THE BEHAVIOR OF THE PRINCIPAL EIGENVALUE OF A MIXED ELLIPTIC PROBLEM WITH RESPECT TO A PARAMETER

EDUARDO COLORADO AND JORGE GARCÍA-MELIÁN

Abstract. In this paper we determine the exact asymptotic behavior of the principal eigenvalue of a mixed elliptic eigenvalue problem which depends on a positive parameter λ when $\lambda \to \infty$. We analyze the case in which the problem is considered in a smooth bounded domain Ω of $\mathbb{R}^N$, and also the case of planar domains which are smooth except for a finite number of corner points.

1. Introduction

Our starting point in this paper is the work [9]. There, to perform the analysis of an elliptic problem with a nonlinear boundary condition, the auxiliary eigenvalue problem

$$\begin{cases} \Delta u = \sigma u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

was considered in a bounded, piece-wise smooth domain Ω of $\mathbb{R}^N$. The main point was to ascertain the behavior of the principal eigenvalue $\sigma_1(\lambda)$ of (1.1), that is, the only eigenvalue associated to a positive eigenfunction, with regard to the positive parameter λ. It was proved that, for a large class of smooth domains,

$$\sigma_1(\lambda) \sim \lambda^2 \qquad \text{as } \lambda \to \infty. \tag{1.2}$$

Moreover, it was noticed that if the domain has corners, then this asymptotic behavior is modified because of them. In the particular case of planar domains which are piece-wise smooth and have outward pointing corners with half angles $\alpha_1, \dots, \alpha_k$, it was conjectured that

$$\sigma_1(\lambda) \sim \max_{1\le j\le k} \{\text{cosec}^2\alpha_j\}\lambda^2 \qquad \text{as } \lambda \to \infty, \tag{1.3}$$

although this conjecture was only proved for triangles.

Besides its use in the study of problems with nonlinear boundary conditions, problem (1.1) also arises in the context of superconductivity in zero-field for a finite size superconducting body (see [7]). We also refer to [3] for a probabilistic interpretation of $\sigma_1(\lambda)$, and to [6], [7], [8] for some monotonicity properties with respect to the domain.

We mention that (1.2) for problem (1.1) has been proved in [12], and we have also learned after the research that led to this paper was completed, that (1.3) has also been proved in [10] for the same problem.

The main purpose of the present work is to prove (1.2) and (1.3) for slightly more general eigenvalue problems. In this regard, we are generalizing (1.1) in two directions: on one hand, we are allowing the presence of weights in the equation; on the other hand, instead of studying a pure Robin-type boundary value problem we are enlarging our scope to deal with a mixed boundary value problem. Thus we will be interested in

$$\begin{cases} \Delta u = \sigma G(x) u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda H(x) u & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_2 \end{cases} \tag{1.4}$$

where $\partial\Omega = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$ and Γ_1 is an open subset of $\partial\Omega$ (hence Γ_2 is closed). We assume for simplicity that Γ_1 and Γ_2 are smooth. The functions G and H will be assumed to be α-Hölder continuous in $\overline{\Omega}$ with $G > 0$ in $\overline{\Omega}$, $H > 0$ on $\partial\Omega$.

It is not hard to show (see the statements in Section 2) that problem (1.4) admits a unique principal eigenvalue $\sigma_1(\lambda)$, which is an increasing, convex function verifying $\sigma_1(\lambda) \to \infty$. Our main task is to show that the asymptotic behavior of $\sigma_1(\lambda)$ as $\lambda \to \infty$ can also be exactly determined.

We consider first the case of smooth domains in $\mathbb{R}^N$.

Theorem 1. *Assume Ω is a C^2 bounded domain of $\mathbb{R}^N$, $N \geq 2$, and $G, H \in C^\alpha(\overline{\Omega})$ verify $G > 0$ in $\overline{\Omega}$, $H > 0$ on $\partial\Omega$. Let $\sigma_1(\lambda)$ be the principal eigenvalue of problem* (1.4). *Then*

$$\lim_{\lambda\to\infty} \frac{\sigma_1(\lambda)}{\lambda^2} = \sup_{x\in\Gamma_1} \frac{H(x)^2}{G(x)}. \tag{1.5}$$

We now turn to the case of piece-wise smooth domains of $\mathbb{R}^2$. Thus we assume that Ω is a bounded domain which verifies:

(D) Ω is piecewise smooth with a finite number of corner points $x_1, \ldots, x_l$, with half inner angles $\alpha_1, \ldots, \alpha_l$ verifying $\alpha_j \in (0, \pi/2)$.

With no loss of generality we may assume $x_1, \ldots, x_m \in \Gamma_1$, that is, the corner points which lie on Γ_1 are exactly the first m, with $m \leq l$ (of course $m = 0$ is possible). In this case we show that the asymptotic behavior of $\sigma_1(\lambda)$ can also be determined, obtaining a generalization of (1.3).

Theorem 2. *Assume Ω is a bounded domain of $\mathbb{R}^2$ verifying (D), and $G, H \in C^\alpha(\overline{\Omega})$ verify $G > 0$ in $\overline{\Omega}$, $H > 0$ on $\partial\Omega$. Let $\sigma_1(\lambda)$ be the principal eigenvalue of problem* (1.4). *Then*

$$\lim_{\lambda\to\infty} \frac{\sigma_1(\lambda)}{\lambda^2} = \max\left\{ \sup_{\Gamma_1} \frac{H(x)^2}{G(x)}, \max_{1\leq j\leq m} \left\{ \frac{H(x_j)^2}{G(x_j)}\ cosec^2 \alpha_j \right\} \right\}, \tag{1.6}$$

where $x_1, \ldots, x_m$ are the corner points of $\partial\Omega$ which belong to Γ_1.

Our proofs rely on an analysis of the eigenfunctions to (1.4) as $\lambda \to \infty$ near the points where they achieve their maxima (in particular the study for planar domains is essentially different than that in [10]). This method has been already used in [4] in the context of mixed problems. The analysis is completed by means of Liouville type theorems both in half-spaces or in cones in $\mathbb{R}^2$. We mention by the way that our proofs can be adapted to cover more general situations, for instance when a divergence type operator replaces the Laplacian in (1.4) (with the corresponding conormal boundary condition on Γ_1). If the equation is not of divergence form, then only the information

$$C_1 \le \frac{\sigma_1(\lambda)}{\lambda^2} \le C_2 \qquad \text{for } \lambda > 0$$

can be obtained, but nevertheless if a lower inequality for the inferior limit in (1.5) or (1.6) can be provided, then the full conclusion follows.

We mention in passing that our proofs also provide some information on the asymptotic behavior for the eigenfunctions: they concentrate on points of Γ_1 where the maximum in (1.5) or (1.6) is achieved and converge to zero exponentially fast outside a neighborhood of these points. But it also follows that the maximum points cannot go "too fast" to the interface $\overline{\Gamma_1} \cap \Gamma_2$ (see Remarks 2 and 3 right after the proofs of Theorems 1 and 2, respectively).

Finally, let us remark that the exact asymptotic behavior of the principal eigenvalue of (1.4) can be used as in [9] to deal with the study of the mixed problem with nonlinear boundary condition

$$\begin{cases} \Delta u = f(x,u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = g(x,u) & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_2 \end{cases}$$

for suitable nonlinearities f and g, but this question will not be pursued here.

The paper is organized as follows: in Section 2 we state without proof some basic properties of the principal eigenvalue $\sigma_1(\lambda)$. Section 3 is dedicated to consider some Liouville type theorems, which will be important in our proofs. Finally Theorems 1 and 2 are proved in Sections 4 and 5, respectively.

2. Preliminaries on the principal eigenvalue

We devote this brief section to state some basic properties of the principal eigenvalue of problem (1.4). Their proofs are all direct adaptations of well-known properties for the pure Dirichlet or Robin eigenvalue problem and will not therefore be given (see for instance [9]). We denote by $H^1_{\Gamma_2}(\Omega)$ the Banach space of functions in $H^1(\Omega)$ which vanish in the sense of traces on Γ_2.

Theorem 3. *Let $G, H \in C^\alpha(\overline{\Omega})$ be such that $G > 0$ in $\overline{\Omega}$, $H > 0$ on $\partial\Omega$. Then for every $\lambda > 0$ problem* (1.4)*:*

$$\begin{cases} \Delta u = \sigma G(x) u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda H(x) u & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_2 \end{cases}$$

admits a unique principal eigenvalue $\sigma_1(\lambda)$, which is simple, and can be variationally characterized as the Rayleigh quotient

$$\sigma_1(\lambda) = \inf_{\substack{u \in H^1_{\Gamma_2}(\Omega) \\ u \neq 0}} \frac{\lambda \displaystyle\int_{\Gamma_1} H(x) u^2 - \int_\Omega |\nabla u|^2}{\displaystyle\int_\Omega G(x) u^2}. \tag{2.1}$$

Moreover, $\sigma_1(\lambda)$ is an analytic function of λ, which is increasing and convex and verifies $\sigma_1(\lambda) \to \infty$ as $\lambda \to \infty$.

