A REMARK ON UNIQUENESS OF LARGE SOLUTIONS FOR ELLIPTIC SYSTEMS OF COMPETITIVE TYPE*†

Jorge García-Melián ‡

Dpto. de Análisis Matemático, Universidad de La Laguna,
C/ Astrofísico Francisco Sánchez s/n, 38271 - La Laguna, SPAIN

Abstract

We prove that the semilinear system $\Delta u = a(x)u^p v^q$, $\Delta v = b(x)u^r v^s$ in a smooth bounded domain $\Omega \subset \mathbb{R}^N$ has a unique positive solution with the boundary condition $u = v = +\infty$ on $\partial\Omega$, provided that $p, s > 1$, $q, r > 0$ and $(p-1)(s-1) - qr > 0$. The main novelty is imposing a growth on the possibly singular weights $a(x)$, $b(x)$ near $\partial\Omega$, rather than requiring them to have a precise asymptotic behavior.

1. Introduction and results

This work is concerned with the uniqueness issue of positive solutions to the boundary blow-up elliptic system

$$\begin{cases} \Delta u = a(x)u^p v^q & \text{in} \quad \Omega \\ \Delta v = b(x)u^r v^s & \text{in} \quad \Omega \\ u = v = +\infty & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $p, s > 1$, $q, r > 0$, Ω is a bounded C^2 domain of $\mathbb{R}^N$, $a, b \in C^\eta(\Omega)$ are positive weight functions, which may be singular on $\partial\Omega$ and $0 < \eta < 1$. The boundary condition is interpreted as $u(x), v(x) \to +\infty$ as $d(x) = \operatorname{dist}(x, \partial\Omega) \to 0+$.

There is a large amount of literature on elliptic problems related to (1.1). Scalar equations have been usually considered, and little work has been devoted to systems. For the particular equation $\Delta u = a(x)u^p$, $p > 1$, which is linked to (1.1) in an important fashion, almost everything is known: existence and nonexistence of solutions, uniqueness, boundary behavior of the solution and its normal derivatives, second order estimates near the boundary, etc. The assumptions on the weight $a(x)$ have varied along the years: first, it was assumed to be bounded and bounded away from zero in $\overline{\Omega}$ in [25], [24], [1], [2], [30], [29], [11] (in this case for the $p-$Laplacian operator) or [10]; then, it was allowed to vanish on $\partial\Omega$ in [15], [18], [27] (see also [14] for the $p-$Laplacian setting); and finally it could also be unbounded on $\partial\Omega$: see [31], [4], [5]. Some recent work trying to find optimal conditions on $a(x)$ to ensure uniqueness has also been done in [7], [8], [6], [28] or [17]. We refer the interested reader to

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‡E-mail address: jjgarmel@ull.es

[16] for an extensive list of references with more general nonlinearities $f(u)$, as the important pioneering works [3] and [22].

As for systems, there are only some particular results. We quote [9] for predator-prey Lotka–Volterra systems, [12], [13], [19], [26] and [21] for competitive type systems and [20] for cooperative systems. In spite of this, large solutions were only explicitly treated in [19], [20] and [21].

Problem (1.1) was deeply analyzed in [21]. Some existence and nonexistence results, together with uniqueness and boundary behavior of solutions were obtained under the assumption $a(x) \sim C_1 d(x)^{\gamma_1}$ and $b(x) \sim C_2 d(x)^{\gamma_2}$ as $d(x) \to 0$ for some $C_1, C_2 > 0$ and $\gamma_1, \gamma_2 > -2$ (cf. Theorem 11 there). We remark that the uniqueness proof there was achieved by means of boundary estimates for the solutions, a procedure which is usual in the literature. For the scalar problem $\Delta u = a(x) u^p$, it has been shown however that the exact boundary behavior of solutions is not needed, but only estimates of this behavior (see [23], [5] and [17]; the method is also used in [16] to deal with some more general nonlinearities).

