MAXIMUM AND COMPARISON PRINCIPLES FOR OPERATORS INVOLVING THE p-LAPLACIAN *

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Abstract

In this paper some characterizations for the validity of both maximum and weak comparison principles for the operator $\mathcal{L}_p u = -\Delta_p u + a(x)|u|^{p-2}u$, under Dirichlet conditions, are given. Some comparison and nonresonance results for sublinear operators of the form $-\Delta_p u + f(x,u)$ are also studied.

1. Introduction

The main objective of this work is to provide a characterization of both the maximum and the comparison principles for the operator $\mathcal{L}_p u := -\Delta_p u + a(x)\varphi_p(u)$. The operator $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$, $p > 1$, $\frac{1}{p} + \frac{1}{p'} = 1$, stands for the well-known p-Laplacian defined as $\langle -\Delta_p u, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v \; dx$, where $W_0^{1,p}(\Omega)$ is the Sobolev space of weakly differentiable functions in $L^p(\Omega)$ which vanish on $\partial\Omega$, with first derivatives also in $L^p(\Omega)$. $W^{-1,p'}(\Omega)$ is its dual space (cf. [13]), and $\varphi_p(z) := |z|^{p-2}z$. It will be assumed henceforth that $\Omega \subset \mathbf{R^n}$ is a bounded domain of class $C^{2+\alpha}$, $0 < \alpha < 1$, with outward unit normal ν on $\partial\Omega$ meanwhile the weight function $a = a(x)$ belongs to $L^\infty(\Omega)$.

Consider $f \in W^{-1,p'}(\Omega)$. As usual, $u \in W^{1,p}_{\rm loc}(\Omega)$ is defined to be a weak solution to the equation $-\Delta_p u + a(x)\varphi_p(u) = f(x)$ if

$$\int_\Omega \left\{ |\nabla u|^{p-2}\nabla u \nabla v + a\varphi_p(u)v \right\} \; dx = \langle f, v\rangle \,, \tag{1.1}$$

for each $v \in C_0^\infty(\Omega)$. A supersolution (respectively, subsolution) $u \in W^{1,p}_{\rm loc}(\Omega)$ is defined in the same way by changing " $=$ " by " $\geq$ " (r. " $\leq$ ") in (1.1) and requiring that the test functions $v \in C_0^\infty(\Omega)$ be nonnegative. However, we are dealing here with solutions, sub and supersolutions $u \in W^{1,p}(\Omega)$ and in this case $C_0^\infty(\Omega)$ can be replaced by $W_0^{1,p}(\Omega)$. On the other hand, we shall normally work with $f \in L^\infty(\Omega)$ (cf. Section 2).

It will be said, in the context of the present work, that the operator $\mathcal{L}_p$ satisfies the maximum principle (MP for short) if every weak solution $u \in W^{1,p}(\Omega)$ to

$$\begin{cases} -\Delta_p u + a(x)\varphi_p(u) = f(x) & \text{in} \quad \Omega \\ u \geq 0 & \text{on} \quad \partial\Omega, \end{cases}$$

*This research was supported in part by a DGICYT project under contract PB930465.

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with $f \geq 0$, verifies $u(x) \geq 0$ in Ω. $\mathcal{L}_p$ satisfies the strong maximum principle (SMP for short) if, in addition, $u(x) > 0$ in Ω whenever $f \neq 0$. Similarly, $\mathcal{L}_p$ is said to satisfy the weak comparison principle (WCP for short) if $\mathcal{L}_p u_1 \leq \mathcal{L}_p u_2$ in Ω together with $u_1 \leq u_2$ on $\partial\Omega$, with $u_i \in W^{1,p}(\Omega)$, $i = 1, 2$, implies $u_1 \leq u_2$ in Ω.