Remarks 1. (a) Problem (1.4) can be also considered for $\lambda \leq 0$. It is well-known that in that case the same properties hold, and it can be further proved that $\sigma_1(\lambda) \to \sigma_D$ as $\lambda \to -\infty$, where σ_D is the principal eigenvalue of the Dirichlet problem

$$\begin{cases} \Delta u = \sigma G(x) u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

(b) It is also easy to establish by means of (2.1) that $\sigma_1(0) = 0$ when $\Gamma_2 = \emptyset$, while $\sigma_1(0) < 0$ if Γ_2 has positive measure.

3. Some Liouville type theorems

In this section we consider some Liouville type theorems which will be used in the proofs of Sections 4 and 5. We consider problems in half-spaces of $\mathbb{R}^N$ and in cones of $\mathbb{R}^2$. We are interested in two types of results: first we determine for which values of a parameter certain boundary value problems admit a positive solution and then we show that some problems have no solutions.

3.1. **Problems in half-spaces.** Let us begin with the case of a half space $\mathbb{R}^N_+ = \{(x_1, x') : \ x_1 > 0\}$. In the sequel, we denote $T_1 = \{x \in \partial\mathbb{R}^N_+ : \ x_N < 0\}$, $T_2 = \{x \in \partial\mathbb{R}^N_+ : \ x_N \geq 0\}$.

We consider first for $a > 0$ the pure Robin type problem

$$\begin{cases} \Delta u = au & \text{in } \mathbb{R}^N_+ \\ \dfrac{\partial u}{\partial \nu} = u & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{3.1}$$

which was analyzed in [12]. The proof of the following theorem can be found there.

Theorem 4. *Let $u \in C^2(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$ be a solution to (3.1) with $0 < u \leq 1$ and $a \geq 1$. Then $a = 1$ and $u \leq e^{-x_1}$ in $\mathbb{R}^N_+$. If moreover $u = 1$ at some point on $\partial\mathbb{R}^N_+$, then $u = e^{-x_1}$.*

Next, we treat a mixed version of problem (3.1), namely

$$\begin{cases} \Delta u = au & \text{in } \mathbb{R}^N_+ \\ \dfrac{\partial u}{\partial \nu} = u & \text{on } T_1 \\ u = 0 & \text{on } T_2 \end{cases} \tag{3.2}$$

With a similar argument as that used in [12], we can prove:

Theorem 5. *Assume there exists $u \in C^2(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$ verifying (3.2) with $0 < u \leq 1$. If u attains the value 1 somewhere on T_1, then $0 < a < 1$ and $u < e^{-\sqrt{a}x_1}$ in $\overline{\mathbb{R}^N_+}$.*

Proof. With a change of variables $y = \sqrt{a}x$, we transform the equation into

$$\begin{cases} \Delta u = u & \text{in } \mathbb{R}^N_+ \\ \dfrac{\partial u}{\partial \nu} = \dfrac{1}{\sqrt{a}} u & \text{on } T_1 \\ u = 0 & \text{on } T_2. \end{cases}$$

Choose $h, r > 0$ and define

$$\begin{aligned}
Q_{h,r} &= \{(y_1, y') \in \mathbb{R}^N_+ : \ 0 < y_1 < h, \ |y'| < r\} \\
T_{1,h,r} &= (\overline{Q_{h,r}} \cap T_1) \setminus (\partial\mathbb{R}^N_+ \cap \{|y'| = r\}) \\
T_{2,h,r} &= \overline{Q_{h,r}} \cap T_2 \\
\Gamma_{h,r} &= (\partial Q_{h,r} \cap \mathbb{R}^N_+) \cup (\partial\mathbb{R}^N_+ \cap \{|y'| = r\}).
\end{aligned}$$

Now choose a nonnegative smooth function ϕ which verifies $0 \le \phi \le 1$ and $\phi = 0$ in $|y'| \le r/3$, $\phi = 1$ in $|y'| \ge 2r/3$. Consider the auxiliary problem

$$\left\{\begin{array}{ll}
\Delta v = v & \text{in } Q_{h,r} \\
\dfrac{\partial v}{\partial \nu} = \dfrac{1}{\sqrt{a}} & \text{on } T_{1,h,r} \\
v = \phi & \text{on } T_{2,h,r} \\
v = 1 & \text{on } \Gamma_{h,r}.
\end{array}\right. \tag{3.3}$$

We claim that this problem admits a positive solution $v_{h,r} \in C^2(Q_{h,r}) \cap C(\overline{Q_{h,r}})$ and that $u \le v_{h,r}$ in $Q_{h,r}$.

Assuming the existence for the moment, let us show that $u \le v_{h,r}$. Indeed, the function $u - v_{h,r}$ attains a nonnegative maximum somewhere on $\partial Q_{h,r}$ thanks to the strong maximum principle and it cannot be achieved at $T_{1,h,r}$ by Hopf's principle. Thus the maximum is attained at $T_{2,h,r} \cup \Gamma_{h,r}$ and then $u - v_{h,r} \le 0$.

Observe that the regularity of $v_{h,r}$ is consequence of standard theory (cf. [5]) and the results in [11]. To prove the existence of such a solution we make

$$v = \frac{1}{\sqrt{a}} z + w$$

where

$$\left\{\begin{array}{ll}
\Delta z = z & \text{in } Q_{h,r} \\
\dfrac{\partial z}{\partial \nu} = 1 & \text{on } T_{1,h,r} \\
z = 0 & \text{on } T_{2,h,r} \cup \Gamma_{h,r}.
\end{array}\right. \tag{3.4}$$

and

$$\left\{\begin{array}{ll}
\Delta w = w & \text{in } Q_{h,r} \\
\dfrac{\partial w}{\partial \nu} = 0 & \text{on } T_{1,h,r} \\
w = \phi & \text{on } T_{2,h,r} \\
w = 1 & \text{on } \Gamma_{h,r}.
\end{array}\right. \tag{3.5}$$

We consider first (3.4). It is easy to see that $\underline{z} = 0$ is a subsolution while $\overline{z} = -y_1 + h$ is a supersolution. Thus there exists a weak solution $z_{h,r}$ which, according to standard regularity, verifies $z_{h,r} \in C^2(Q_{h,r})$ and also $z_{h,r} \in C(\overline{Q_{h,r}})$ thanks to the results in [11]. Uniqueness follows by comparison as before.

Now let $\psi_{h,r}$ be the unique solution to

$$\left\{\begin{array}{ll}
\Delta \psi = \psi & \text{in } Q_{h,r} \\
\dfrac{\partial \psi}{\partial \nu} = 1 & \text{on } T_{1,h,r} \cup T_{2,h,r} \\
\psi = 0 & \text{on } \Gamma_{h,r},
\end{array}\right.$$

which is constructed in [12]. Be obtain by a similar comparison that $z_{h,r} \le \psi_{h,r}$.

We now intend to let $r \to \infty$. Notice that it follows again by comparison that $z_{h,r}$ is increasing in r and since it is bounded we deduce $z_{h,r} \to z_h$ uniformly on compacts of $Q_h = \{0 < y_1 < h\}$. We deduce that z_h is a solution to

$$\begin{cases} \Delta z = z & \text{in } Q_h \\ \dfrac{\partial z}{\partial \nu} = 1 & \text{on } T_{1,h} \\ z = 0 & \text{on } T_{2,h} \cup \Gamma_h \end{cases}$$

where $T_{1,h} = \partial Q_h \cap T_1$, $T_{2,h} = \partial Q_h \cap T_2$, $\Gamma_h = \partial Q_h \cap \mathbb{R}^N_+$. Moreover

$$z_h \le \frac{e^{-h} e^{y_1} + e^h e^{-y_1}}{e^h + e^{-h}}, \tag{3.6}$$

since $\psi_{h,r}$ converges to the right-hand side of (3.6) (cf. [12]). In the same way, we can obtain a sequence $h_n \to \infty$ such that $z_{h_n} \to z$, which is a solution to

$$\begin{cases} \Delta z = z & \text{in } \mathbb{R}^N_+ \\ \dfrac{\partial z}{\partial \nu} = 1 & \text{on } T_1 \\ z = 0 & \text{on } T_2 \end{cases}$$

verifying $0 < z \le e^{-y_1}$.

We now consider (3.5). It is similarly shown that it admits a unique positive solution $w_{h,r} \in C^2(Q_{h,r}) \cap C(\overline{Q_{h,r}})$, which in addition verifies $w_{h,r} \to w_h$ as $r \to \infty$ where w_h satisfies

$$w_h \le \frac{e^{y_1} + e^{-y_1}}{e^h + e^{-h}},$$

hence $w_h \to 0$ as $h \to \infty$.

