if $p > 1$, and the lemma is thus proved. □

The bounds provided by Lemma 7 can be improved to reach the critical value $\frac{2}{p-1}$ in the exponents if we can compare the values of $h(2R)$ and $h(R)$ for large R (this will be indeed the situation in Section 3). In the next result we only state the bounds we are going to need later.

Lemma 8. *Let $q > 1$, $p > 0$ and $h(R)$ be a positive decreasing function defined for $R > R_0$ and verifying*

$$h(R)^p \leq C\left(\frac{h(R)}{R^2} + \frac{h(R)^q}{R^q}\right) \tag{2.7}$$

for $R > R_0$ and some positive constant $C > 0$. Then:

(a) *If $1 < q < 2$ and $p = \frac{q}{2-q}$, then there exists a constant C such that*

$$h(R) \leq CR^{-\frac{2}{p-1}}$$

for $R > R_0$.

(b) *If $q > \frac{N}{N-1}$ and $p = \frac{N}{N-2}$, then there exists a constant C such that*

$$h(R) \leq CR^{-(N-2)}$$

for $R > R_0$.

Proof. The proof is based in the same iterative argument as in Lemma 7, but having a precise control on the constants appearing in each step.

Let us proceed with the proof of part (a). Without loss of generality we assume $R_0 > 1$, and also that the constant C in (2.7) is such that $2C > 1$. We have initially $h(R) \leq C_0 R^{-\gamma_0}$ for $R > R_0$, where $\gamma_0 = \frac{q}{p}$. Taking C_0 larger if necessary we may also assume

$$C_0 > (2C)^{\frac{1}{p-q}} > 1$$

(observe that $p > q$ in this case). Then

$$\begin{aligned} h(R)^p &\leq C\left(\frac{C_0 R^{-\gamma_0}}{R^2} + \frac{C_0^q R^{-\gamma_0 q}}{R^q}\right) \leq CC_0^q (R^{-\gamma_0 - 2} + R^{-\gamma_0 q - q}) \\ &\leq CC_0^q R^{-\gamma_0 q - q}(R^{\gamma_0 q + q - \gamma_0 - 2} + 1) \leq 2CC_0^q R^{-\gamma_0 q - q}, \end{aligned}$$

since $\gamma_0 = \frac{q}{p} = 2 - q < \alpha = \frac{2-q}{q-1}$ and $R > 1$. We deduce that $h(R) \leq C_1 R^{-\gamma_1}$ for $R > R_0$, where $C_1 = (2CC_0^q)^{\frac{1}{p}}$ and $\gamma_1 = \frac{\gamma_0 q + q}{p}$. Observe that $(2C)^{\frac{1}{p-q}} < C_1 < C_0$ and $\gamma_1 > \gamma_0$.

We can iterate this procedure to obtain a decreasing sequence $\{C_k\}$ with $C_k > (2C)^{\frac{1}{p-q}}$ and an increasing sequence $\{\gamma_k\}$ with $\gamma_k \leq \alpha$ such that $h(R) \leq C_k R^{-\gamma_k}$ for $R > R_0$. It is not hard to show that $C_k \to \bar{C} = (2C)^{\frac{1}{p-q}}$, $\gamma_k \to \alpha$ as k goes to infinity. Passing to the limit we achieve $h(R) \leq \bar{C} R^{-\alpha}$ for $R > R_0$, and the proof concludes by noticing that $\alpha = \frac{2}{p-1}$.

To show (b), observe that, since h is bounded, (2.7) implies

$$h(R)^p \leq C\left(\frac{h(R)}{R^2} + \frac{h(R)^{q'}}{R^{q'}}\right)$$

for $R > R_0$ and every $q' < q$ (enlarging R_0 if necessary). Hence we can always assume that q is close to $\frac{N}{N-1}$, so that $p > q$ and $\frac{q}{p} < \alpha$. Thus the first part of the proof applies and we obtain that $h(R) \le CR^{-\alpha}$ for some $C>0$.

Now we iterate as in Lemma 7 to obtain sequences τ_j and C_j such that $h(R) \le C_j R^{-\tau_j}$, where $\tau_{j+1} = \tau_1 + (\tau_1 - \tau_0)\sum_{k=1}^{j} p^{-k}$ and $C_j = (2CC_{j-1}^q)^{\frac{1}{p}}$. Thus $\tau_j \to \frac{2}{p-1} = N-2$ while $C_j \to \bar{C} = (2C)^{\frac{1}{p-q}}$. This concludes the proof. □

3. Proof of Theorems 1, 2 and 3

In this section we collect the proof of our main results dealing with positive supersolutions which do not blow up at infinity. We begin by considering the "subcritical" case contained in Theorem 1, which is considerably easier than those covered by Theorems 2 and 3.

Proof of Theorem 1. We analyze separately the cases: $1 < q < \frac{N}{N-1}$ and $0 < p < \frac{q}{2-q}$, and $q \ge \frac{N}{N-1}$ and $0 < p < \frac{N}{N-2}$.

Case: $1 < q < \frac{N}{N-1}$ *and* $0 < p < \frac{q}{2-q}$. We first remark that the function $\Phi(x) = A|x|^{-\alpha}$ is a subsolution to the equation $-\Delta v + |\nabla v|^q = 0$ in $\mathbb{R}^N \setminus \{0\}$, where

$$\alpha := \frac{2-q}{q-1} \quad \text{and} \quad 0 < A \le A_0 = \frac{1}{\alpha}(\alpha + 2 - N)^{\frac{1}{q-1}}.$$