The operator $\mathcal{L}_p$ under Dirichlet conditions has a first eigenvalue $\lambda_{1,p}(a)$ which is variationally defined as,

$$\lambda_{1,p}(a) = \inf \int_\Omega \{|\nabla u|^p + a|u|^p\} \, dx, \tag{1.2}$$

the infimum being extended to all $u \in W_0^{1,p}(\Omega)$ under the restriction $\int_\Omega |u|^p \, dx = 1$ (cf. Section 2) and we are characterizing the MP and SMP in terms of the sign of $\lambda_{1,p}(a)$ and related facts (cf. Theorem 2 below). The WCP for $\mathcal{L}_p$ can also be characterized in terms of the sign of $a(x)$ when $p > 2$.

Let us give now an overview of the work. First, we are summarizing some recent results, which are here extended to the operator $\mathcal{L}_p$, concerning linear elliptic equations. Consider the elliptic operator $\mathcal{L}u := -\sum_{i,j=1}^n a_{ij}\partial_{ij}u + \sum_{i=1}^n a_i\partial_i u + au$, where ellipticity is understood as $\sum_{i,j=1}^n a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2$ for certain positive constant λ and every $\xi \in \mathbf{R^n}$, $x \in \overline{\Omega}$, and assume that the coefficients $a_{ij}, a_i, a \in C^\alpha(\overline{\Omega})$, Ω being a $C^{2+\alpha}$ bounded domain of $\mathbf{R^n}$. Then the operator $\mathcal{L}$, endowed with Dirichlet conditions, admits a simple eigenvalue $\lambda_1(a)$ characterized as the unique eigenvalue associated with a one-signed eigenfunction (cf. [1], [17]). The following is a first result, due to Walter ([25]), relating the MP with $\lambda_1(a)$ in an implicit way.

Theorem (Walter)*Assume that there exists a positive* $\phi \in C^2(\Omega) \cap C(\overline{\Omega})$ *satisfying*

$$\begin{cases} \mathcal{L}\phi \geq 0 & in \quad \Omega \\ \phi \geq 0 & on \quad \partial\Omega, \end{cases} \tag{1.3}$$

and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *be Lipschitzian in* $\overline{\Omega}$ *such that,*

$$\begin{cases} \mathcal{L}u \geq 0 & in \quad \Omega \\ u \geq 0 & on \quad \partial\Omega. \end{cases}$$

Then, either $u > 0$ *in* Ω, *or* $u = 0$ *in* Ω, *or* $u = c\,\phi$ *for some* $c < 0$

Observe that $\mathcal{L}$ fails to exhibit the MP when the last option in Walter's theorem holds. This necessarily implies $\mathcal{L}u = \mathcal{L}\phi = 0$ in Ω with $u = \phi = 0$ on $\partial\Omega$, i. e. $\lambda_1(a) = 0$ with ϕ as an eigenfunction. On the other hand, the MP holds if one of the inequalities in (1.3) is strict. In this sense, Walter's is a sharp version of known cases where the MP holds true: if $a(x) \geq 0$ by setting $\phi = 1$ (cf. [18], [13]) or if $\phi > 0$ up to $\partial\Omega$ (cf. [18], theorem 10, p. 73). Finally, Walter's result remains true for strong solutions (cf. [13], Chaper 9) if the regularity of the coefficients is considerably relaxed ([20]). Our generalization of Walter's theorem can be stated as follows.

Theorem 1 *Let* $\phi \in W_0^{1,p}(\Omega)$, $\mathcal{L}_p\phi \in L^\infty(\Omega)$ *be a supersolution of the equation* $\mathcal{L}_p v = 0$ *and let* $u \in W^{1,p}(\Omega)$ *satisfy,*

$$\begin{cases} -\Delta_p u + a(x)\varphi_p(u) = f(x) & in \quad \Omega \\ u \geq 0 & on \quad \partial\Omega, \end{cases} \tag{1.1}$$

with $f \in W^{-1,p'}(\Omega)$, $f \geq 0$. *Then, either* $u \geq 0$ *in* Ω, *or* $u = c\phi$ *for some negative* c. *If, in addition,* $f \in L^\infty_{\text{loc}}(\Omega)$ *then either* $u > 0$ *in* Ω, *or* $u = 0$ *in* Ω, *or* $u = c\phi$ *for some negative* c.