To summarize, we have

$$u \le \frac{1}{\sqrt{a}} z_{h,r} + w_{h,r} \qquad \text{in } Q_{h,r}$$

for every $h, r > 0$. We can let $r \to \infty$ and then put $h = h_n$ and let n go to infinity to arrive at

$$u \le \frac{1}{\sqrt{a}} z \qquad \text{in } \mathbb{R}^N_+.$$

Finally, since $z \le e^{-y_1}$, Hopf's principle implies $z < e^{-y_1}$ on T_1 and taking into account that $u = 1$ somewhere on T_1 we obtain $a < 1$. This concludes the proof. $\square$

Finally let us briefly consider the problems

$$\begin{cases} \Delta u = u & \text{in } \mathbb{R}^N_+ \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial \mathbb{R}^N_+ \end{cases} \tag{3.7}$$

and

$$\begin{cases} \Delta u = u & \text{in } \mathbb{R}^N_+ \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } T_1 \\ u = 0 & \text{on } T_2. \end{cases} \tag{3.8}$$

Notice that if u is a bounded positive solution either to (3.7) which achieves its maximum at a point on $\partial\mathbb{R}^N_+$ or to (3.8) whose maximum is attained on T_1, then Hopf's principle is contradicted. Hence we have

Theorem 6. *Problems* (3.7) *and* (3.8) *do not admit positive bounded solutions* $u \in C^2(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$ *which achieve their maxima on* $\partial\mathbb{R}^N_+$.

3.2. **Problems in planar cones.** We next consider some problems in cones of $\mathbb{R}^2$. For a fixed $\alpha \in (0, \frac{\pi}{2})$, let $Q_\alpha = \{(x,y) \in \mathbb{R}^2 :\ x > 0,\ |y| < (\tan\alpha)x\}$, $T_{1,\alpha} = \{(x,y) \in \mathbb{R}^2 :\ x > 0,\ y = -(\tan\alpha)x\}$, $T_{2,\alpha} = \{(x,y) \in \mathbb{R}^2 :\ x \geq 0,\ y = (\tan\alpha)x\}$, so that $\partial Q_\alpha = T_{1,\alpha} \cup T_{2,\alpha}$. We first deal with

$$\begin{cases} \Delta u = au & \text{in } Q_\alpha \\ \dfrac{\partial u}{\partial \nu} = u & \text{on } \partial Q_\alpha, \end{cases} \tag{3.9}$$

for some $a > 0$.

Theorem 7. *Let* $u \in C^2(Q_\alpha) \cap C(\overline{Q_\alpha})$ *be a positive solution to* (3.9) *for some* $a > 0$ *with* $0 < u \leq 1$. *Then* $a = cosec^2\alpha$, *and there exists* $\kappa \in (0, 1]$ *such that* $u = \kappa e^{-(cosec\,\alpha)x}$.

The proof of Theorem 7 is based on a comparison lemma, whose proof is based on an idea in [2]. We consider it first.

Lemma 8. *Let* $a \geq b > 0$, $d \geq c \geq 0$. *Assume* $u, v \in C^2(Q_\alpha) \cap C(\overline{Q_\alpha})$ *verify*

$$\begin{cases} \Delta u \geq au & \text{in } Q_\alpha \\ \dfrac{\partial u}{\partial \nu} \leq cu & \text{on } \partial Q_\alpha \end{cases}$$

and

$$\begin{cases} \Delta v \leq bv & \text{in } Q_\alpha \\ \dfrac{\partial v}{\partial \nu} \geq dv & \text{on } \partial Q_\alpha \end{cases}$$

together with $0 < u \leq 1$ *and* $v > 0$ *(not necessarily bounded). Then* $a = b$, $c = d$ *and* $u = \kappa v$ *for some* $\kappa > 0$.

Proof. Let $\varepsilon > 0$ and define $w_\varepsilon = u/(v + \varepsilon)$ (observe that $v > 0$ in $\overline{Q_\alpha} \setminus \{0\}$, but Hopf's principle cannot be applied at 0). A straightforward calculation shows that $(v + \varepsilon)\Delta w_\varepsilon + 2\nabla v \nabla w_\varepsilon \geq (a - b)v w_\varepsilon + \varepsilon a w_\varepsilon \geq 0$ in Q_α, which can be written in divergence form as

$$w_\varepsilon \operatorname{div}((v + \varepsilon)^2 \nabla w_\varepsilon) \geq 0. \tag{3.10}$$

Choose a test function $\xi \in C_0^\infty(B_2)$ with $0 \leq \xi \leq 1$, $\xi = 1$ in B_1, where B_1, B_2 denote the balls with radius 1 and 2, respectively, centered at the origin. Set

$$\xi_R(x,y) = \xi\left(\frac{x}{R}, \frac{y}{R}\right).$$

By multiplying (3.10) by ξ_R^2 and integrating in Q_α, we obtain, after an integration by parts:

$$\int_{Q_\alpha} (v + \varepsilon)^2 \xi_R^2 |\nabla w_\varepsilon|^2 \leq \int_{\partial Q_\alpha} \xi_R^2 (v + \varepsilon)^2 w_\varepsilon \frac{\partial w_\varepsilon}{\partial \nu} - 2\int_{Q_\alpha} \xi_R w_\varepsilon (v + \varepsilon)^2 \nabla \xi_R \nabla w_\varepsilon.$$

Now notice that on ∂Q_α we have $(v+\varepsilon)\frac{\partial w_\varepsilon}{\partial \nu} \le c(v+\varepsilon)w_\varepsilon - dvw_\varepsilon \le \varepsilon c w_\varepsilon$, and hence

$$\int_{Q_\alpha} (v+\varepsilon)^2 \xi_R^2 |\nabla w_\varepsilon|^2 \le \varepsilon c \int_{\partial Q_\alpha} \xi_R^2 (v+\varepsilon) w_\varepsilon^2 - 2 \int_{Q_\alpha} \xi_R w_\varepsilon (v+\varepsilon)^2 \nabla \xi_R \nabla w_\varepsilon. \tag{3.11}$$

Our intention is letting $\varepsilon \to 0$ in (3.11). Since $v > 0$ on $\partial Q_\alpha \setminus \{0\}$ we have $\varepsilon c \xi_R^2 (v+\varepsilon) w_\varepsilon^2 \to 0$ on $\partial Q_\alpha \setminus \{0\}$. Thus dominated convergence gives

$$\varepsilon c \int_{\partial Q_\alpha} \xi_R^2 (v+\varepsilon) w_\varepsilon^2 \to 0$$

as $\varepsilon \to 0$. Also, since $\nabla \xi_R = 0$ near zero, it is immediate that

$$\int_{Q_\alpha} \xi_R w_\varepsilon (v+\varepsilon)^2 \nabla \xi_R \nabla w_\varepsilon \to \int_{Q_\alpha} \xi_R w v^2 \nabla \xi_R \nabla w$$

where $w = u/v$. However, we cannot pass to the limit directly in the first integral of (3.11), but we observe that for every small positive δ we could pass to the limit if the integration were only performed in $Q_\alpha \setminus B_\delta$. Hence

$$\int_{Q_\alpha \setminus B_\delta} v^2 \xi_R^2 |\nabla w|^2 \le -2 \int_{Q_\alpha} \xi_R w v^2 \nabla \xi_R \nabla w,$$

and letting $\delta \to 0$, we obtain by monotone convergence

$$\int_{Q_\alpha} v^2 \xi_R^2 |\nabla w|^2 \le -2 \int_{Q_\alpha} \xi_R w v^2 \nabla \xi_R \nabla w. \tag{3.12}$$

Next notice that the integral in the right-hand side of (3.12) is only taken in $Q_{2R} \setminus Q_R$, where $Q_R := Q_\alpha \cap B_R$, $Q_{2R} := Q_\alpha \cap B_{2R}$. Applying Cauchy-Schwarz inequality:

$$\int_{Q_{2R}} v^2 \xi_R^2 |\nabla w|^2 \le 2 \left(\int_{Q_{2R} \setminus Q_R} v^2 \xi_R^2 |\nabla w|^2 \right)^{1/2} \left(\int_{Q_{2R} \setminus Q_R} v^2 w^2 |\nabla \xi_R|^2 \right)^{1/2}, \tag{3.13}$$

and the last integral in (3.13) is bounded:

$$\int_{Q_{2R} \setminus Q_R} v^2 w^2 |\nabla \xi_R|^2 = \frac{1}{R^2} \iint_{Q_{2R} \setminus Q_R} u(x,y)^2 \left| \nabla \xi \left(\frac{x}{R}, \frac{y}{R} \right) \right|^2 dxdy \le C,$$

since $|Q_{2R} \setminus Q_R| \le |B_{2R}| \le CR^2$ (we denote the letter C to denote constants not depending on R). Thus (3.13) implies

$$\int_{Q_{2R}} v^2 \xi_R^2 |\nabla w|^2 \le C \left(\int_{Q_{2R} \setminus Q_R} v^2 \xi_R^2 |\nabla w|^2 \right)^{1/2} \tag{3.14}$$