Our intention in the present work is to extend the idea in [23] to deal with problem (1.1). Since only global estimates for the solutions are then needed, it is not necessary to impose a boundary behavior to the weights $a(x)$ and $b(x)$, but only a control on their growth near $\partial\Omega$. Thus we are assuming that $a, b \in C^\eta(\Omega)$ for some $\eta \in (0,1)$ and there exist $\gamma_1, \gamma_2 \in \mathbb{R}$ and positive constants C_1, C_2, C_1', C_2' such that

$$\left.\begin{array}{l} C_1 d(x)^{\gamma_1} \le a(x) \le C_2 d(x)^{\gamma_1} \\ C_1' d(x)^{\gamma_2} \le b(x) \le C_2' d(x)^{\gamma_2} \end{array}\right. \quad \text{for } x \in \Omega. \tag{W}$$

We remark that in this case weak solutions to (1.1) are indeed classical, i. e. $u, v \in C^{2,\eta}(\Omega)$. However, the smoothness assumption could be weakened to have $a, b \in L^\infty_{\rm loc}(\Omega)$, and in that case solutions would belong to $W^{1,p}_{\rm loc}(\Omega) \cap C^{1,\nu}(\Omega)$ for all $p > 1$ and $\nu \in (0,1)$.

We state next our main result, which is an extension of Theorem 11 of [21]. Although our main interest is uniqueness, we are also including the issues of existence and nonexistence for the sake of completeness.

Theorem 1. *Assume $p, s > 1$, $q, r > 0$ and $(p-1)(s-1) - qr > 0$, and let $a, b \in C^\eta(\Omega)$ for some $\eta \in (0,1)$ verify hypothesis (W). Then problem (1.1) admits a positive classical solution (u,v) if and only if $\gamma_1, \gamma_2 > -2$ and*

$$\frac{q}{s-1} < \frac{2+\gamma_1}{2+\gamma_2} < \frac{p-1}{r}. \tag{1.2}$$

This solution is in addition unique, and verifies

$$\left.\begin{array}{l} D_1 d(x)^{-\alpha} \le u(x) \le D_2 d(x)^{-\alpha} \\ D_1' d(x)^{-\beta} \le v(x) \le D_2' d(x)^{-\beta} \end{array}\right. \quad \textit{for } x \in \Omega$$

for some positive constants D_1, D_2, D_1', D_2', where

$$\alpha = \frac{(\gamma_1+2)(s-1) - (\gamma_2+2)q}{(p-1)(s-1) - qr}, \quad \beta = \frac{(\gamma_2+2)(p-1) - (\gamma_1+2)r}{(p-1)(s-1) - qr}.$$

Organization of the paper: in Section 2 we collect some preliminary results which are essential for the proof of Theorem 1, which will be performed in Section 3.

2. Preliminaries

This section is dedicated to recall some already known results which will be used in the proof of Theorem 1. They all deal with the reference equation

$$\begin{cases} \Delta u = d(x)^\gamma u^p & \text{in} \quad \Omega, \\ u = +\infty & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where $d(x)$ stands for the distance of a point $x \in \Omega$ to the boundary $\partial\Omega$, $\gamma \in \mathbb{R}$ and $p > 1$. We are not including most of the proofs, but refer the reader to [21]. Only the case $\gamma < 0$ was considered there, but this is precisely the most interesting for our purposes.

Lemma 2. *Problem (2.1) has a positive classical solution if and only if $\gamma > -2$. In that case, the solution is unique and will be denoted by $U_{p,\gamma}$. This solution is obtained as the limit as $n \to +\infty$ of the solution U_n to the problem*

$$\begin{cases} \Delta u = d(x)^\gamma u^p & \textit{in} \quad \Omega, \\ u = n & \textit{on} \quad \partial\Omega. \end{cases}$$

Moreover,

$$\lim_{x \to x_0} d(x)^\alpha U_{p,\gamma}(x) = (\alpha(\alpha+1))^{\frac{1}{p-1}}$$

for every $x_0 \in \partial\Omega$, where $\alpha = (2+\gamma)/(p-1)$.