The next result is a particular case for scalar equations of theorem 2.1 in [15] (cf. also [14], [5] and [20], [14], [26] for the extension to cooperative systems). The MP is linked there, among other facts, to the sign of $\lambda_1(a)$ in a precise way.

Theorem *Under the conditions on $\mathcal{L}$ and Ω stated above the following assertions are equivalent,*

i) *There exists $\varphi \in C^{2+\alpha}(\overline{\Omega})$, $\mathcal{L}\varphi \geq 0$, $\mathcal{L}\varphi \neq 0$ in Ω and $\varphi = 0$ on $\partial\Omega$.*

ii) *(Maximum Principle) If $u \in C^{2+\alpha}(\overline{\Omega})$, $\mathcal{L}u \in C^{\alpha}(\overline{\Omega})$ satisfies $\mathcal{L}u \geq 0$ in Ω and $u = 0$ on $\partial\Omega$ then $u \geq 0$ in Ω.*

iii) *(Strong Maximum Principle) If, under the conditions of ii), $u \neq 0$ in Ω then $u(x) > 0$ for each $x \in \Omega$ and $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$.*

iv) *The operator $\mathcal{L}^{-1} : C^{\alpha}(\overline{\Omega}) \to C_0^{2+\alpha}(\Omega) := \{u \in C^{2+\alpha}(\Omega) | u = 0 \text{ on } \partial\Omega\}$ is compact and strongly order preserving* (cf. [1]).

v) $\lambda_1(a) > 0$.

The corresponding extension here shown for the p-Laplacian case is stated as follows.

Theorem 2 *Assume that $\Omega \subset \mathbf{R^n}$ is a bounded domain of class $C^{2+\alpha}$, $0 < \alpha < 1$ and suppose that $a = a(x) \in L^\infty(\Omega)$. Then the following assertions are equivalent,*

i) *$\mathcal{L}_p := -\Delta_p + a\varphi_p$ satisfies the maximum principle.*

ii) *$\mathcal{L}_p := -\Delta_p + a\varphi_p$ satisfies the strong maximum principle.*

iii) $\lambda_{1,p}(a) > 0$.

iv) *There exists a positive strict supersolution $\phi \in W_0^{1,p}(\Omega)$, $\mathcal{L}_p\phi \in L^\infty(\Omega)$, i. e. $\mathcal{L}_p\phi = f$ in Ω, $f \in L^\infty(\Omega)$, $f \geq 0$, $f \neq 0$ with $\phi = 0$ on $\partial\Omega$.*

v) *For each nonnegative $f \in L^\infty(\Omega)$ there exists a unique weak solution $u \in W_0^{1,p}(\Omega)$ to the equation $\mathcal{L}_p u = f$ which is also nonnegative.*

The condition $\phi = 0$ on $\partial\Omega$ imposed to the supersolution ϕ in Theorem 1 and in iv) of Theorem 2 can be weakened if one assumes certain regularity of ϕ on the boundary $\partial\Omega$. More precisely, the following results hold.

Theorem 3 *Assume that $\phi = \phi(x)$ in Theorem 1 is positive in Ω and satisfies instead: $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\mathcal{L}_p\phi \in L^\infty(\Omega)$ and $\phi|_{\partial\Omega} \in C^{1+\alpha}(\partial\Omega)$. Then, the conclusion of Theorem 1 remains true.*

Theorem 4 *Under the hypotheses of Theorem 2, the statements i) to v) are equivalent to,*

iv)' *There exists a positive strict supersolution $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $\mathcal{L}_p\phi \in L^\infty(\Omega)$ and $\phi|_{\partial\Omega} \in C^{1+\alpha}(\partial\Omega)$.*