Let u be a positive classical supersolution to (1.1) in $\mathbb{R}^N \setminus B_{R_0}$ which does not blow up at infinity and denote, as in Section 2, $m(R) = \min_{|x|=R} u(x)$. By Lemma 6, $m(R)$ is decreasing for large R and $m(R) \to 0$ as $R \to \infty$. We claim that

$$\liminf_{R\to\infty} m(R) R^{\alpha} > 0. \tag{3.1}$$

Assuming the contrary, there exists a sequence $R_k \to \infty$ with $m(R_k)R_k^{\alpha} \to 0$. Fix k_0 large enough so that $m(R_{k_0})R_{k_0}^{\alpha} < A_0$. The function

$$\Psi(x) = \frac{m(R_{k_0}) - m(R_k)}{R_{k_0}^{-\alpha} - R_k^{-\alpha}}(|x|^{-\alpha} - R_k^{-\alpha}) + m(R_k)$$

is a subsolution to $-\Delta v + |\nabla v|^q = 0$ in the annulus $A(R_{k_0}, R_k)$ provided that k is sufficiently large. Since $\Psi \le u$ on $\partial A(R_{k_0}, R_k)$, by comparison (see for example [18] or [28]) we have $\Psi \le u$ in $A(R_{k_0}, R_k)$. Letting $k \to \infty$ we achieve

$$u(x) \ge m(R_{k_0}) R_{k_0}^{\alpha} |x|^{-\alpha} \quad \text{in } |x| > R_{k_0}.$$

In particular, taking x such that $|x| = R_j$, we get $m(R_j)R_j^{\alpha} \ge m(R_{k_0})R_{k_0}^{\alpha}$ if $R_j > R_{k_0}$. Letting $j \to \infty$ we arrive at a contradiction with $m(R_j)R_j^{\alpha} \to 0$. This shows the claim.

To conclude the proof in this case, we just notice that, thanks to Lemmas 6 (a) and 7 (a) and (b), we have

$$m(R) \le CR^{-\theta} \quad \text{for large } R,$$

where $\theta > 0$ is arbitrary if $0 < p \le 1$ and $0 < \theta < \frac{2}{p-1}$ if $1 < p < \frac{q}{2-q}$. When combined with (3.1), we have an immediate contradiction if $0 < p \le 1$ and also when $1 < p < \frac{q}{2-q}$, since this implies that $\alpha < \frac{2}{p-1}$.

Case: $q \geq \frac{N}{N-1}$ *and* $0 < p < \frac{N}{N-2}$. It can be easily checked in this case that the function $\widetilde{\Phi}(x) = A|x|^{2-N-\varepsilon}$ is a subsolution to the equation $-\Delta v + |\nabla v|^q = 0$ in $|x| > 1$ when ε is small enough and

$$0 < A \leq \widetilde{A}_0 = \frac{\varepsilon^{\frac{1}{q-1}}}{N-2+\varepsilon}.$$

If u is a positive classical supersolution to (1.1), we claim that

$$\liminf_{R\to\infty} m(R) R^{N-2+\varepsilon} > 0. \tag{3.2}$$

If this were not true, there would exist a sequence $R_k \to \infty$ such that $m(R_k)R_k^{N-2+\varepsilon} \to 0$. In this case, choosing k_0 such that $m(R_{k_0})R_{k_0}^{N-2+\varepsilon} < \widetilde{A}_0$ and then a large k, the function

$$\widetilde{\Psi}(x) = \frac{m(R_{k_0}) - m(R_k)}{R_{k_0}^{2-N-\varepsilon} - R_k^{2-N-\varepsilon}} (|x|^{2-N-\varepsilon} - R_k^{2-N-\varepsilon}) + m(R_k)$$

is a subsolution to $-\Delta v + |\nabla v|^q = 0$ in the annulus $A(R_{k_0}, R_k)$ with $\widetilde{\Psi} \leq u$ on $\partial A(R_{k_0}, R_k)$. By comparison we get $\widetilde{\Psi} \leq u$ in $A(R_{k_0}, R_k)$. Letting $k \to \infty$, we obtain $u(x) \geq m(R_{k_0})R_{k_0}^{N-2+\varepsilon}|x|^{2-N+\varepsilon}$ in $|x| > R_{k_0}$, and we reach a contradiction when we take x with $|x| = R_j$ and let $j \to \infty$. This shows the claim.

Observe next that $0 < p < \frac{q}{2-q}$ when $\frac{N}{N-1} \leq q < 2$, so that Lemma 6 together with Lemma 7 can be applied to obtain $m(R) \leq CR^{-\theta}$, where $\theta > 0$ is arbitrary for $0 < p \leq 1$ and $\frac{N}{N-1} \leq q < 2$, while $\theta \in (0, \frac{2}{p-1})$ for $1 < p < \frac{N}{N-2}$ and $q \geq \frac{N}{N-1}$. Thanks to (3.2), we obtain a contradiction if $N - 2 + \varepsilon < \theta$, which is certainly possible if ε is selected small enough. This concludes the proof. □

The first critical case: $1 < q < \frac{N}{N-1}$ *and* $p = \frac{q}{2-q}$.

To deal with the critical cases, some extra work needs to be done. The first step in proving Theorem 2 is to show that the function $m(R)R^\alpha$ is increasing, where α is as in the proof of Theorem 1. This is not so direct as in the case where the gradient term is absent, due to the non homogeneity of the operator. Recall that we are assuming that f is nondecreasing and verifies (1.3).

Notice that thanks to (1.3), there exists $\delta > 0$ such that $f(s) \geq (\gamma_1 - \eta)s^p$ for $0 < s \leq \delta$ if $\eta > 0$ is sufficiently small. Now let us fix $\eta > 0$ sufficiently small and observe that Lemma 6 implies that $m(R)$ is decreasing for large R and $m(R) \to 0$ as $R \to \infty$, so that enlarging R_0 if necessary, we may always have $m(R) \leq m(R_0) \leq \delta$ if $R > R_0$.

Let us begin by assuming that λ has been chosen so that

$$-\alpha(\alpha + 2 - N)s + \alpha^q s^q < \lambda(\gamma_1 - \eta)s^p \quad \text{for every } s > 0 \text{ and } \eta \text{ small}. \tag{3.3}$$

It is not hard to check that (3.3) holds provided

$$\lambda > \frac{2-q}{2\gamma_1} \left(\frac{q(2-q)}{2(N - q(N-1))} \right)^{\frac{q}{2-q}}. \tag{3.4}$$

and η is small enough (cf. [10]). Recall that we are considering $p = \frac{q}{2-q} > q$.