As a consequence of Lemma 2, the quantities

$$A_{p,\gamma} := \sup_{x\in\Omega} d(x)^\alpha U_{p,\gamma}(x), \qquad B_{p,\gamma} := \inf_{x\in\Omega} d(x)^\alpha U_{p,\gamma}(x) \tag{2.2}$$

are finite and positive. We are quoting one of their important properties.

Lemma 3. *The quantities $A_{p,\gamma}$ and $B_{p,\gamma}$ are bounded and bounded away from zero when γ is bounded and bounded away from -2. Also,*

$$\lim_{\gamma \to -2+} A_{p,\gamma} = \lim_{\gamma \to -2+} B_{p,\gamma} = 0\ .$$

We are stating next a comparison lemma related to problem (2.1). Its proof follows thanks to uniqueness, the method of sub and supersolutions and a suitable rescaling.

Lemma 4. *Let $u \in C^2(\Omega)$ verify $\Delta u \le C d(x)^\gamma u^p$ in Ω for some positive constant C, and $u = +\infty$ on $\partial\Omega$. Then $u(x) \ge C^{-\frac{1}{p-1}} U_{p,\gamma}(x)$. Similarly, if $\Delta u \ge C d(x)^\gamma u^p$ in Ω, then $u(x) \le C^{-\frac{1}{p-1}} U_{p,\gamma}(x)$.*

We close this section by proving that subsolutions to problem (2.1) are bounded in the regime $\gamma \le -2$. This will also be used in the nonexistence proof.

Lemma 5. *Let $u \in C^2(\Omega)$ verify $\Delta u \ge C d^\gamma u^p$ in Ω, for some $\gamma \le -2$. Then u is bounded in Ω.*

Proof. Let $x \in \Omega$, and introduce the function $v(y) = d(x)^\alpha u(x + d(x)y)$, for $y \in B_{1/2}(0)$ with $\alpha = (2+\gamma)/(p-1) \le 0$. It is not hard to see that v verifies $\Delta v \ge Cv^p$ in $B_{1/2}(0)$, and by Lemma 4 $v \le C^{-\frac{1}{p-1}} U_{p,0}$ in $B_{1/2}(0)$, where $U_{p,0}$ denotes in this context the solution to (2.1) in $B_{1/2}(0)$ with $\gamma = 0$. In particular, for $y = 0$, we have

$$u(x) = v(0) d(x)^{-\alpha} \le C^{-\frac{1}{p-1}} U_{p,0}(0) d(x)^{-\alpha}$$

in Ω, which implies that u is bounded since $\alpha \le 0$. This concludes the proof. □

Remark 1. Observe that the proof of Lemma 5 also shows that when $\gamma < -2$ then necessarily $u = 0$ on $\partial\Omega$.

3. Proof of Theorem 1

We are now ready to prove Theorem 1, whose proof will be split in several lemmas. We begin by showing that condition (1.2) is sufficient for the existence of positive solutions. Throughout the whole section, we are assuming without loss of generality that $C_1 = C_1'$ and $C_2 = C_2'$ in condition (W).

Lemma 6. *Assume $\gamma_1, \gamma_2 > -2$ and (1.2) holds. Then problem (1.1) admits at least one positive classical solution (u, v).*

Proof. We use the method of sub and supersolutions. We remark that the system in (1.1) is of competitive type, and hence the sub and supersolutions are to be considered with one of the inequalities reversed (see the Appendix in [21]). We are searching for the sub and supersolutions with the aid of Lemma 2.