It should be remarked that in contrast to the case of linear operators as $\mathcal{L}$ above, there is not an equivalence between the MP and the complete invertibility of $\mathcal{L}_p$. In other words, the result in v) of Theorem 2 is the best possible and the condition $f \geq 0$ there can not in general be relaxed. In fact, examples of f have been constructed in the one-dimensional case with $a = -\lambda$, $0 < \lambda < \lambda_{1,p}(0)$ where $\mathcal{L}_p u = f$ exhibits two solutions in $W_0^{1,p}(\Omega)$, for $p \neq 2$ (cf. [9], [8]). On the contrary, if the condition $a(x) \geq 0$ holds, then the weak comparison principle in [23] (cf. Section 5) ensures that $\mathcal{L}_p u = f$ has a unique solution $u \in W_0^{1,p}(\Omega)$ for each $f \in L^\infty(\Omega)$. Moreover, it will be proven in Section 5 (Corollary 8) that $a(x) \geq 0$ is also a necessary condition in order that $\mathcal{L}_p$ be invertible. On the other hand, as a nonlinear complement of the assertion iv), it is proven in Section 4 (cf. Theorem 6) that sublinear problems of the form $-\Delta_p u + f(x,u) = g(x)$ admit a solution in $W_0^{1,p}(\Omega)$, for each $g \in L^\infty(\Omega)$, if the nonresonance conditions $\lambda_{1,p}(a_{\pm\infty}) > 0$, $a_{\pm\infty} = \liminf_{u\to\pm\infty} f(x,u)/\varphi_p(u)$, hold.

Another important difference with respect to the linear theory is that the relation between the MP and the comparison principle, even in the weak sense, is still not well understood for the case of the p-Laplacian. The only worthy known situation where $\mathcal{L}_p u_1 \leq \mathcal{L}_p u_2$ in Ω together with $u_1 \leq u_2$ on $\partial\Omega$ implies $u_1 \leq u_2$ in Ω, free of additional requirements, is the case $a(x) \geq 0$ (cf. Section 5). Another more restrictive known case where comparison holds is $a(x) = -\lambda$, $\lambda < \lambda_{1,p}(0)$, $0 \leq \mathcal{L}_p u_1 \leq \mathcal{L}_p u_2$, $u_i = 0$ on $\partial\Omega$ and $\mathcal{L}_p u_i \in L^\infty(\Omega)$, $i = 1,2$ (cf. [8]). A slight extension of the last result, dropping the condition on the sign of both u_1 and $\mathcal{L}_p u_1$, is provided in the next theorem.

Theorem 5 *Assume that* $\lambda_{1,p}(a) > 0$, $u_i \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *satisfying* $\mathcal{L}_p u_i \in L^\infty(\Omega)$, $u_i|_{\partial\Omega} \in C^{1+\alpha}(\partial\Omega)$, $i = 1,2$ *together with the inequalities,*

$$\begin{cases} \mathcal{L}_p u_1 \leq \mathcal{L}_p u_2 & in \quad \Omega \\ u_1 \leq u_2 & on \quad \partial\Omega. \end{cases} \tag{1.4}$$

Assume in addition that $\mathcal{L}_p u_2 \geq 0$, *in* Ω, *with* $u_2 \geq 0$ *on* $\partial\Omega$. *Then,*

$$u_1(x) \leq u_2(x) \qquad for\ each \quad x \in \Omega.$$

If in (1.4) $u_1 = u_2 = 0$ *on* $\partial\Omega$, *then the same conclusion holds under the less restricitive assumptions:* $u_i \in W_0^{1,p}(\Omega)$, $\mathcal{L}_p u_i \in L^\infty(\Omega)$, $i = 1,2$ *and* $\mathcal{L}_p u_2$ *nonnegative in* Ω.

Finally, some results concerning the WCP are contained in Section 5. It was shown in [23] that the operators $-\Delta_p + f(x,\cdot)$ with f nondecreasing in u satisfy the WCP. We shall state that this condition turns out to be also necessary in order to have the WCP, if f satisfies adecuate growth and nonresonance conditions (Theorem 7). As a consequence, the sign condition $a(x) \geq 0$ characterizes both the WCP and invertibility for $\mathcal{L}_p$ (Corollary 8) when $p > 2$.