Lemma 9. *Let* $1 < q < \frac{N}{N-1}$ *and* u *be a positive classical supersolution to* (1.1) *which does not blow-up at infinity, where* f *is nondecreasing and verifies* (1.3) *with* $p = \frac{q}{2-q}$. *Assume* (3.3) *holds. Then the function* $m(R)R^\alpha$ *is increasing.*

Proof. Let $A_0 = 0$, and for every positive k define A_k to be the unique positive solution of the equation

$$-\alpha(\alpha+2-N)A_k + \alpha^q A_k^q = \lambda(\gamma_1-\eta)A_{k-1}^p.$$

It easily follows that $\{A_k\}$ is an increasing sequence, since $1 < q < \frac{N}{N-1}$ implies that $\alpha+2-N > 0$. Moreover, $A_k \to \infty$, since if it were bounded we would have $A_k \to \bar{A}$, which should be a solution to $-\alpha(\alpha+2-N)\bar{A}+\alpha^q\bar{A}^q = \lambda(\gamma_1-\eta)\bar{A}^p$, contradicting (3.3).

Let us see that $m(R)R^\alpha$ is increasing as long as $0 < m(R)R^\alpha < A_1$. Indeed, if $R_1 > R_0$ is such that $m(R_1)R_1^\alpha < A_1$, then the function

$$\Psi(x) = \frac{m(R_1)-m(R_2)}{R_1^{-\alpha}-R_2^{-\alpha}}(|x|^{-\alpha}-R_2^{-\alpha}) + m(R_2)$$

verifies

$$-\Delta\Psi + |\nabla\Psi|^q = \alpha C_{R_1,R_2}\left((\alpha C_{R_1,R_2})^{q-1} - \alpha - 2 + N\right)|x|^{-\alpha-2} < 0$$

for large R_2, where $C_{R_1,R_2} = \frac{m(R_1)-m(R_2)}{R_1^{-\alpha}-R_2^{-\alpha}}$, since $0 < C_{R_1,R_2} < A_1$ for sufficiently large R_2, and

$$A_1^{q-1} = \frac{\alpha+2-N}{\alpha^{q-1}}.$$

Also note that $\Psi \le u$ on $\partial A(R_1,R_2)$. On the other hand, since

$$-\Delta u + |\nabla u|^q \ge 0 \quad \text{in } A(R_1,R_2),$$

by comparison we get $u \ge \Psi$ in $A(R_1,R_2)$. Letting $R_2 \to \infty$ we have $u(x) \ge m(R_1)R_1^\alpha|x|^{-\alpha}$ in $|x| > R_1$. In particular $m(R)R^\alpha \ge m(R_1)R_1^\alpha$ if $R > R_1$. This shows that $m(R)R^\alpha$ is increasing as long as $m(R)R^\alpha < A_1$, and also by continuity that if $m(R_1)R_1^\alpha = A_1$ for some $R_1 > R_0$, then $m(R)R^\alpha \ge A_1$ for every $R > R_1$.

In the latter case we would have $u(x) \ge A_1|x|^{-\alpha}$ for $|x| > R_1$ and since f is nondecreasing, we obtain $f(u(x)) \ge f(A_1|x|^{-\alpha})$. Observe that $A_1|x|^{-\alpha} \le A_1R_1^{-\alpha} = m(R_1) \le \delta$, so that u verifies $-\Delta u + |\nabla u|^q \ge \lambda(\gamma_1-\eta)A_1^p|x|^{-\alpha p}$ if $|x| > R_1$. Now assume that $R_2 > R_1$ is such that $m(R_2)R_2^\alpha < A_2$. Consider the function

$$\widetilde{\Psi}(x) = \frac{m(R_2)-m(R_3)}{R_2^{-\alpha}-R_3^{-\alpha}}(|x|^{-\alpha}-R_3^{-\alpha}) + m(R_3)$$

for $R_3 > R_2 > R_1$. If R_3 is large enough, we have $\widetilde{\Psi}$ verifies $-\Delta\widetilde{\Psi}+|\nabla\widetilde{\Psi}|^q \le \lambda(\gamma_1-\eta)A_1^p|x|^{-\alpha p}$. By comparison, $u \ge \widetilde{\Psi}$ in $A(R_2,R_3)$, and letting $R_3 \to \infty$ we obtain as before that $m(R)R^\alpha \ge m(R_2)R_2^\alpha$ if $R > R_2$.

The same argument shows that, whenever $A_{k-1} < m(R)R^\alpha < A_k$ for some R, then $m(R)R^\alpha$ is increasing as long as it is less than A_k, and if it reaches this value, it stays above this value. Since $A_k \to \infty$, we have that the function $m(R)R^\alpha$ is increasing, as we wanted to prove. □

Once the monotonicity of $m(R)R^\alpha$ has been established, the proof of nonexistence can be carried out. We remark again that our proof is completely different than that of the critical case in [11].

Lemma 10. *Let* $1 < q < \frac{N}{N-1}$ *and assume that* f *is nondecreasing and verifies* (1.3) *with* $p = \frac{q}{2-q}$. *Suppose in addition that* (3.3) *holds. Then there are no positive supersolutions to* (1.1) *which do not blow-up at infinity.*

Proof. According to Lemma 9, the function $m(R)R^\alpha$ is increasing for large R. In particular, we deduce that $m(R) \le 2^\alpha m(2R)$ for large R, and then Lemma 6 (a) gives that $m(R)$ verifies (2.7) for some positive constant C and large R. By Lemma 8 (a), we deduce that $m(R)R^\alpha$ is bounded. Denote

$$l = \lim_{R\to\infty} m(R)R^\alpha.$$

Choose a small $\varepsilon > 0$. There exists $R_{1,\varepsilon} > R_0$ such that $m(R)R^\alpha \ge l-\varepsilon$ if $R > R_{1,\varepsilon}$, so that $u(x) \ge (l-\varepsilon)|x|^{-\alpha}$ if $|x| > R_{1,\varepsilon}$. Take a point $x_{0,\varepsilon} \in \mathbb{R}^N$ with $|x_{0,\varepsilon}| > 2R_{1,\varepsilon}$ and $u(x_{0,\varepsilon}) = m(|x_{0,\varepsilon}|)$. Introduce the function

$$v_\varepsilon(y) = |x_{0,\varepsilon}|^\alpha u(x_{0,\varepsilon} + |x_{0,\varepsilon}|y) \quad y \in B_{1/2},$$