We look for a subsolution of the form $(\underline{u}, \underline{v}) = (\varepsilon U_{p,\tau}, \varepsilon^{-\delta} U_{s,\sigma})$, where we are using the notation of Section 2, $\varepsilon > 0$ is small and $\tau, \sigma, \delta > 0$ are to be chosen. Then $(\underline{u}, \underline{v})$ will be a subsolution provided

$$\varepsilon^{p-\delta q-1} C_2 d^{\gamma_1 - \tau} U_{s,\sigma}^q \le 1, \qquad \varepsilon^{r-\delta(s-1)} C_1 d^{\gamma_2-\sigma} U_{p,\tau}^r \ge 1. \tag{3.1}$$

It is not hard to show that condition (1.2) allows us to select $\tau, \sigma > -2$ so that

$$\frac{\gamma_1 - \tau}{q} = \frac{\sigma + 2}{s-1}, \qquad \frac{\gamma_2 - \sigma}{r} = \frac{\tau + 2}{p-1}$$

and hence inequality (3.1) will hold for small ε if $p - \delta q - 1 > 0$, $r - \delta s + \delta < 0$, thanks to Lemma 2. Thus, fixing $r/(s-1) < \delta < (p-1)/q$, we obtain a subsolution. It is similarly checked that $(\overline{u}, \overline{v}) = (MU_{p,\tau}, M^{-\delta} U_{s,\sigma})$ is a supersolution when $M > 0$ is large enough. Since $\underline{u} \le \overline{u}$, $\underline{v} \ge \overline{v}$, Theorem A.2 in the Appendix of [21] implies the existence of a positive solution (u, v) to (1.1) (we mention that that theorem was considered there for the case $a(x) = b(x) = 1$, but its generalization is straightforward). □

Before proving uniqueness to (1.1), we need to obtain rough estimates for the boundary behavior of all possible positive solutions (the estimates are indeed global, since solutions are smooth in Ω). This is the content of the next result.

Lemma 7. *Let (u,v) be a positive classical solution to* (1.1)*. Then there exist constants D_1, D_2, D_1', D_2' such that*

$$\begin{aligned} D_1 d(x)^{-\alpha} \le u(x) \le D_2 d(x)^{-\alpha} \\ D_1' d(x)^{-\beta} \le v(x) \le D_2' d(x)^{-\beta} \end{aligned} \qquad \text{for } x\in\Omega,$$

where

$$\alpha = \frac{(\gamma_1+2)(s-1)-(\gamma_2+2)q}{(p-1)(s-1)-qr}, \quad \beta = \frac{(\gamma_2+2)(p-1)-(\gamma_1+2)r}{(p-1)(s-1)-qr}. \tag{3.2}$$

Proof. We use the iterative argument in [21]. Since v is strictly positive, it follows that $\Delta u \ge C_1(\inf_\Omega v^q) d(x)^{\gamma_1} u^p$ in Ω, and by Lemma 4 and the definition (2.2)

$$u \le (C_1 \inf_\Omega v^q)^{-\frac{1}{p-1}} U_{p,\gamma_1} \le (C_1 \inf_\Omega v^q)^{-\frac{1}{p-1}} A_{p,\gamma_1} d(x)^{-\alpha_1},$$

where $\alpha_1 = (2+\gamma_1)/(p-1)$. Inserting this inequality in the second equation of (1.1), we obtain $\Delta v \le C_2 (C_1 \inf_\Omega v^q)^{-\frac{r}{p-1}} A^r_{p,\gamma_1} d(x)^{\gamma_2 - \alpha_1 r} v^s$ and thus again by Lemma 4 and (2.2):

$$v \ge (C_2 (C_1 \inf_\Omega v^q)^{-\frac{r}{p-1}} A^r_{p,\gamma_1})^{-\frac{1}{s-1}} B_{s,\gamma_2-\alpha_1 r} d(x)^{-\beta_1},$$

where $\beta_1 = (2+\gamma_2-\alpha_1 r)/(s-1)$. We now iterate this procedure to obtain that:

$$\begin{aligned} u(x) &\le M_n d(x)^{-\alpha_n} \\ v(x) &\ge N_n d(x)^{-\beta_n}, \end{aligned} \tag{3.3}$$