The contents of the paper is organized as follows. Section 2 introduces some basic facts concerning the regularity of solutions to $\mathcal{L}_p u = f(x)$ and the main properties satisfied by $\lambda_{1,p}(a)$. The proofs of Theorems 1 to 5 are contained in Section 3. Section 4 is dedicated to the proof of a nonresonance result. Finally, our results concerning the WCP are contained in Section 5.

2. Preliminary results

In the framework of the present paper we are dealing in some cases with solutions $u \in W_0^{1,p}(\Omega)$ to the problem,

$$\begin{cases} -\Delta_p u + a\varphi_p(u) = f & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{2.1}$$

with $a \in L^\infty(\Omega)$, $f \in L^\infty(\Omega)$. Due to Serrin's L^∞ local "a priori" estimates ([19]) it follows that any solution to (2.1) satisfies $u \in L^\infty_{\text{loc}}(\Omega)$. Moreover, by reflecting u with regard to $\partial\Omega$ in a small ball $B_\epsilon(x_0)$ of any point $x_0 \in \partial\Omega$ (cf. [23], [3]) the same estimates also give $u \in L^\infty(B_\epsilon(x_0) \cap \Omega)$. Thus $u \in L^\infty(\Omega)$. On the other hand, the $C^{1+\gamma}$ local estimates in [4], [22] provide $u \in C^{1+\gamma_0}(\overline{\Omega}')$ for any $\Omega' \subset\subset \Omega$ and some $0 < \gamma_0 < 1$, $\gamma_0 = \gamma_0(\Omega')$. Again, local reflection on $\partial\Omega$ near any $x_0 \in \partial\Omega$ gives $u \in C^{1+\gamma}(B_\epsilon(x_0) \cap \overline{\Omega})$ for $0 < \gamma < 1$ and $\gamma = \gamma(x_0)$. Summing up, we can state the following Lemma.

Lemma 1 *Let $u \in W_0^{1,p}(\Omega)$ be any solution to (2.1) with $f \in L^\infty(\Omega)$. Then $u \in C^{1+\beta}(\overline{\Omega})$ for some $0 < \beta < 1$.*

We will also be faced with the nonhomogeneous problem,

$$\begin{cases} -\Delta_p u + a\varphi_p(u) = f & x \in \Omega \\ u = f_1 & x \in \partial\Omega, \end{cases} \tag{2.2}$$

with $f \in L^\infty(\Omega)$, and we will need as much information as possible about the regularity of its weak solutions up to the boundary. The next result, stated as a Lemma for later use, is a direct consequence of Theorem 1 in [16].

Lemma 2 *Let $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a weak solution to (2.2) where Ω is a bounded domain in R^n of class $C^{2+\alpha}$ and where $f_1 \in C^{1+\alpha}(\partial\Omega)$ with $0 < \alpha < 1$. Then, there exists $0 < \beta < 1$, $\beta = \beta(\|u\|_{\infty,\Omega}, \|f\|_{\infty,\Omega}, \alpha, p, n)$ such that $u \in C^{1+\beta}(\overline{\Omega})$. Moreover, $\|u\|_{1+\beta,\Omega} \le C$ for some positive $C = C(\|u\|_{\infty,\Omega}, \|f\|_{\infty,\Omega}, \|f_1\|_{1+\alpha,\partial\Omega}, \alpha, p, \Omega)$.*

Let us put in order now some basic facts concerning the first eigenvalue $\lambda_{1,p}(a)$ of $\mathcal{L}_p$ which will be used in the sequel. The next statement follows step by step the ideas in [2] where existence and, more importantly, uniqueness and isolation of the first *positive* eigenvalue to the weighted eigenvalue problem $-\Delta_p u = \lambda m(x) u$ in Ω, $u = 0$ on $\partial\Omega$, $m \in L^\infty(\Omega)$ and meas $\{m > 0\} > 0$, are proved. Its proof is included below for completeness.