where $B_{1/2}$ stands for the ball of radius $1/2$ centered at zero. It is not hard to see that the function v_ε verifies $v_\varepsilon(0) = |x_{0,\varepsilon}|^\alpha u(x_{0,\varepsilon}) = |x_{0,\varepsilon}|^\alpha m(|x_{0,\varepsilon}|) \le l$, together with

$$v_\varepsilon(y) \ge (l-\varepsilon)|x_{0,\varepsilon}|^\alpha |x_{0,\varepsilon} + |x_{0,\varepsilon}|y|^{-\alpha} = (l-\varepsilon)|e_\varepsilon + y|^{-\alpha} \quad y \in B_{1/2},$$

and $-\Delta v_\varepsilon + |\nabla v_\varepsilon|^q \ge \lambda(\gamma_1-\eta)(l-\varepsilon)^p|e_\varepsilon+y|^{-\alpha p}$ in $B_{1/2}$, where $e_\varepsilon = x_{0,\varepsilon}/|x_{0,\varepsilon}|$ (notice that $(l-\varepsilon)R_{1,\varepsilon}^{-\alpha} \le m(R_{1,\varepsilon}) \le \delta$). In particular, v_ε is a supersolution of the problem

$$(3.5) \qquad \left\{ \begin{array}{ll} -\Delta w + |\nabla w|^q = \lambda(\gamma_1-\eta)(l-\varepsilon)^p|e_\varepsilon+y|^{-\alpha p} & \text{in } B_{1/2} \\ w = (l-\varepsilon)|e_\varepsilon+y|^{-\alpha} & \text{on } \partial B_{1/2}. \end{array} \right.$$

Since $\underline{v}(y) = (l-\varepsilon)|e_\varepsilon + y|^{-\alpha}$ is a subsolution to the same problem, we deduce by using the standard method of sub and supersolutions (cf. [1]) that there exists a classical solution w_ε to (3.5), verifying in addition $w_\varepsilon(y) \ge (l-\varepsilon)|e_\varepsilon+y|^{-\alpha}$ in $B_{1/2}$, $l-\varepsilon \le w_\varepsilon(0) \le v_\varepsilon(0) \le l$.

Our next intention is to pass to the limit as $\varepsilon \to 0$. For this aim, we are obtaining appropriate bounds for the solutions w_ε. Notice that since $|e_\varepsilon| = 1$, we have $|e_\varepsilon + y| \ge \frac{1}{2}$ for every $y \in B_{1/2}$. Thus $-\Delta w_\varepsilon \le 2^{\alpha p}\gamma_1 l^p \lambda$ in $B_{1/2}$, with $w_\varepsilon \le 2^\alpha l$ on $\partial B_{1/2}$ and by comparison we obtain that w_ε is uniformly bounded.

On the other hand, all first derivatives of the right-hand side in (3.5) are also uniformly bounded, so that we can use Theorem A.1 in [22] to obtain uniform local bounds for $|\nabla w_\varepsilon|$ in $B_{1/2}$. This provides with uniform local bounds for $|\Delta w_\varepsilon|$, so that by standard regularity (cf. [18]) we get local bounds for $|w_\varepsilon|_{C^{1,\gamma}}$ for every $\gamma \in (0,1)$. These entail local bounds for $|\Delta w_\varepsilon|_{C^\gamma}$, which provide with local bounds for $|w_\varepsilon|_{C^{2,\gamma}}$. In particular, we have

$$|w_\varepsilon|_{C^{2,\gamma}(\overline{B_{1/4}})} \le C$$

where C is independent of ε. Thus there exists a sequence $\varepsilon_n \to 0$ such that $w_{\varepsilon_n} \to w$ in $C^2(\overline{B_{1/4}})$. By passing to a further subsequence we may also assume that $e_{\varepsilon_n} \to e_0$ where $|e_0| = 1$. Then w verifies:

$$-\Delta w + |\nabla w|^q = \lambda l^p (\gamma_1 - \eta)|e_0 + y|^{-\alpha p} \qquad \text{in } B_{1/4},$$

together with $w(y) \geq l|e_0 + y|^{-\alpha}$ in $B_{1/4}$ and $w(0) = l$. If we let $z = |e_0 + y|^\alpha w(y)$, it easily follows that $z \geq l$ in $B_{1/4}$, $z(0) = l$ and

$$\begin{aligned} &-\alpha(\alpha + 2 - N)z + 2\alpha(e_0 + y)\nabla z - |e_0 + y|^2 \Delta z \\ &\quad + \big||e_0 + y|\nabla z - \alpha |e_0 + y|^{-1}(e_0 + y)z\big|^q = \lambda(\gamma_1 - \eta)l^p. \end{aligned} \tag{3.6}$$

To conclude just notice that z has a minimum at $y = 0$, so that $\Delta z(0) \geq 0$, $\nabla z(0) = 0$, and we obtain from (3.6):

$$-\alpha(\alpha + 2 - N)l + \alpha^q l^q \geq \lambda(\gamma_1 - \eta)l^p,$$

which is a contradiction to the hypothesis (3.3) since $l > 0$. This contradiction proves the lemma. □

Proof of Theorem 2. The inequality (1.5) follows from (3.4) and Lemma 10 and the last assertion of the theorem is a direct consequence of Proposition 5.5 in [10]. □

The second critical case: $q > \frac{N}{N-1}$ *and* $p = \frac{N}{N-2}$.

Now we turn our attention to the second critical case, that contained in Theorem 3. Recall that we are assuming f to be nondecreasing and verifying (1.3) with $p = \frac{N}{N-2}$, while $q > \frac{N}{N-1}$. There are some differences in the proof depending on whether $p \geq q$ or $p < q$, and this entails that the monotonicity of $m(R)R^{N-2}$ is difficult to prove. However, it suffices to have less information.