We deduce first from (3.14) that

$$\int_{Q_{2R}} v^2 \xi_R^2 |\nabla w|^2 \le C,$$

so that letting $R \to \infty$ we obtain

$$\int_{Q_\alpha} v^2 |\nabla w|^2 \le C,$$

and hence (3.14) implies

$$\int_{Q_\alpha} v^2 |\nabla w|^2 = 0,$$

so that w is a positive constant κ. Hence $u = \kappa v$ and this finally implies $a = b$, $c = d$. This concludes the proof. □

Proof of Theorem 7. The proof consists in comparing the solution u with an explicit solution to a similar problem. Let $v(x,y) = e^{-(\operatorname{cosec}\alpha)x}$. Since $\nu = (-\sin\alpha, -\cos\alpha)$ on $T_{1,\alpha}$, while $\nu = (-\sin\alpha, \cos\alpha)$ on $T_{2,\alpha}$, it is not difficult to see that v solves:

$$\begin{cases} \Delta v = (\operatorname{cosec}^2\alpha)\, v & \text{in } Q_\alpha \\ \dfrac{\partial v}{\partial \nu} = v & \text{on } \partial Q_\alpha. \end{cases}$$

It follows immediately from Lemma 8 that $a = \operatorname{cosec}^2\alpha$ and $u = \kappa v$ for some $\kappa > 0$. Since $0 < u, v \le 1$ we also have $\kappa \in (0,1]$. This finishes the proof. □

Our next problem will be of mixed type in Q_α. We consider

$$\begin{cases} \Delta u = au & \text{in } Q_\alpha \\ \dfrac{\partial u}{\partial \nu} = u & \text{on } T_{1,\alpha} \\ u = 0 & \text{on } T_{2,\alpha}. \end{cases} \tag{3.15}$$

For this problem, we have the following property:

Theorem 9. *Assume $u \in C^2(Q_\alpha) \cap C(\overline{Q_\alpha})$ is a positive solution to (3.15) for some $a > 0$ with $0 < u \le 1$. Then $a < 1$.*

We will need a variant of Lemma 8. The proof is a straightforward modification of that one and therefore it will be omitted. Observe that no boundary condition is needed for v on $T_{2,\alpha}$.

Lemma 10. *Let $a \ge b > 0$, $d \ge c \ge 0$. Assume $u, v \in C^2(Q_\alpha) \cap C(\overline{Q_\alpha})$ verify*

$$\begin{cases} \Delta u \ge au & \text{in } Q_\alpha \\ \dfrac{\partial u}{\partial \nu} \le cu & \text{on } T_{1,\alpha} \\ u = 0 & \text{on } T_{2,\alpha} \end{cases}$$

and

$$\begin{cases} \Delta v \le bv & \text{in } Q_\alpha \\ \dfrac{\partial v}{\partial \nu} \ge dv & \text{on } T_{1,\alpha} \end{cases}$$

together with $0 < u \le 1$ and $v > 0$ (not necessarily bounded). Then $a = b$, $c = d$ and $u = \kappa v$ for some $\kappa > 0$.

Proof of Theorem 9. Let $v(x,y) = e^{-(\sin\alpha)x - (\cos\alpha)y}$. It is not difficult to show that $\Delta v = v$ in Q_α and $\frac{\partial v}{\partial \nu} = v$ on $T_{1,\alpha}$ (but not on $T_{2,\alpha}$!). If we had $a \ge 1$, then Lemma 10 would imply $a = 1$ and $u = \kappa v$ for some $\kappa > 0$. Of course, this is impossible since $u = 0$ on $T_{2,\alpha}$, while $v \ne 0$ there. Thus $a < 1$, as we wanted to show. □

Finally we need to show that the problems

$$\begin{cases} \Delta u = u & \text{in } Q_\alpha \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial Q_\alpha \end{cases} \tag{3.16}$$

and

$$\begin{cases} \Delta u = u & \text{in } Q_\alpha \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } T_{1,\alpha} \\ u = 0 & \text{on } T_{2,\alpha} \end{cases} \tag{3.17}$$

have no positive solutions. The proof also relies in a comparison with suitable explicit solutions to related problems. Notice that in this case Hopf's principle cannot be applied at the corner of the cone.

Theorem 11. *Let* $u \in C^2(Q_\alpha) \cap C(\overline{Q_\alpha})$ *be a bounded nonnegative solution to* (3.16) *or to* (3.17). *Then* $u \equiv 0$.

Proof. Let $v(x,y) = e^{-x}$. Then $\Delta v = v$ in Q_α, $\frac{\partial v}{\partial \nu} = (\sin\alpha)v$ on ∂Q_α. If we assume that u is a positive bounded solution to (3.16), Lemma 8 implies $\sin\alpha = 0$, which is clearly impossible. If u solves (3.17) instead, then Lemma 10 can be used to obtain the same contradiction. Thus $u \equiv 0$ in either case. This concludes the proof. □

4. Smooth domains

This section is dedicated to the proof of Theorem 1. Thus we are assuming that Ω is a C^2 bounded domain of $\mathbb{R}^N$ and $\sigma_1(\lambda)$ is the principal eigenvalue of (1.4) for $\lambda > 0$. We first show the lower inequality in (1.5). From now on, we denote $d(x) = \operatorname{dist}(x, \partial\Omega)$.

Lemma 12. *Assume* Ω *is a* C^2 *bounded domain of* $\mathbb{R}^N$ *and let* $\sigma_1(\lambda)$ *be the principal eigenvalue of* (1.4) *for* $\lambda > 0$. *Then*

$$\liminf_{\lambda\to\infty} \frac{\sigma_1(\lambda)}{\lambda^2} \ge \sup_{x\in\Gamma_1} \frac{H(x)^2}{G(x)}. \tag{4.1}$$

Proof. Fix a point $x_0 \in \Gamma_1$. For every $\varepsilon > 0$ small enough, there exists $r > 0$ such that $H(x) \ge H(x_0) - \varepsilon$ if $x \in B_r(x_0) \cap \partial\Omega$ and $G(x) \le G(x_0) + \varepsilon$ if $x \in B_r(x_0) \cap \Omega$. By diminishing r if necessary we may assume $B_r(x_0) \cap \Gamma_2 = \emptyset$.

Choose a cut-off function $\psi \in C_0^\infty(B_r(x_0))$ with $0 \le \psi \le 1$ and define

$$u(x) = \psi(x) e^{-\lambda\gamma d(x)},$$

where $\gamma > 0$ will be chosen later on. Since $d \in C^2(B_r(x_0))$ and $u = 0$ in $\Omega \setminus B_r(x_0)$, it is clear that $u \in H^1_{\Gamma_2}(\Omega)$ (the Sobolev space which consists of functions in $H^1(\Omega)$ with zero trace on Γ_2). Thus we may use u as a test function in the variational formulation (2.1) of $\sigma_1(\lambda)$ to obtain

$$\sigma_1(\lambda) \ge \frac{\lambda \displaystyle\int_{\Gamma_1} H(x)u^2 - \int_\Omega |\nabla u|^2}{\displaystyle\int_\Omega G(x)u^2} \ge \frac{\lambda(H(x_0)-\varepsilon) \displaystyle\int_{\Gamma_1} u^2 - \int_\Omega |\nabla u|^2}{(G(x_0)+\varepsilon) \displaystyle\int_\Omega u^2}. \tag{4.2}$$

We now notice that

$$\int_\Omega |\nabla u|^2 = \int_{B_r(x_0)\cap\Omega} (|\nabla\psi|^2 - 2\lambda\gamma\nabla\psi\nabla d + \lambda^2\gamma^2\psi^2) e^{-2\lambda\gamma d},$$

and our next task will be to estimate each of these integrals. First observe that if ν is the outward unit normal at $\partial\Omega$, we may write:

$$\begin{aligned}\lambda^2\gamma^2\int_{B_r(x_0)\cap\Omega}\psi^2 e^{-2\lambda\gamma d} &= \lambda^2\gamma^2\int_0^r\int_{\partial\Omega}\psi(y-\tau\nu(y))^2e^{-2\lambda\gamma\tau}dS(y)d\tau\\ &= \lambda\gamma^2\int_0^{\lambda r}\int_{\partial\Omega}\psi(y-\frac{z}{\lambda}\nu(y))^2e^{-2\gamma z}dS(y)dz\\ &= \lambda\gamma^2\left(\int_0^\infty e^{-2\gamma z}dz\right)\int_{\partial\Omega}\psi(y)^2dS(y)+o(\lambda)\\ &= \frac{\lambda\gamma}{2}\int_{\partial\Omega}\psi^2+o(\lambda),\end{aligned}$$

and it is similarly shown that

$$2\lambda\gamma\int_\Omega\nabla\psi\nabla d\, e^{-2\lambda\gamma d}=O(1)$$

and

$$\int_\Omega|\nabla\psi|^2e^{-2\lambda\gamma d}=O\left(\frac{1}{\lambda}\right).$$