Lemma 3 *Let $\Omega \subset \mathbf{R^n}$ be a $C^{2+\alpha}$ domain, $0 < \alpha < 1$ and $a \in L^\infty(\Omega)$. Then, the eigenvalue problem,*

$$\begin{cases} -\Delta_p u + a\varphi_p(u) = \lambda\varphi_p(u) & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{2.3}$$

has a unique eigenvalue $\lambda = \lambda_{1,p}(a)$ which is defined by (1.2), with the property of exhibiting a positive eigenfunction $\phi \in W_0^{1,p}(\Omega)$. Moreover, $\lambda_{1,p}(a)$ is simple and isolated. The positive eigenfunctions satisfy $\phi \in C^{1+\beta}(\overline{\Omega})$ for some $0 < \beta < 1$ together with $\frac{\partial\phi}{\partial\nu} < 0$ on $\partial\Omega$.

The simplicity, uniqueness and isolation assertions in Lemma 2 are a consequence of the following result, here included for later use. It was first shown by Díaz & Saa in [11] and sharpened later by Anane in [2].

Lemma 4 (Díaz & Saa, Anane) *Consider the functional:*

$$I(u,v) = \left\langle -\Delta_p u, \frac{u^p - v^p}{u^{p-1}} \right\rangle - \left\langle -\Delta_p v, \frac{u^p - v^p}{v^{p-1}} \right\rangle,$$

defined in $\mathcal{D}(I) = \left\{ (u,v) \in \left(W_0^{1,p}(\Omega) \right)^2 |u \ge 0, v \ge 0 \text{ in } \Omega, \frac{u}{v}, \frac{v}{u} \in L^\infty(\Omega) \right\}$.

Then, $I(u,v) \ge 0$ for each $(u,v) \in \mathcal{D}(I)$. Moreover, $I(u,v) = 0$ implies $v = cu$ for some positive c.

Sketch of the proof of Lemma 3. It can be assumed without loss of generality that $a(x) \geq 0$ since otherwise it suffices with replacing $a(x)$ and λ in (2.3) by $a_M(x) := a(x) + M$ and $\lambda_M := \lambda + M$ respectively, M suitably chosen so that $a_M \geq 0$ in Ω. If $a \geq 0$, the functional $J(u) = \int_\Omega \{|\nabla u|^p + a|u|^p\}\ dx$ is sequentially weakly lower semicontinuos in $W_0^{1,p}(\Omega)$ and coercive in $\mathcal{V} := \{u \in W_0^{1,p}(\Omega) | \int_\Omega |u|^p\ dx = 1\}$. Therefore (cf. [21]) the infimum $\lambda_{1,p}(a)$ in (1.3) is attained for some $\phi \in \mathcal{V}$ which is a weak solution to $\mathcal{L}_p\phi = \lambda_{1,p}(a)\varphi_p(\phi)$.

It follows from Lemma 1, after normalizing ϕ so as $\|\phi\|_{\infty,\Omega} = 1$ that $\phi \in C^{1+\beta}(\overline{\Omega})$ for some $0 < \beta < 1$. A direct computation shows that $\phi^+(x) := \max\{\phi(x), 0\}$ also solves $\mathcal{L}_p\phi^+ = \lambda_{1,p}(a)\varphi_p(\phi^+)$ and hence $-\Delta_p\phi^+ + \|a\|_{\infty,\Omega}\varphi_p(\phi^+) \geq 0$ in Ω. Thus the strong maximum principle in [24] yields either $\phi^+(x) > 0$ in Ω with $\frac{\partial\phi^+}{\partial\nu} < 0$ on Ω, or $\phi^+ = 0$. In any case ϕ is one-signed.