Lemma 11. *Let* $q > \frac{N}{N-1}$. *Assume* u *is a positive supersolution to* (1.1) *which does not blow up at infinity and* f *is nondecreasing and verifies* (1.3) *with* $p = \frac{N}{N-2}$. *Then the function* $m(R)R^{N-2}$ *is bounded from below.*

Proof. Since $q > \frac{N}{N-1}$, we can choose a small ε so that $q(N-1) - N - \varepsilon\frac{N}{N-2} > 0$. Observe that $m(R) \geq C_0 R^{2-N-\varepsilon}$ for some positive constant C_0, by (3.2) in the proof of Theorem 1, so that $u(x) \geq C_0|x|^{2-N-\varepsilon}$ and since f is nondecreasing then u is a supersolution to

$$-\Delta u + |\nabla u|^q = C_1|x|^{-N-\varepsilon\frac{N}{N-2}} \tag{3.7}$$

if $|x| > R$ and R is large, for some $C_1 > 0$. Assume that there exists a sequence $R_n \to \infty$ such that $m(R_n)R_n^{N-2} \to 0$. Then we may choose and fix a large enough k to have $m(R_k)R_k^{N-2} < \frac{C_1^{1/q}}{N-2}$ (we may as well suppose $R_k > 1$), and this yields

$$\frac{m(R_k) - m(R_n)}{R_k^{2-N} - R_n^{2-N}} < \frac{C_1^{1/q}}{N-2}$$

for large enough n. This and our choice of ε entail that the function

$$\Psi(x) = \frac{m(R_k) - m(R_n)}{R_k^{2-N} - R_n^{2-N}}(|x|^{2-N} - R_n^{2-N}) + m(R_n)$$

is a subsolution to (3.7) in $A(R_k, R_n)$ (recall that $|x| > 1$ there). Since $\Psi \le u$ on $\partial A(R_k, R_n)$, we have by comparison that $\Psi \le u$ in $A(R_k, R_n)$. Letting $n \to \infty$ we arrive at $u(x) \ge m(R_k)R_k^{N-2}|x|^{2-N}$ when $|x| > R_k$. Taking x with $|x| = R_j$ for large enough j, we arrive at

$$m(R_j)R_j^{N-2} \ge m(R_k)R_k^{N-2}.$$

We arrive at a contradiction when we let again $j \to \infty$. This contradiction concludes the proof. □

Lemma 12. *Under the same hypotheses of Lemma 11, for every sufficiently small $\nu > 0$, there exists $\widetilde{R} > R_0$ such that the function $m(R)R^{N-2+\nu}$ is increasing for $R > \widetilde{R}$.*

Proof. For $R_1, B > 0$, denote by $S(R_1, B)$ the unique solution A to

$$-\nu(N-2+\nu)AR_1^{\theta} + A^q(N-2+\nu)^q = \lambda(\gamma_1 - \eta)B^pR_1^{\theta - \frac{2\nu}{N-2}}, \tag{3.8}$$

where $\theta = q(N-1+\nu) - N - \nu$ (taking ν sufficiently small we can achieve $\theta > 0$ due to $q > \frac{N}{N-1}$) and define recursively $A_0(R_1) = 0$, $A_{k+1}(R_1) = S(R_1, A_k(R_1))$. Then it can be easily checked that the sequence $A_k(R_1)$ is increasing in k. When $p \ge q$, if R_1 is taken large enough, it follows that $A_k(R_1) \to \infty$ as $k \to \infty$, while for $p < q$ we have $A_k(R_1) \to \bar{A}(R_1)$, which is the smallest solution of the equation

$$-\nu(N-2+\nu)AR_1^{\theta} + A^q(N-2+\nu)^q = \lambda(\gamma_1 - \eta)A^pR_1^{\theta - \frac{2\nu}{N-2}}. \tag{3.9}$$

It is important to remark that, as a direct consequence of (3.9), $\bar{A}(R_1) \ge CR_1^{\frac{\theta - \frac{2\nu}{N-2}}{q-p}}$, where C is a positive constant which does not depend on R_1.

Next, observe that if we had $u(x) \ge B|x|^{2-N-\nu}$ for $|x| > R_1 \ge R_0$, then as in the previous subsection:

$$-\Delta u + |\nabla u|^q \ge \lambda(\gamma_1 - \eta)B^p|x|^{-N-\nu\frac{N}{N-2}} \tag{3.10}$$

if $|x| > R_1$, where η is small. Notice that the function $\underline{u} = A|x|^{2-N-\nu}$ is a subsolution to (3.10) if $|x| \ge R_1$ (we may assume with no loss of generality that $R_0 > 1$) provided that (3.8) holds.

When $p \ge q$, the proof is just an adaptation of that of Lemma 9, so we skip it. Hence we will assume in the rest of the proof that $p < q$. Observe that, thanks to Lemmas 6 (a) and 7 (b) and (d), we have $m(R) \le CR^{2-N+\nu}$, so that $m(R)R^{N-2+\nu} \le CR^{2\nu}$. By diminishing ν if necessary, we may also achieve $2\nu < (\theta - \frac{2\nu}{N-2})/(q-p)$, so that for large R_1, $m(R_1)R_1^{N-2+\nu} < \bar{A}(R_1)$. We assume in what follows that R_1 is so large as to have this inequality.

Arguing similarly as in Lemma 9 we obtain that $m(R)R^{N-2+\nu}$ is increasing as long as $R > R_1$ and $A_k(R_1) < m(R)R^{N-2+\nu} < A_{k+1}(R_1)$, this yielding that $m(R)R^{N-2+\nu}$ is increasing as long as $R > R_1$ and $0 < m(R)R^{N-2+\nu} < \bar{A}(R_1)$.

Define

$$A_\infty = \sup\left\{\bar{A} : \ m(R)R^{N-2+\nu} \text{ increases in the range } (m(R_1)R_1^{N-2+\nu}, \bar{A})\right\}.$$

If $A_\infty = \infty$ then we are done. If $A_\infty < \infty$, then we have two situations: (i) $m(R)R^{N-2+\nu} < A_\infty$ for every R, in which case the conclusion is trivial, or (ii) $m(\bar{R})\bar{R}^{N-2+\nu} = A_\infty$ for some $\bar{R}$, where we can repeat the above argument by setting $A_0(\bar{R}) = A_\infty$ to deduce that $m(R)R^{N+2-\nu}$ is increasing as long as $A_\infty < m(R)R^{N+2-\nu} < \bar{A}(\bar{R})$, which contradicts the maximality of A_∞. Hence this last possibility cannot occur, and this concludes the proof of the lemma. □

The actual proof of Theorem 3 is in spirit very close to that of Lemma 10. There is one important difference when dealing with the method of sub and supersolutions, since $q > 2$ is possible here and the standard method does not work unless the sub and the supersolution are chosen properly.