Thus we obtain by plugging everything into (4.2):

$$\begin{aligned}\sigma_1(\lambda)\ &\geq \frac{\lambda(H(x_0)-\varepsilon)\displaystyle\int_{\partial\Omega}\psi^2-\frac{\lambda\gamma}{2}\int_{\partial\Omega}\psi^2+o(\lambda)}{\dfrac{G(x_0)+\varepsilon}{2\gamma\lambda}\displaystyle\int_{\partial\Omega}\psi^2+o\left(\frac{1}{\lambda}\right)}\\ &= \lambda^2\frac{H(x_0)-\varepsilon-\dfrac{\gamma}{2}+o(1)}{\dfrac{G(x_0)+\varepsilon}{2\gamma}+o(1)}.\end{aligned}$$

Letting $\lambda\to\infty$ and then $\varepsilon\to 0$, we arrive at

$$\liminf_{\lambda\to\infty}\frac{\sigma_1(\lambda)}{\lambda^2}\geq\frac{(2H(x_0)-\gamma)\gamma}{G(x_0)}. \tag{4.3}$$

This inequality is valid for all positive γ, so that setting $\gamma=H(x_0)$ to maximize the right hand-side in (4.3) we obtain

$$\liminf_{\lambda\to\infty}\frac{\sigma_1(\lambda)}{\lambda^2}\geq\frac{H(x_0)^2}{G(x_0)}.$$

Since $x_0\in\Gamma_1$ is arbitrary we finally have (4.1). This concludes the proof. □

We now come to the proof of Theorem 1. As noted in the introduction, the proof is based on a scaling argument near the points where the eigenfunctions achieve their maxima.

Proof of Theorem 1. Let us prove first that

$$\limsup_{\lambda\to\infty}\frac{\sigma_1(\lambda)}{\lambda^2}<\infty. \tag{4.4}$$

Assume by contradiction that there exists a sequence $\lambda_k\to\infty$ such that $\sigma_k/\lambda_k^2\to\infty$, where $\sigma_k=\sigma_1(\lambda_k)$. Choose a positive eigenfunction u_k associated to λ_k normalized by $|u_k|_\infty = 1$, and let $x_k \in \overline{\Omega}$ be a point where u_k achieves its maximum. Thanks to the strong maximum principle we have $x_k \in \Gamma_1$.

By passing to a subsequence, we may assume $x_k \to x_0 \in \overline{\Gamma_1}$. Now two options may occur:

(a) $x_0 \in \Gamma_1$;
(b) $x_0 \in \Gamma = \overline{\Gamma_1} \cap \Gamma_2$.

Suppose first we are in case (b). With no loss of generality we may assume that $x_0 = 0$, that the unit normal at x_0 is $\nu = -e_1$ and that the unit normal to Γ in the hyperplane $x_1 = 0$ is $-e_N$. This implies that $\partial\Omega$ can be locally parameterized as $x_1 = \varphi(x')$, where $\varphi(0) = 0$, $\nabla\varphi(0) = 0$. Likewise we can parameterize Γ as $x_N = \psi(x_2, \dots, x_{N-1})$, where $\psi(0) = 0$, $\nabla\psi(0) = 0$. We remark that the gradient of φ is taken with respect to the variables $x_2, \dots, x_N$ while that of ψ is with respect to $x_2, \dots, x_{N-1}$.

We change variables in (1.4) by setting $y = h(x)$, where

$$\begin{cases} h_1(x) = x_1 - \varphi(x') \\ h_i(x) = x_i, \quad i = 2, \dots, N-1 \\ h_N(x) = x_N - \psi(x_2, \dots, x_{N-1}). \end{cases}$$

In a neighborhood V of zero, problem (1.4) is transformed into

(4.5)
$$\begin{cases} \Delta u + \sum_{i=2}^{N} a_i(y) \dfrac{\partial^2 u}{\partial y_1 \partial y_i} + \sum_{i=1}^{N-1} b_i(y) \dfrac{\partial^2 u}{\partial y_i \partial y_N} + |\nabla\varphi|^2 \dfrac{\partial^2 u}{\partial y_1^2} & \\ \quad + |\nabla\psi|^2 \dfrac{\partial^2 u}{\partial y_N^2} + c(y) \dfrac{\partial u}{\partial y_1} + d(y) \dfrac{\partial u}{\partial y_N} = \sigma_k G(y) u & \text{in } V \cap \mathbb{R}^N_+ \\ \nabla u \cdot \nu - \nabla\varphi \cdot \nu' \dfrac{\partial u}{\partial y_1} - \nabla\psi \cdot \nu'' \dfrac{\partial u}{\partial y_N} = \lambda_k H(y) u & \text{on } V \cap \partial\mathbb{R}^N_+ \cap \{y_N < 0\} \\ u = 0 & \text{on } V \cap \partial\mathbb{R}^N_+ \cap \{y_N \geq 0\} \end{cases}$$

where $a_i(y) = -2\partial\varphi/\partial x_i$, $i = 2, \dots, N$, $b_1(y) = 2\sum_{i=2}^N \partial\varphi/\partial x_i \partial\psi/\partial x_i$, $b_i(y) = -2\partial\psi/\partial x_i$, $i = 2, \dots, N-1$, $c(y) = -\Delta\varphi$, $d(y) = -\Delta\psi$, $\nu = (\nu_1, \nu') = (\nu_1, \nu'', \nu_N)$ (observe that, with a slight abuse of notation, we are still denoting by u, G, H the functions obtained after the change of variables).

It is not difficult to see that in case (a) exactly the same equations are obtained provided we set $\psi = 0$, the Robin-type boundary condition holds on $V \cap \partial\mathbb{R}^N_+$ and the Dirichlet boundary condition is absent.

Define

$$v_k(z) = u_k(y_k + \sigma_k^{-1/2} z), \quad z \in V_k := \{z \in \mathbb{R}^N : \ y_k + \sigma_k^{-1/2} z \in V \cap \mathbb{R}^N_+\}.$$

Then it is easily seen that v_k verifies the problem

$$\begin{cases} \Delta v_k + \sum_{i=2}^{N} a_i \dfrac{\partial^2 v_k}{\partial z_1 \partial z_i} + \sum_{i=1}^{N-1} b_i \dfrac{\partial^2 v_k}{\partial z_i \partial z_N} + |\nabla\varphi|^2 \dfrac{\partial^2 v_k}{\partial z_1^2} & \\ \quad + |\nabla\psi|^2 \dfrac{\partial^2 v_k}{\partial z_N^2} + \sigma_k^{-1/2} c \dfrac{\partial v_k}{\partial z_1} + \sigma_k^{-1/2} d \dfrac{\partial v_k}{\partial z_N} = G v_k & \text{in } V_k \\ \nabla v_k \cdot \nu - \nabla\varphi \cdot \nu' \dfrac{\partial v_k}{\partial z_1} - \nabla\psi \cdot \nu'' \dfrac{\partial v_k}{\partial z_N} = \sigma_k^{-1/2} \lambda_k H v_k & \text{on } T_{1,k} \\ v_k = 0 & \text{on } T_{2,k}, \end{cases}$$

where all coefficients are evaluated at $y_k + \sigma_k^{-1/2} z$, $y_k = h(x_k) = (y_k^1, \dots, y_k^N)$ and $T_{1,k} = V_k \cap \{z_1 = 0\} \cap \{z_N < -\sigma_k^{1/2} y_N^k\}$, $T_{2,k} = V_k \cap \{z_1 = 0\} \cap \{z_N \geq -\sigma_k^{1/2} y_N^k\}$.

Now we are in a position to pass to the limit. According to standard interior estimates (cf. [1], [5]), we have that the sequence $\{v_k\}$ is bounded in $C^{2,\alpha}_{\rm loc}(\mathbb{R}^N_+)$, and further applying Theorem 1 in [11] we also get boundedness in $C^{\alpha}_{\rm loc}(\overline{\mathbb{R}^N_+})$. Thus we may assume that $v_k \to v$ in v in $C^2_{\rm loc}(\mathbb{R}^N_+) \cap C_{\rm loc}(\overline{\mathbb{R}^N_+})$, where v verifies $0 \leq v \leq 1$, $v(0) = 1$.