For obtaining the simplicity observe that $I(\phi_0, \hat{\phi}_0) = 0$ for any pair of positive eigenfunctions of $\lambda_{1,p}(a)$. Lemma 4 then yields $\hat{\phi}_0 = c\phi_0$. As for the uniqueness let $\lambda \neq \lambda_{1,p}(a)$ (and so $\lambda > \lambda_{1,p}(a)$) be an eigenvalue with $\mathcal{L}_p\phi = \lambda\varphi_p(\phi)$ and $\phi > 0$ in Ω. A suitable scaling of ϕ leads to $I(\phi_0, \phi) = (\lambda_{1,p}(a) - \lambda)(\lambda^{-1} - \lambda_{1,p}(a)^{-1}) < 0$, which is not possible. Hence the uniqueness (recall that the problem can be shifted to have $\lambda_{1,p}(a) > 0$).

Finally, to get the isolation assume that $\lambda_{1,p}(a) = \lim \lambda_n$, $\lambda_n > \lambda_{1,p}(a)$ with $\mathcal{L}_p\phi_n = \lambda_n\varphi_p(\phi_n)$. By a standard compactness argument it is possible to extract a conveniently normalized subsequence $\phi_{n'}$, so that $\lim \phi_{n'} = \phi_0$ in $W_0^{1,p}(\Omega)$ with $\mathcal{L}_p\phi_0 = \lambda_{1,p}(a)\varphi_p(\phi_0)$ and $\phi_0 > 0$ in Ω. On the other hand, an adaptation of Prop. 2 in [2] permits to show that meas $\{\phi_{n'} < 0\} \geq k > 0$, k not depending on n'. This contradicts the positivity of ϕ_0 and the proof of the isolation is finished.

It should be remarked that the strategy of the proof of Lemma 2 can be adapted word for word to deal with the eigenvalue problem $-\Delta_p u + a\varphi_p(u) = \lambda m\varphi_p(u)$ in Ω, $u = 0$ on $\partial\Omega$, $m \in L^\infty(\Omega)$, $m > 0$ in a set with positive measure, provided that $\lambda_{1,p}(a) > 0$. Thus one can obtain the existence of a first *positive* eigenvalue $\lambda_{1,p}(a, m)$ with the same properties as $\lambda_{1,p}(a)$ in Lemma 2.

3. Proof of theorems 1 to 5

Proof of Theorem 2. i) $\Rightarrow$ ii). Assume that the MP holds and let $u \in W^{1,p}(\Omega)$ be a weak solution to $-\Delta_p u + a\varphi_p(u) = f$ in Ω with $f \in L^\infty(\Omega)$, $f \geq 0$, $f \not\equiv 0$ in Ω, such that $u \geq 0$ on $\partial\Omega$. Then, since $u \geq 0$ in Ω, $-\Delta_p u + \|a\|_{\infty,\Omega}\varphi_p(u) \geq 0$ in Ω. Taking into account that $u \in C^1(\Omega)$ then the strong maximum principle (cf. Theorem 5 in [24] implies $u(x) > 0$ in Ω together with $\frac{\partial u}{\partial\nu} < 0$ at any $x_0 \in \partial\Omega$ where $u(x_0) = 0$, provided that such a derivative exists.

ii) $\Rightarrow$ iii). Let $\phi \in C^{1+\beta}(\overline{\Omega})$ be an eigenfunction and assume that $\lambda_{1,p}(a) \leq 0$. Then, $\phi_1 = -\phi$ satisfies $-\Delta_p\phi_1 + a\varphi_p(\phi_1) \geq 0$ that implies $\phi \leq 0$, against the choice of ϕ. Therefore $\lambda_{1,p}(a) > 0$.

iii) $\Rightarrow$ iv). It is immediate.

iv) $\Rightarrow$ iii). Consider a positive eigenfunction $\phi_1 \in C^{1+\beta}(\overline{\Omega})$. Since ϕ is a strict supersolution then $(\phi_1, \phi) \in \mathcal{D}(I)$. In particular, $\frac{\phi}{\phi_1} \in L^\infty(\Omega)$. Set $\phi_2 = c\phi_1$, $c > 0$ chosen so that $c \geq \|\phi/\phi_1\|_{\infty,\Omega}$.