where M_n, N_n, are recursively defined as

$$\begin{aligned} M_n &= (C_1 N^q_{n-1})^{-\frac{1}{p-1}} A_{p,\gamma_1-\beta_{n-1}q} \\ N_n &= (C_2 M^r_n)^{-\frac{1}{s-1}} B_{s,\gamma_2-\alpha_n r} \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \alpha_n &= \frac{2+\gamma_1-\beta_{n-1}q}{p-1} \\ \beta_n &= \frac{2+\gamma_2-\alpha_n r}{s-1}. \end{aligned} \tag{3.5}$$

It is then easily seen that

$$\alpha_n = \frac{2+\gamma_1}{p-1} - \frac{q}{p-1}\frac{2+\gamma_2}{s-1} + \frac{qr}{(p-1)(s-1)}\,\alpha_{n-1},$$

and since $qr/(p-1)(s-1) < 1$ the sequence α_n converges to α, given by (3.2). This implies that $\beta_n \to \beta$, also given by (3.2).

On the other hand, we have for M_n that:

$$M_n = \left(C_1 C_2^{-\frac{q}{s-1}} B^q_{s,\gamma_2-\alpha_{n-1}r} \right)^{-\frac{1}{p-1}} A_{p,\gamma_1-\beta_{n-1}q} M_{n-1}^{\frac{qr}{(p-1)(s-1)}}. \tag{3.6}$$

Now since $\gamma_1 - \beta_{n-1} q$ and $\gamma_2 - \alpha_{n-1} r$ are bounded and bounded away from -2, we obtain from Lemma 3 the existence of a constant $K > 0$ such that

$$M_n \leq K M_{n-1}^{\delta},$$

where $\delta = \frac{qr}{(p-1)(s-1)} < 1$. Iterating this inequality we arrive at

$$M_n \leq K^{1+\delta+\cdots+\delta^{n-2}} M_1^{\delta^{n-1}}.$$

Letting $n \to \infty$, we have that M_n is bounded from above by a positive constant D_2, and it is similarly shown that N_n is bounded from below by a positive constant D_1'. This implies, passing to the limit in (3.3) that $u \leq D_2 d(x)^{-\alpha}$, $v \geq D_1' d(x)^{-\beta}$. The symmetric reasoning gives positive constants D_1 and D_2' such that $u \geq D_1 d(x)^{-\alpha}$ and $v \leq D_2' d(x)^{-\beta}$. This proves the lemma. □

We now proceed to prove uniqueness of positive solutions to (1.1). The proof is an adaptation of that in Lemma 10 of [21], but substituting the use of the maximum principle by an adaptation of the device in [23] (see also [17], where a refinement was already made).

Lemma 8. *Problem* (1.1) *admits at most one positive classical solution.*

Proof. Let (u_1, v_1), (u_2, v_2) be positive solutions to (1.1). We first remark that, thanks to Lemma 7, the quotients u_1/u_2 and v_1/v_2 are bounded and bounded away from zero. Thus let $\theta = \sup_\Omega u_1/u_2$, and assume $\theta > 1$.

Let $\delta > 0$ be such that

$$\frac{r}{s-1} < \delta < \frac{p-1}{q}.$$

We claim that $v_2 \leq \theta^\delta v_1$ in Ω. Suppose not. Then there exists a point $x_0 \in \Omega$ such that $v_2(x_0) > \theta^\delta v_1(x_0)$. Let $\Omega_0 = \{v_2 > \theta^\delta v_1\} \cap B_\rho(x_0)$, where $\rho = d(x_0)/2$. Then in Ω_0 we have

$$\Delta(v_2 - \theta^\delta v_1) = b(x)(u_2^r v_2^s - \theta^\delta u_1^r v_1^s) > \theta^\delta b(x)(\theta^{\delta(s-1)-r} - 1) u_1^r v_1^s.$$