Passing to the limit in the equation satisfied by v_k we obtain

$$\Delta v = G(0) v \quad \text{in } \mathbb{R}^N_+.$$

We next notice that the boundary condition satisfied by v depends on two different cases which may occur, depending on whether the sequence $\{\sigma_k^{1/2} y_k^N\}$ is bounded or not. Passing to a further subsequence we may assume that we have either

$$-\sigma_k^{1/2} y_k^N \to \infty$$

or

$$-\sigma_k^{1/2} y_k^N \to s \geq 0.$$

The first case gives rise to

$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial \mathbb{R}^N_+,$$

(notice that we interpret the boundary condition in the weak sense) and the second to

$$\begin{cases} \dfrac{\partial v}{\partial \nu} = 0 & \text{on } T_{1,s}, \\ v = 0 & \text{on } T_{2,s} \end{cases}$$

where $T_{1,s} = \{z_1 = 0\} \cap \{z_n < s\}$ and $T_{2,s} = \{z_1 = 0\} \cap \{z_N \geq s\}$. By a translation in the variable z_N we may set $s = 0$, and Theorem 6 is contradicted in either case. This shows (4.4).

Finally let $\lambda_k \to \infty$ be arbitrary. Thanks to (4.4) we can assume, by passing to a subsequence, that $\sigma_k / \lambda_k^2 \to \sigma_0$, and Lemma 12 gives

$$\sigma_0 \geq \sup_{\Gamma_1} \frac{H(x)^2}{G(x)}. \tag{4.6}$$

Choose eigenfunctions u_k and points $x_k \in \Gamma_1$ as before, and introduce the same change of variables to arrive at (4.5). First assume that $x_k \to 0 \in \overline{\Gamma_1} \cap \Gamma_2$. This time we define:

$$w_k(z) = u_k(y_k + \lambda_k^{-1} z), \quad z \in V_k' := \{z \in \mathbb{R}^N : \ y_k + \lambda_k^{-1} z \in V \cap \mathbb{R}^N_+\},$$

and the equation satisfied by w_k will be

$$\begin{cases} \Delta w_k + \displaystyle\sum_{i=2}^N a_i \frac{\partial^2 w_k}{\partial z_1 \partial z_i} + \sum_{i=1}^{N-1} b_i \frac{\partial^2 w_k}{\partial z_i \partial z_N} + |\nabla \varphi|^2 \frac{\partial^2 w_k}{\partial z_1^2} & \\ \qquad + |\nabla \psi|^2 \dfrac{\partial^2 w_k}{\partial z_N^2} + \lambda_k^{-1} c \dfrac{\partial w_k}{\partial z_1} + \lambda_k^{-1} d \dfrac{\partial w_k}{\partial z_N} = \dfrac{\sigma_k}{\lambda_k^2} G w_k & \text{in } V_k' \\ \nabla w_k \cdot \nu - \nabla \varphi \cdot \nu' \dfrac{\partial w_k}{\partial z_1} - \nabla \psi \cdot \nu'' \dfrac{\partial w_k}{\partial z_N} = H w_k & \text{on } \widetilde{T}_{1,k} \\ w_k = 0 & \text{on } \widetilde{T}_{2,k}, \end{cases}$$

where $\widetilde{T}_{1,k} = V_k \cap \{z_1 = 0\} \cap \{z_N < -\lambda_k y_N^k\}$, $\widetilde{T}_{2,k} = V_k \cap \{z_1 = 0\} \cap \{z_N \geq -\lambda_k y_N^k\}$ and all coefficients are evaluated at $y_k + \lambda_k^{-1} z$. As before we may assume $w_k \to w$ in $C^2_{\rm loc}(\mathbb{R}^N_+) \cap C_{\rm loc}(\overline{\mathbb{R}^N_+})$, where w verifies $0 \leq w \leq 1$, $w(0) = 1$ and

$$\Delta w = \sigma_0 G(0) w \quad \text{in } \mathbb{R}^N_+,$$

and one of the boundary conditions

$$\frac{\partial w}{\partial \nu} = H(0) w \quad \text{on } \partial\mathbb{R}^N_+$$

or

$$\begin{cases} \dfrac{\partial w}{\partial \nu} = H(0) w & \text{on } T_{1,s}, \\ w = 0 & \text{on } T_{2,s} \end{cases}$$

for some $s \geq 0$, depending on whether $-\lambda_k y_k^N \to \infty$ or $-\lambda_k y_k^N \to s$, respectively. In the second case we also may assume by a translation that $s = 0$. With a further change of independent variable we arrive at one of the problems

$$\begin{cases} \Delta w = \sigma_0 \dfrac{G(0)}{H(0)^2} w & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial w}{\partial \nu} = w & \text{on } \partial\mathbb{R}^N_+ \end{cases} \tag{4.7}$$

or

$$\begin{cases} \Delta w = \sigma_0 \dfrac{G(0)}{H(0)^2} w & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial w}{\partial \nu} = w & \text{on } T_1, \\ w = 0 & \text{on } T_2. \end{cases} \tag{4.8}$$

In case of (4.8), Theorem 5 implies $\sigma_0 G(0)/H(0)^2 < 1$, which is in contradiction with (4.6). Thus (4.7) must hold and Theorem 4 implies $\sigma_0 G(0)/H(0)^2 = 1$, that is

$$\sigma_0 = \frac{H(0)^2}{G(0)}, \tag{4.9}$$

which implies that (4.6) is in fact an equality.

In case $x_k \to 0 \in \Gamma_1$, we argue similarly as in the first part of the proof and Theorem 4 gives again (4.9). The conclusion of the theorem follows since the sequence $\{\lambda_k\}$ is arbitrary. □

Remark 2. Notice that the above proof also shows that for every sequence $\lambda_k \to \infty$, if $x_k \in \Gamma_1$ is a point where an eigenfunction u_k attains its maximum, then $\lambda_k d_{\Gamma_2}(x_k) \to \infty$, where $d_{\Gamma_2}(x)$ stands for the distance from a point x to Γ_2. That is, the maximum points "stay away" from the interface $\Gamma = \overline{\Gamma_1} \cap \Gamma_2$.

5. Planar domains with corners

We finally consider the case where Ω is a planar domain whose boundary is piecewise smooth in the sense of condition (D) in the introduction. We begin with the lower inequality.

Lemma 13. *Assume Ω is a bounded domain of $\mathbb{R}^2$ verifying (D), and let $\sigma_1(\lambda)$ be the principal eigenvalue of problem* (1.4)*. Then*

$$\liminf_{\lambda\to\infty} \frac{\sigma_1(\lambda)}{\lambda^2} \geq \max\left\{\sup_{\Gamma_1} \frac{H(x)^2}{G(x)}, \max_{1\leq j\leq m}\left\{\frac{H(x_j)^2}{G(x_j)}\ cosec^2\alpha_j\right\}\right\}, \tag{5.1}$$

where $x_1,\dots,x_m$ are the corner points of $\partial\Omega$ which belong to Γ_1.

Proof. Arguing exactly as in the proof of Lemma 12, but now choosing $x_0 \in \Gamma_1 \setminus \{x_1,\dots,x_m\}$, we obtain

$$\liminf_{\lambda\to\infty} \frac{\sigma_1(\lambda)}{\lambda^2} \geq \sup_{\Gamma_1} \frac{H(x)^2}{G(x)}. \tag{5.2}$$

Now choose $j \in \{1,\dots,m\}$ and denote for simplicity $\alpha = \alpha_j$. With no loss of generality we may assume that $x_j = 0$ and the domain Ω is (locally) contained in $x > 0$ so that the bisectrix of the angle which $\partial\Omega$ forms at 0 is the axis $y = 0$. In a neighborhood of the origin, $\partial\Omega$ can be parameterized by means of two curves $y = f(x)$, $y = g(x)$ where $g(x) > f(x)$ for $x > 0$, $g(0) = f(0) = 0$, $g'(0) = -f'(0) = \tan\alpha$ (see Figure 1).

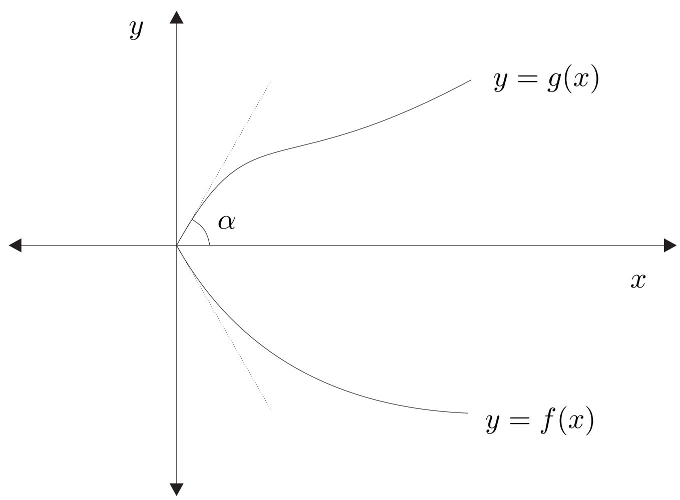

FIGURE 1. The domain $\partial\Omega$ in the neighborhood of a corner point.