Proof of Theorem 3. Choose and fix a small $\nu > 0$ (less than one, say) such that Lemma 12 holds. We then have $m(R) \le 2^{N-2+\nu} m(2R) \le 2^{N-1} m(2R)$ if $R > R_1$. Since $m(R) > 0$ for every R, the inequality $m(R) \le Cm(2R)$ is valid for every R and a suitable positive constant C. Hence we may use Lemmas 6 (a) and 8 (b) to achieve that $m(R)R^{N-2}$ is bounded. Since it is also bounded away from zero by Lemma 11, we can define

$$l = \liminf_{R\to\infty} m(R)R^{N-2} > 0.$$

For fixed small $\varepsilon > 0$, there exists $R_{1,\varepsilon} > R_0$ such that $m(R)R^{N-2} \ge l - \varepsilon$ if $R \ge R_{1,\varepsilon}$. Hence $u(x) \ge (l-\varepsilon)|x|^{2-N}$ if $|x| \ge R_{1,\varepsilon}$. Choose $x_{0,\varepsilon} \in \mathbb{R}^N$ with $|x_{0,\varepsilon}| \ge \max\{2R_{1,\varepsilon}, \varepsilon^{-1}\}$ (so that $|x_{0,\varepsilon}| \to \infty$ when $\varepsilon \to 0$) and $u(x_{0,\varepsilon}) = m(|x_{0,\varepsilon}|) \le l + \varepsilon$. Define

$$v_\varepsilon(y) = |x_{0,\varepsilon}|^{N-2} u(x_{0,\varepsilon} + |x_{0,\varepsilon}|y) \quad y \in B_{1/2}.$$

Then we have $v_\varepsilon(y) \ge (l-\varepsilon)|e_\varepsilon + y|^{2-N}$, for some unit vector e_ε, and

$$-\Delta v_\varepsilon + |x_{0,\varepsilon}|^{N-q(N-1)} |\nabla v_\varepsilon|^q \ge \lambda(\gamma_1 - \eta)(l-\varepsilon)^p |e_\varepsilon + y|^{-N} \qquad \text{in } B_{1/2}.$$

Also $v_\varepsilon(0) \le l + \varepsilon$.

Next, if $\delta > 0$ is fixed, we have $|x_{0,\varepsilon}|^{N-q(N-1)} \le \delta$ if ε is small enough, so that v_ε is a supersolution to the problem

$$\begin{cases} -\Delta w + \delta|\nabla w|^q = \lambda(\gamma_1 - \eta)(l-\varepsilon)^p |e_\varepsilon + y|^{-N} & \text{in } B_{1/2} \\ w = (l-\varepsilon)|e_\varepsilon + y|^{2-N} & \text{on } \partial B_{1/2}. \end{cases} \tag{3.11}$$

Also $\underline{v} := (l-\varepsilon)|e_\varepsilon + y|^{2-N}$ is a subsolution and they are ordered, but we cannot directly apply the method of sub and supersolutions since $q > 2$ is possible. Thus we need to work as in [23]. Let $\bar{v}$ be the unique solution to

$$\begin{cases} -\Delta v = \lambda(\gamma_1 - \eta)(l-\varepsilon)^p |e_\varepsilon + y|^{-N} & \text{in } B_{1/2} \\ v = (l-\varepsilon)|e_\varepsilon + y|^{2-N} & \text{on } \partial B_{1/2}, \end{cases}$$

which is a supersolution to (3.11). By comparison $\bar{v} \ge \underline{v}$ in $B_{1/2}$. Thus we may use Theorem III.1 in [23] (we remark that the proof there can be adapted to deal with nonhomogeneous Dirichlet problems, as long as the sub and supersolution coincide on the boundary, as in our case) to obtain that (3.11) has a solution w_ε with $\underline{v} \le w_\varepsilon \le \overline{v}$ in $B_{1/2}$. By comparison it also follows that $w \le v_\varepsilon$. As a consequence, $l - \varepsilon \le w_\varepsilon(0) \le l + \varepsilon$.

We can now pass to the limit exactly as in the proof of Lemma 10, to arrive at a solution w to the equation

$$-\Delta w + \delta |\nabla w|^q = (\gamma_1 - \eta) l^p |e_0 + y|^{-N} \quad \text{in } B_{1/4}$$

verifying $w \geq l|e_0 + y|^{2-N}$, $w(0) = l$, for some unit vector e_0. Setting $z = |e_0 + y|^{N-2} w$, we obtain a function which has a minimum at $y = 0$ and verifies

$$\begin{aligned} &2(N-2)|e_0 + y|^{-N}(e_0 + y)\nabla z - |e_0 + y|^{2-N} \Delta z \\ &\quad + \delta \left| |e_0 + y|^{2-N} \nabla z - (N-2)|e_0 + y|^{-N}(e_0 + y) z \right|^q = (\gamma_1 - \eta) l^p |e_0 + y|^{-N} \end{aligned}$$

in $B_{1/4}$. Using $\Delta z(0) \geq 0$, $\nabla z(0) = 0$, $z(0) = l$ we obtain

$$\delta (N-2)^q l^q \geq (\gamma_1 - \eta) l^p.$$

Letting $\delta \to 0$ we deduce $l = 0$, a contradiction. The proof is concluded. □

4. Supersolutions which blow up at infinity

We turn now to the question of positive supersolutions to (1.1) which blow up at infinity. It is worth mentioning that the approach to prove Theorem 4 is essentially different to the one followed in the previous theorems. The proof actually relies in reducing the problem to a one-dimensional situation, where the condition $p > q$ is easily seen to be responsible for the nonexistence.

Lemma 13. *Assume $q > 1$, $p > 0$, and that there exists a positive function $u \in C^2(\mathbb{R}^N \setminus B_{R_1})$ verifying*

$$-\Delta u + |\nabla u|^q \geq C u^p \quad \text{in } \mathbb{R}^N \setminus B_{R_1}$$

and $\lim_{x \to \infty} u(x) = +\infty$. Then there exists an increasing positive function $v \in C^2(R_1, \infty)$ such that $\lim_{r \to \infty} v(r) = +\infty$ and

$$-v''(r) + v'(r)^q \geq C v(r)^p \tag{4.1}$$

for $r > R_1$.