Thanks to the choice of δ and the estimates furnished by Lemma 7, taking into account that $\rho/2 \leq d(x) \leq 3\rho/2$ in Ω_0, we obtain

$$\Delta(v_2 - \theta^\delta v_1) > C\theta^\delta \rho^{\gamma_2 - \alpha r - \beta s} = C\theta^\delta \rho^{-\beta-2} \tag{3.7}$$

(in the rest of the proof, C will denote a positive constant, not depending on x_0). Now let ϕ be the solution to $-\Delta\phi = 1$ in $B_\rho(x_0)$, with $\phi = 0$ on $\partial B_\rho(x_0)$. It is immediately seen that $\phi(x) = C(\rho^2 - |x - x_0|^2)$ for some positive constant C. Thanks to (3.7), we have $\Delta(v_2 - \theta^\delta v_1 + C\theta^\delta \rho^{-\beta-2}\phi) > 0$ in Ω_0, and the maximum principle gives a point $x_1 \in \partial\Omega_0$ such that

$$v_2(x_0) - \theta^\delta v_1(x_0) + C\theta^\delta \rho^{-\beta-2}\phi(x_0) < v_2(x_1) - \theta^\delta v_1(x_1) + C\theta^\delta \rho^{-\beta-2}\phi(x_1). \tag{3.8}$$

If we assume that $v_2(x_1) = \theta^\delta v_1(x_1)$, we obtain that $\phi(x_0) < \phi(x_1)$, which is a contradiction, since $\phi(x_0) = \sup_{B_\rho(x_0)} \phi$. Thus $v_2(x_1) > \theta^\delta v_1(x_1)$ and this implies $x_1 \in \partial B_\rho(x_0)$. Hence $\phi(x_1) = 0$ and (3.8) gives

$$C\theta^\delta \rho^{-\beta} < v_2(x_1) - \theta^\delta v_1(x_1).$$

We now use again the estimates of Lemma 7, and find that $\rho^{-\beta} \geq C v_1(x_1)$. This entails $v_2(x_1) > (1+C)\theta^\delta v_1(x_1)$, where C is a constant which does not depend on x_0 or on x_1. We can iterate this procedure to obtain a sequence $\{x_n\} \subset \Omega$ such that

$$v_2(x_n) > (1+C)^n \theta^\delta v_1(x_n),$$

and this clearly implies that v_2/v_1 is not bounded, which is impossible. This proves our claim.

Now let $\theta_n = \theta - 1/n$, and choose a sequence $\{x_n\} \subset \Omega$ such that $u_1(x_n) > \theta_n u_2(x_n)$. Introduce the set $\Omega_n = \{u_1 > \theta_n u_2\} \cap B_{\rho_n}(x_n)$, where $\rho_n = d(x_n)/2$. Then in Ω_n:

$$\Delta(u_1 - \theta_n u_2) > \theta_n a(x)(\theta_n^{p-1}\theta^{-\delta q} - 1)u_2^p v_2^q.$$

Since $\theta_n^{p-1}\theta^{-\delta q} \to \theta^{p-1-\delta q} > 1$, by the choice of δ, we have for a positive constant C, as in the first part of the proof:

$$\Delta(u_1 - \theta_n u_2) > C\theta_n \rho_n^{\gamma_1 - \alpha p - \beta q} = C\theta_n \rho_n^{-\alpha-2}$$

in Ω_n, for large n. Introducing the function ϕ as before, but in the ball $B_{\rho_n}(x_n)$ instead, and applying the maximum principle, we get the existence of points $y_n \in \partial\Omega_n$ such that

$$u_1(x_n) - \theta_n u_2(x_n) + C\theta_n \rho_n^{-\alpha-2}\phi(x_n) < u_1(y_n) - \theta_n u_2(y_n) + C\theta_n \rho_n^{-\alpha-2}\phi(y_n).$$

It follows in the same manner that $\phi(y_n) = 0$, and then $C\theta_n\rho_n^{-\alpha} < u_1(y_n) - \theta_n u_2(y_n)$. By means of the estimates for u_1 we finally arrive at

$$u_1(y_n) > (1+C)\theta_n u_2(y_n).$$

Thanks to the definition of θ as the supremum of u_1/u_2, we then have $\theta > (1+C)\theta_n$, and we arrive at a contradiction when we let n go to infinity.