Take $\varepsilon > 0$ and $r > 0$ small enough so that $H(x) \geq H(0) - \varepsilon$ if $x \in B_r(0)\cap\partial\Omega$ and $G(x) \leq G(0) + \varepsilon$ if $x \in B_r(0)\cap\Omega$. Choose $\psi \in C_0^\infty(B_r(0))$ with $0 \leq \psi \leq 1$ and define

$$u(x,y) = \psi(x,y)e^{-\lambda\gamma x}$$

for $\gamma > 0$ to be specified later. Thanks to the variational characterization (2.1) we arrive at

$$\sigma_1(\lambda) \geq \frac{\lambda(H(0)-\varepsilon)\displaystyle\int_{\Gamma_1} u^2 - \int_\Omega |\nabla u|^2}{(G(0)+\varepsilon)\displaystyle\int_\Omega u^2}. \tag{5.3}$$

Moreover, we have $|\nabla u|^2 = (\lambda^2\gamma^2\psi^2 - 2\lambda\gamma\psi_x + |\nabla\psi|^2)e^{-2\lambda\gamma x}$, so we need to examine every integral in turn. We notice that, after a change of variables, and thanks to the mean value theorem for integrals:

$$\begin{aligned}
\lambda^2\gamma^2\int_\Omega \psi^2 e^{-2\lambda\gamma x} &= \lambda^2\gamma^2\int_0^r\int_{f(x)}^{g(x)} \psi(x,y)^2 e^{-2\lambda\gamma x} dy\, dx \\
&= \lambda\gamma^2\int_0^{\lambda r}\left(\int_{f(\frac{z}{\lambda})}^{g(\frac{z}{\lambda})} \psi\left(\frac{z}{\lambda},y\right)^2 dy\right) e^{-2\gamma z} dz \\
&= \gamma^2\int_0^{\lambda r} z\psi\left(\frac{z}{\lambda},y_{\lambda,z}\right)^2 \int_0^1 \left[g'\left(\frac{sz}{\lambda}\right) - f'\left(\frac{sz}{\lambda}\right)\right] e^{-2\gamma z} ds\, dz \\
&= \frac{g'(0)}{2}\psi(0,0)^2 + o(1),
\end{aligned}$$

where $y_{\lambda,z}$ verifies $f(\frac{z}{\lambda}) < y_{\lambda,z} < g(\frac{z}{\lambda})$. In a similar way

$$2\lambda\gamma\int_\Omega \psi_x e^{-2\lambda\gamma x} = \frac{g'(0)}{\gamma\lambda}\psi_x(0,0) + o\left(\frac{1}{\lambda}\right)$$

and

$$\int_\Omega |\nabla\psi|^2 e^{-2\lambda\gamma x} = \frac{g'(0)}{2\gamma^2\lambda^2}|\nabla\psi(0,0)|^2 + o\left(\frac{1}{\lambda^2}\right).$$

Next let us estimate the surface integrals. We have

$$\begin{aligned}
\int_{\Gamma_1} \psi^2 e^{-2\lambda\gamma x} &= \int_0^r \psi(x,g(x))^2 e^{-2\lambda\gamma x}\sqrt{1+g'(x)^2}dx \\
&\quad + \int_0^r \psi(x,f(x))^2 e^{-2\lambda\gamma x}\sqrt{1+f'(x)^2}dx,
\end{aligned}$$

and we are only considering the first one, the other being treated exactly the same way. It follows similarly as before that:

$$\begin{aligned}
&\int_0^r \psi(x,g(x))^2 e^{-2\lambda\gamma x}\sqrt{1+g'(x)^2}dx \\
&\quad = \frac{1}{\lambda}\int_0^{\lambda r} \psi\left(\frac{z}{\lambda}, g\left(\frac{z}{\lambda}\right)\right) e^{-2\gamma z}\sqrt{1+g'\left(\frac{z}{\lambda}\right)^2}dz \\
&\quad = \frac{\sqrt{1+g'(0)^2}}{2\gamma\lambda}\psi(0,0)^2 + o\left(\frac{1}{\lambda}\right).
\end{aligned}$$

Hence we obtain from (5.3) that:

$$\sigma_1(\lambda) \ge \frac{(H(x_0)-\varepsilon)\dfrac{\sqrt{1+g'(0)^2}}{\gamma}\psi(0,0)^2 - \dfrac{g'(0)}{2}\psi(0,0)^2 + o(1)}{(G(x_0)+\varepsilon)\dfrac{g'(0)}{2\gamma^2\lambda^2}\psi(0,0)^2 + o\left(\dfrac{1}{\lambda^2}\right)}.$$

Dividing by λ^2, letting $\lambda \to \infty$ and then $\varepsilon \to 0$, we arrive at

$$\liminf_{\lambda\to\infty}\frac{\sigma_1(\lambda)}{\lambda^2} \ge \frac{2\gamma H(0)\sqrt{1+g'(0)^2} - g'(0)\gamma^2}{G(0)g'(0)}.$$

We choose $\gamma = H(0)\sqrt{1+g'(0)^2}/g'(0)$ and use that $(1+g'(0)^2)/g'(0)^2 = \text{cosec}^2\alpha$ to obtain

$$\liminf_{\lambda\to\infty} \frac{\sigma_1(\lambda)}{\lambda^2} \geq \frac{H(0)^2}{G(0)}\text{cosec}^2\alpha.$$

Since this inequality is valid for all points $x_1, \dots, x_m$, we deduce

$$\liminf_{\lambda\to\infty} \frac{\sigma_1(\lambda)}{\lambda^2} \geq \max_{1\leq i\leq m} \left\{ \frac{H(x_i)^2}{G(x_i)}\text{cosec}^2\alpha_i \right\}. \tag{5.4}$$

Finally, (5.1) follows thanks to (5.2) and (5.4). This concludes the proof. □

Our last step is the proof of Theorem 2.

Proof of Theorem 2. The idea is the same as in Theorem 1. We need to prove first (4.4). Thus assume there exists a sequence $\lambda_k \to \infty$ such that $\sigma_k/\lambda_k^2 \to \infty$, and let u_k be corresponding positive normalized eigenfunctions which achieve their maxima at points $x_k \in \Gamma_1$. With no loss of generality assume $x_k \to x_0 \in \overline{\Gamma_1}$.

If x_0 is a regular point of $\partial\Omega$ we reach a contradiction as in Theorem 1. Thus we suppose x_0 is a corner point x_j, $j \in \{1, \dots, m\}$, and make the same simplifying assumptions as in the proof of Lemma 13: $x_0 = 0$, and the boundary $\partial\Omega$ can be parameterized in a neighborhood of the origin by means of two curves $y = f(x)$, $y = g(x)$ where $g(x) > f(x)$ for $x > 0$, $g(0) = f(0) = 0$, $g'(0) = -f'(0) = \tan\alpha$.

We assume for simplicity that $0 \in \overline{\Gamma_1} \cap \Gamma_2$, the other case being treated similarly. For a small fixed $\delta > 0$, we introduce the change of variables

$$\begin{cases} \xi = g(x) - f(x) \\ \eta = \tan\alpha(2y - g(x) - f(x)) \end{cases}$$

where $x \in (0, \delta)$. By means of this change the curve Γ_1 is transformed into $\eta = -(\tan\alpha)\xi$, while Γ_2 gets mapped into $\eta = (\tan\alpha)\xi$. Moreover, the domain Ω is given near $(0,0)$ by $0 < \xi < \delta$, $|\eta| < (\tan\alpha)\xi$ if δ is small enough. It is not difficult to show that after the change of variables an eigenfunction u verifies the equation

$$\begin{aligned} &u_{\xi\xi}(g'-f')^2 - 2u_{\xi\eta}\tan\alpha((g')^2 - (f')^2) + u_{\eta\eta}\tan^2\alpha((g'+f')^2 + 4) \\ &\quad + u_\xi(g''-f'') - u_\eta \tan\alpha(g''+f'') = \sigma G u \end{aligned}$$

in $0 < \xi < \delta$, $|\eta| < (\tan\alpha)\xi$ (possibly we need to further reduce δ a little bit). The boundary conditions become

$$\frac{1}{\sqrt{1+(f')^2}}\left(u_\xi(g'-f')f' - u_\eta\tan\alpha(g'+f')f' - 2(\tan\alpha)u_\eta\right) = \lambda u$$

for $0 < \xi < \delta$, $\eta = -(\tan\alpha)\xi$ and

$$u(\xi, (\tan\alpha)\xi) = 0$$

if $0 < \xi < \delta$.