Proof. Choose $R_2 > R_1$ and consider the problem

$$\begin{cases} -\Delta z + |\nabla z|^q = C m(|x|)^p & \text{in } A(R_1, R_2) \\ z = 0 & \text{on } \partial A(R_1, R_2), \end{cases} \tag{4.2}$$

where the function m is given by (2.1) and $A(R_1, R_2)$ is the annulus $\{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$. Since $\underline{z} = 0$ is clearly a subsolution to this problem, we may apply Theorem III.1 in [24] to deduce that (4.2) has a unique (strong) solution $z \in W^{2,\theta}(A(R_1, R_2))$ for every $\theta > N$. Setting $w = z + m(R_1)$, we have that w is the unique strong solution to

$$\begin{cases} -\Delta w + |\nabla w|^q = C m(|x|)^p & \text{in } A(R_1, R_2) \\ w = m(R_1) & \text{on } \partial A(R_1, R_2). \end{cases} \tag{4.3}$$

Let us denote this unique solution by w_{R_2}. It is important to notice that w_{R_2} is radially symmetric.

Next, observe that $m(R)$ is an increasing function by Lemma 6 (b). Then, since $-\Delta u + |\nabla u|^q \geq C u^p \geq C m(|x|)^p$ together with $u \geq m(R_1)$ in $A(R_1, R_2)$, the comparison principle implies $u \geq w_{R_2} \geq m(R_1)$ in $A(R_1, R_2)$.

Our intention is letting $R_2 \to \infty$. Observe that we have uniform local bounds for w_{R_2}. Using part 2 in Theorem A.1 in [22], we also obtain local bounds for $|\nabla w_{R_2}|$ in L^θ for every $\theta > 1$. It follows from here that Δw_{R_2} is locally bounded in L^θ for every $\theta > 1$, and hence by classical regularity w_{R_2} is locally bounded in $C^{1,\alpha}$ for every $\alpha \in (0,1)$ (cf. [18]).

Hence, we may choose a sequence $R_n \to \infty$ such that $w_{R_n} \to w$ in $C^1_{\rm loc}(\mathbb{R}^N \setminus B_{R_1})$, where w is a radially symmetric weak solution to $-\Delta w + |\nabla w|^q = Cm(|x|)^p$ in $|x| > R_1$, with

$$m(R_1) \le w(x) \le u(x). \tag{4.4}$$

It follows by bootstrapping and the fact that w is radially symmetric that w is indeed a classical solution.

Setting $w(x) = v(r)$, where $r = |x|$, we see that (4.4) implies $m(R_1) \le v(r) \le m(r)$, so that v verifies

$$- v'' - \frac{N-1}{r} v' + |v'|^q \ge C v^p. \tag{4.5}$$

Note that $v(R) = \min_{|x|=R} v(|x|) =: m_v(R)$. Then $0 < m(R_1) \le m_v(R)$ for any $R \ge R_1$. Applying Lemma 6 to $m_v(R)$, we deduce that $m_v(R)$ is unbounded and increasing. Therefore, v is increasing and $\lim_{r\to\infty} v(r) = \infty$. The proof concludes by observing that (4.5) implies (4.1). □

Let us finally prove Theorem 4. The main point is to reduce the one-dimensional problem to the inequality $v' \ge K v^{p/q}$ for some $K > 0$. This inequality cannot have global solutions if $p > q$.

Proof of Theorem 4. Thanks to condition (1.6), there exist $R_1 > R_0$ and $C > 0$ such that $-\Delta u + |\nabla u|^q \ge Cu^p$ in $\mathbb{R}^N \setminus B_{R_1}$. We may then use Lemma 13 to obtain an increasing, positive C^2 function v verifying $-v'' + (v')^q \ge Cv^p$ for $r > R_1$. Since $v' \ge 0$, if we had $v' = 0$ at some point then v' would reach a minimum at this point so that $v'' = 0$, contradicting the inequality. Hence $v' > 0$.

Inspired by [29], we consider the function $S(r) = Cv(r)^p - 2v'(r)^q$ for $r > R_1$. Then it is easily seen that

$$S'(r) \ge pCv(r)^{p-1} v'(r) + 2q(Cv(r)^p - v'(r)^q) v'(r)^{q-1}.$$

In particular, if we had $S(r_0) = 0$ for some $r_0 > R_1$, then $S'(r_0) \ge pCv(r_0)^{p-1} v'(r_0) + 2qv'(r_0)^{2q-1} > 0$. This means that S has at most one zero for $r > R_1$, and in particular $S(r)$ keeps sign for large enough r.

Assume first that S is positive for large r. Then it also follows that $-v'' \ge Cv^p/2$ for large r. Multiplying by v' and integrating in (r_1, r) for some large enough r_1 we obtain

$$-(v')^2 \ge \frac{C}{p+1} v^{p+1} + D$$

for some constant D. However, this implies that v is bounded, which is impossible.

Thus S is negative for large r. Then $v' \ge K v^{\frac{p}{q}}$ for some positive constant K and large enough r. Since $p > q$, it is well-known that this inequality implies that v blows up in finite time, which is also impossible. Thus no nonnegative supersolutions to (1.1) blowing up at infinity may exist. The proof is concluded. □

5. A related problem

In this final section, we will show that similar ideas as the ones used throughout the paper apply to deal with the related problem

$$-\Delta u - |\nabla u|^q = f(u) \quad \text{in } \mathbb{R}^N \setminus B_{R_0}, \tag{5.1}$$

where $q > 1$ and f is a continuous, positive function, verifying

$$\liminf_{s\to 0} \frac{f(s)}{s^p} > 0, \tag{5.2}$$

for some $p > 0$. One important difference with respect to (1.1) is that supersolutions to (5.1) are superharmonic, hence they cannot blow up at infinity.

Notice by the way that supersolutions to (5.1) are also supersolutions to $-\Delta u = f(u)$ in $\mathbb{R}^N \setminus B_{R_0}$, so that when $p \le \frac{N}{N-2}$ they cannot exist, according to previous works (cf. [3]). Hence we may restrict our discussion to $p > \frac{N}{N-2}$.