This contradiction shows that $\theta \leq 1$, that is, $u_1 \leq u_2$. The symmetric argument proves then that $u_1 = u_2$, and the first equation in (1.1) finally implies $v_1 = v_2$, as we wanted to prove. $\square$

We finally show that the conditions $\gamma_1, \gamma_2 > -2$ and (1.2) are necessary for (1.1) to have a positive solution.

Lemma 9. *Assume problem* (1.1) *has a positive classical solution* (u, v). *Then* $\gamma_1, \gamma_2 > -2$ *and* (1.2) *holds.*

Proof. Assume (1.1) admits a positive classical solution (u, v) with $\gamma_1 \leq -2$. Since v is bounded from below, we have $\Delta u \geq C_1(\inf_\Omega v^q) d(x)^{\gamma_1} u^p$ in Ω, and it follows from Lemma 5 that u is bounded, a contradiction. Thus $\gamma_1 > -2$ and it is similarly proved that $\gamma_2 > -2$.

Now assume (1.2) does not hold. With no loss of generality, we may assume

$$\frac{2+\gamma_1}{2+\gamma_2} \leq \frac{q}{s-1} < \frac{p-1}{r}, \tag{3.9}$$

the other case being handled similarly (the second inequality always holds thanks to our assumption $(p-1)(s-1) - qr > 0$). We are using an iterative procedure similar to the one already employed in the proof of Lemma 7 to arrive at a contradiction (cf. also [21]).

To this aim, assume initially that the first inequality in (3.9) is strict. Notice that inequalities (3.3) with M_n, N_n given by (3.4) and α_n, β_n by (3.5) still hold, but $\alpha_n \to \alpha < 0$. Since $\alpha_1 > 0$, we can select an appropriate n so that $\alpha_n > 0$, $\alpha_{n+1} \le 0$, which is equivalent to $\beta_{n-1}q < 2+\gamma_1$ and $\beta_n q \ge 2+\gamma_1$. Since, according to (3.3), $v \ge N_n d^{-\beta_n}$, we have that

$$\Delta u \ge C_1 N_n^q d^{\gamma_1-\beta_n q} u^p$$

in Ω, and since $\gamma_1 - \beta_n q \le -2$ Lemma 5 implies that u is bounded, a contradiction.

Now consider the remaining case $(\gamma_1+2)(s-1)-(\gamma_2+2)q = 0$. The iterative procedure and estimates (3.3) are valid, with $\alpha_n \to \alpha = 0$ while $\beta_n \to \beta > 0$. Now notice that $\gamma_1 - \beta_{n-1}q \to -2$, while $\gamma_2 - \alpha_{n-1}r$ is bounded and bounded away from -2. Thus by Lemma 3, $A_{p,\gamma_1-\beta_{n-1}q} \to 0$ while $B_{s,\gamma_2-\alpha_{n-1}r}$ is bounded and bounded away from zero. Hence (3.6) implies that for $\varepsilon > 0$

$$M_n \le \varepsilon M_{n-1}^{\delta},$$

where $\delta = qr/(p-1)(s-1) < 1$, provided that n is large enough. Proceeding as in the proof of Lemma 7, it is easily seen that this entails

$$\limsup_{n\to\infty} M_n \le \varepsilon^{\frac{1}{1-\delta}},$$

and since ε is arbitrary, this shows that $M_n \to 0$. Thus (3.3) gives that $u \equiv 0$, which is not possible. This concludes the proof. $\square$

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