Define

$$v_k(z,t) = u_k(\xi_k + \sigma_k^{-1/2}z, \eta_k + \sigma_k^{-1/2}t), \quad (z,t) \in Q_k$$

where (ξ_k, η_k) is the point where u_k achieves the maximum after the change of variables (observe that $\eta_k = -(\tan\alpha)\xi_k$) and

$$Q_k := \{-\sigma_k^{1/2}\xi_k < z < \sigma_k^{1/2}(\delta - \xi_k),\ -(\tan\alpha)z < t < \tan\alpha(2\sigma_k^{1/2}\xi_k + z)\}.$$

It is easily seen that v_k verifies:

$$v_{zz}(g'-f')^2 - 2v_{zt}\tan\alpha((g')^2-(f')^2) + v_{tt}\tan^2\alpha((g'+f')^2+4) \\ +\sigma_k^{-1/2}v_z(g''-f'') - \sigma_k^{-1/2}v_t\tan\alpha(g''+f'') = Gv$$

where all functions are evaluated at $(X(\xi_k+\sigma_k^{-1/2}z,\eta_k+\sigma_k^{-1/2}t),Y(\xi_k+\sigma_k^{-1/2}z,\eta_k+\sigma_k^{-1/2}t))$ and $(X(\xi,\eta),Y(\xi,\eta))$ is the inverse function of $(\xi(x,y),\eta(x,y))$. The boundary condition is

$$\begin{cases} \dfrac{1}{\sqrt{1+(f')^2}}\big(v_z(g'-f')f' - v_t\tan\alpha(g'+f')f' - 2(\tan\alpha)v_t\big) & \\ \qquad\qquad = \sigma_k^{-1/2}\lambda_k v & \text{on } \partial Q_k\cap\Gamma_1 \\ v=0 & \text{on } \partial Q_k\cap\Gamma_2. \end{cases}$$

Now we have to distinguish two further cases: (a) $\sigma_k^{1/2}\xi_k\to\infty$ or (b) $\sigma_k^{1/2}\xi_k\to s\ge 0$. In the former we have $Q_k\to Q_\infty := \{(z,t)\in\mathbb{R}^2 :\ t>-(\tan\alpha)z\}$ (a half-plane), while in the latter $Q_k\to Q_{\alpha,s} := \{(z,t)\in\mathbb{R}^2 :\ z\ge -s,\ -\tan\alpha<t<(2\tan\alpha)s+(\tan\alpha)z\}$. Notice that $Q_{\alpha,s}$ is nothing but a cone of angle α with vertex at $(-s,-(\tan\alpha)s)$.

The necessary compactness is achieved as in Theorem 1 by means of the estimates in Theorem 1 of [11], hence we may assume $v_k\to v$ in $C^2_{\rm loc}(Q_\infty)\cap C_{\rm loc}(\overline{Q_\infty})$ in case (a) or in $C^2_{\rm loc}(Q_{\alpha,s})\cap C_{\rm loc}(\overline{Q_{\alpha,s}})$ in case (b). In any case the limit v verifies $0\le v\le 1$, $v(0)=1$ and

$$\Delta v = \frac{G(0)}{4\tan^2\alpha}v$$

in Q_∞ or $Q_{\alpha,s}$. Assume first case (a) holds. Then the equation which v verifies is easily seen to be

$$\begin{cases} \Delta v = \dfrac{G(0)}{4\tan^2\alpha}v & \text{in } Q_\infty \\ \dfrac{\partial v}{\partial\nu}=0 & \text{on } \partial Q_\infty \end{cases}$$

where ν is the outward unit normal to ∂Q_∞. With a further rotation and scaling, we can reduce the situation to the case where the domain is $\mathbb{R}^2_+$ and the equation $\Delta v = v$, and we reach a contradiction with Theorem 6.

If case (b) holds that v verifies

$$\begin{cases} \Delta v = \dfrac{G(0)}{4\tan^2\alpha}v & \text{in } Q_{\alpha,s} \\ \dfrac{\partial v}{\partial\nu}=0 & \text{on } T_{1,\alpha,s} \\ v=0 & \text{on } T_{2,\alpha,s}, \end{cases}$$

where $T_{1,\alpha,s}=\{(z,t) :\ z>-s,\ t=-(\tan\alpha)z\}$ and $T_{2,\alpha,s}=\partial Q_{\alpha,s}\setminus T_{1,\alpha,s}$. By means of a translation and a scaling we arrive to a contradiction with Theorem 11.

As we have already said, the case $0\in\Gamma_1$ is dealt with similarly, the necessary contradiction furnished by Theorems 6 and 11. Thus we have proved (4.4).

Now let $\lambda_k \to \infty$ be an arbitrary sequence. Passing to a subsequence we have $\sigma_k/\lambda_k^2 \to \sigma_0$, where

$$\sigma_0 \geq \max\left\{\sup_{\Gamma_1} \frac{H(x)^2}{G(x)}, \max_{1\leq j\leq m}\left\{\frac{H(x_j)^2}{G(x_j)}\operatorname{cosec}^2\alpha_j\right\}\right\} \tag{5.5}$$

thanks to Lemma 13. Let u_k be the corresponding eigenfunctions, achieving maxima in $x_k \in \Gamma_1$, and assume again $x_k \to x_0 = 0$. If x_0 is a smooth point of $\partial\Omega$, it follows as in the proof of Theorem 1 that

$$\sigma_0 = \frac{H(0)^2}{G(0)}.$$

On the other hand, if x_0 is a corner point of $\partial\Omega$, we can argue as in the first part of the proof (assuming $x_0 \in \overline{\Gamma_1} \cap \Gamma_2$) and find that the functions

$$w_k(z,t) = u_k(\xi_k + \lambda_k^{-1}z, \eta_k + \lambda_k^{-1}t)$$

verify $w_k \to w$, where w is a solution either to

$$\begin{cases} \Delta w = \sigma_0 \dfrac{G(0)}{4\tan^2\alpha} w & \text{in } \mathbb{R}^N_+ \\ \dfrac{\partial w}{\partial \nu} = \dfrac{H(0)}{2\tan\alpha} w & \text{on } \partial\mathbb{R}^N_+ \end{cases}$$

or to

$$\begin{cases} \Delta w = \sigma_0 \dfrac{G(0)}{4\tan^2\alpha} w & \text{in } Q_\alpha \\ \dfrac{\partial w}{\partial \nu} = \dfrac{H(0)}{2\tan\alpha} w & \text{on } T_{1,\alpha} \\ w = 0 & \text{on } T_{2,\alpha}. \end{cases}$$

A further scaling reduces these two to

$$\begin{cases} \Delta w = \sigma_0 \dfrac{G(0)}{H(0)^2} w & \text{in } \mathbb{R}^N_+ \\ \dfrac{\partial w}{\partial \nu} = w & \text{on } \partial\mathbb{R}^N_+ \end{cases} \tag{5.6}$$

and

$$\begin{cases} \Delta w = \sigma_0 \dfrac{G(0)}{H(0)^2} w & \text{in } Q_\alpha \\ \dfrac{\partial w}{\partial \nu} = w & \text{on } T_{1,\alpha} \\ w = 0 & \text{on } T_{2,\alpha}. \end{cases}$$

The second case is impossible since, according to Theorem 9 we would have $\sigma_0 < H(0)^2/G(0)$, which is in contradiction with (5.5). In the first case, Theorem 4 would give $\sigma_0 = H(0)^2/G(0)$.

Finally, in the case where $x_0 \in \Gamma_1$ is a corner point, we would obtain the problems (5.6) or

$$\begin{cases} \Delta w = \sigma_0 \dfrac{G(0)}{H(0)^2} w & \text{in } Q_\alpha \\ \dfrac{\partial w}{\partial \nu} = w & \text{on } \partial Q_\alpha. \end{cases}$$

The first case gives $\sigma_0 = H(0)^2/G(0)$, while the second, by means of Theorem 7 gives $\sigma_0 = H(0)^2 \text{cosec}^2\alpha/G(0)$. In any case, (5.5) implies

$$\sigma_0 = \max\left\{\sup_{\Gamma_1} \frac{H(x)^2}{G(x)}, \max_{1\le j\le m}\left\{\frac{H(x_j)^2}{G(x_j)}\,\text{cosec}^2\alpha_j\right\}\right\},$$

and since the sequence $\{\lambda_k\}$ is arbitrary we have finally shown (1.6). This concludes the proof. □

Remark 3. As in the case of smooth domains, it is also a consequence of this proof that for every sequence $\lambda_k \to \infty$, if $x_k \in \Gamma_1$ is a point where an eigenfunction u_k attains its maximum, then $\lambda_k d_{\Gamma_2}(x_k) \to \infty$, where $d_{\Gamma_2}(x)$ stands for the distance from a point x to Γ_2.

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E. Colorado
Departamento de Matemáticas, Universidad Carlos III de Madrid
Avda. Universidad nº 30, 28911 – Leganés, Madrid, SPAIN
E-mail address: ecolorad@math.uc3m.es

J. García-Melián
Departamento de Análisis Matemático, Universidad de La Laguna.
C/. Astrofísico Francisco Sánchez s/n, 38271 – La Laguna, SPAIN
and
Instituto Universitario de Estudios Avanzados (IUdEA) en Física Atómica, Molecular y Fotónica, Facultad de Física, Universidad de La Laguna
C/. Astrofísico Francisco Sánchez s/n, 38203 – La Laguna , SPAIN
E-mail address: jjgarmel@ull.es