Our result for problem (5.2) can be summarized as follows:

Theorem 14. *Assume $f : (0,\infty) \to \mathbb{R}$ is a continuous positive function verifying (5.2), and that $p > \frac{N}{N-2}$, $1 < q \le \frac{N}{N-1}$. Then problem (5.1) does not admit positive classical supersolutions.*

It can be easily seen that, when $q > \frac{N}{N-1}$ and f verifies

$$\limsup_{s\to 0} \frac{f(s)}{s^p} < \infty$$

for some $p > \frac{N}{N-2}$, supersolutions to (5.1) for large enough R_0 can be constructed in the form $u(x) = A|x|^{-\alpha}$, where A is positive and small enough and

$$\max\left\{\frac{2}{p-1}, \frac{2-q}{q-1}\right\} < \alpha < N-2.$$

Thus Theorem 14 is optimal.

We now give the proof of Theorem 14. Since it is very similar to the previous ones, we do not give full details, only sketch the relevant points.

Lemma 15. *Let $u \in C^2(\mathbb{R}^N \setminus B_{R_0})$ verify $-\Delta u - |\nabla u|^q \ge f(u)$ in $\mathbb{R}^N \setminus B_{R_0}$. If $1 < q < \frac{N}{N-1}$, then for every $\alpha \in (0, N-2)$ the function $m(R)R^\alpha$ is increasing for large R. When $q = \frac{N}{N-1}$, either $m(R)R^\alpha$ is increasing for large R and every $\alpha \in (0, N-2)$ or $m(R)R^{N-2}$ is bounded.*

Sketch of proof. Let us begin by observing that for $\alpha \in (0, N-2]$, the function $v = A|x|^{-\alpha}$ verifies $-\Delta v - |\nabla v|^q \le 0$ for $|x| > R_1$ provided that

$$A \ge \frac{(N-2-\alpha)^{\frac{1}{q-1}}}{\alpha} R_1^{\alpha - \frac{2-q}{q-1}}. \tag{5.3}$$

We also notice that $-\Delta u \geq f(u)$, hence $m(R)R^{N-2}$ is always increasing and $m(R) \to 0$ as $R \to \infty$ (cf. for instance [3], [11]). Thus

$$m(R_1)R_1^\alpha \geq CR_1^{\alpha-N+2}. \tag{5.4}$$

If $1 < q < \frac{N}{N-1}$ and $0 < \alpha < N-2$, we have $\frac{2-q}{q-1} > N-2$, then (5.4) implies that $A = m(R_1)R_1^\alpha$ verifies (5.3) for large R_1. Then for large R_1 and R, the function

$$\Psi(x) = \frac{m(R_1) - m(R)}{R_1^{-\alpha} - R^{-\alpha}}(|x|^{-\alpha} - R^{-\alpha}) + m(R)$$

verifies $-\Delta\Psi - |\nabla\Psi|^q \leq 0$ and by comparison $u \geq \Psi$ in $A(R_1, R)$. Letting $R \to \infty$ we arrive at $u(x) \geq m(R_1)R_1^\alpha|x|^{-\alpha}$ in $|x| > R_1$, which implies that $m(R)R^\alpha$ is increasing if $R > R_1$.

When $q = \frac{N}{N-1}$, if we assume that $m(R)R^{N-2}$ is not bounded, then $A = m(R_1)R_1^\alpha$ always verifies (5.3) for every α and large enough R_1. Thus the same reasoning implies that $m(R)R^\alpha$ is increasing for large R. □

Sketch of the proof of Theorem 14. We notice that supersolutions to (5.1) verify $-\Delta u \geq f(u)$. Thus by the results in [3] or [11] we obtain $m(R) \leq CR^{-\frac{2}{p-1}}$ and $m(R)R^{N-2}$ is increasing.

When $1 < q < \frac{N}{N-1}$, by Lemma 15, $m(R) \geq CR^{-\alpha}$ for every $\alpha \in (0, N-2)$, so that $R^{-\alpha} \leq CR^{-\frac{2}{p-1}}$, and we arrive at a contradiction by choosing α small enough.

If $q = \frac{N}{N-1}$ and $m(R)R^{N-2}$ is not bounded, we reach the same contradiction as before, so we may assume that $m(R)R^{N-2}$ is bounded. The rest of the proof is now similar to those of Lemma 10 and Theorem 3.

For large enough R_ε, we have $u(x) \geq (l-\varepsilon)|x|^{2-N}$ if $|x| > R_\varepsilon$, where $l = \lim_{R\to\infty} m(R)R^{N-2} > 0$. Taking $x_{0,\varepsilon}$ with $u(x_{0,\varepsilon}) = m(|x_{0,\varepsilon}|)$ and introducing the function

$$v(y) = |x_{0,\varepsilon}|^{N-2}u(x_{0,\varepsilon} + |x_{0,\varepsilon}|y), \quad |y| < \frac{1}{2},$$

we have $-\Delta v - |\nabla v|^q \geq 0$ in $B_{1/2}$, with $v \geq (l-\varepsilon)|e_\varepsilon + y|^{2-N}$ in $B_{1/2}$, where e_ε is some unit vector. We deduce the existence of a solution $z = z_\varepsilon$ to the problem

$$\begin{cases} -\Delta z - |\nabla z|^q = 0 & \text{in } B_{1/2} \\ z = (l-\varepsilon)|e_\varepsilon + y|^{2-N} & \text{on } \partial B_{1/2}, \end{cases}$$

verifying $z \geq (l-\varepsilon)|e_\varepsilon + y|^{2-N}$, $z(0) \leq l$. We may pass to the limit to obtain $z_\varepsilon \to w$, which verifies $-\Delta w - |\nabla w|^q = 0$ in $B_{1/4}$, $w \geq l|e_0 + y|^{2-N}$, $w(0) = l$, for some unit vector e_0. Setting $w = |e_0 + y|^{2-N}W$, we get that W has a minimum at 0, so that $\Delta W(0) \geq 0$, $\nabla W(0) = 0$. Hence

$$0 \geq l^q(N-2)^q,$$

which is a contradiction. □

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