Assume now that $\lambda_{1,p}(a) \leq 0$. We have then from $f \geq 0$ in Ω,

$$\left\langle -\Delta_p\phi, \frac{\phi_2^p - \phi}{\phi^{p-1}} \right\rangle \geq \int_\Omega -a(\phi_2^p - \phi^p)\ dx,$$

and so,

$$I(\phi_2, \phi) = \int_\Omega (\lambda_{1,p}(a) - a)(\phi_2^p - \phi^p)\ dx \leq 0.$$

It follows from Lemma 4 that $I(\phi_2, \phi) = 0$ and then $\phi_2 = \gamma\phi$ for some $\gamma > 0$. However, $-\Delta_p\phi_2 + a\varphi_p(\phi_2) = \gamma^{p-1}f$. Since $f \geq 0$ and $f \not\equiv 0$ this is in contradiction with the assumption $\lambda_{1,p}(a) \leq 0$. Thus $\lambda_{1,p}(a) > 0$.

iii) $\Rightarrow$ *i)*. Assume, more generally, that $u \in W^{1,p}(\Omega)$ satisfies $-\Delta_p u + a\varphi_p(u) = f$ in Ω, with $f \in W^{-1,p'}(\Omega)$ and $\langle f, v\rangle \geq 0$ for each nonnegative $v \in W_0^{1,p}(\Omega)$, together with $u \geq 0$ on $\partial\Omega$. Since $u^-(x) := \min\{u(x), 0\} \in W_0^{1,p}(\Omega)$ then, by inserting $v = u^-$ in (1.1) we arrive at,

$$\int_\Omega \{|\nabla u^-|^p + a|u^-|^p\} \ dx = \langle f, u^-\rangle \leq 0.$$

Thus u^- must necessarily vanish in Ω. Otherwise,

$$\frac{\int_\Omega \{|\nabla u^-|^p + a|u^-|^p\} \ dx}{\int_\Omega |u^-|^p \ dx} \leq 0 < \lambda_{1,p}(a),$$

and this contradicts (1.2).

This completes the equivalence of assertions i) to iv). Since v) implies iv) it only remains to show, for instance, that iii) implies v).

iii) $\Rightarrow$ *v)*. The condition $\lambda_{1,p}(a) > 0$ implies the coercivity in $W_0^{1,p}(\Omega)$, for each $f \in W^{-1,p'}(\Omega)$, of the functional,

$$J_f(u) = \int_\Omega \{|\nabla u|^p + a|u|^p - pfu\} \ dx.$$

In particular, problem (2.1) admits at least a weak solution $u \in W_0^{1,p}(\Omega)$ for each $f \in L^\infty(\Omega)$, which will be nonnegative provided $f \geq 0$ in Ω. Let us consider $f \in L^\infty(\Omega)$, $f \geq 0$ and $f \neq 0$. We are going to show now that (2.1) admits a unique positive solution. In fact, let $u, v \in W_0^{1,p}(\Omega)$ two nonegative solutions to (2.1). The strong maximum principle implies $u > 0$ and $v > 0$ in Ω meanwhile $\frac{\partial u}{\partial \nu} < 0$, $\frac{\partial v}{\partial \nu} < 0$ on $\partial\Omega$. This implies that $(u, v) \in \mathcal{D}(I)$. Thus,

$$0 \leq \left\langle -\Delta_p u, \frac{u^p - v^p}{u^{p-1}} \right\rangle - \left\langle -\Delta_p v, \frac{u^p - v^p}{v^{p-1}} \right\rangle = \int_\Omega f \ \frac{(v^{p-1} - u^{p-1})}{u^{p-1}v^{p-1}} \ (u^p - v^p) \ dx.$$

The last expression being nonpositive it follows $I(u, v) = 0$. Hence, $v = \gamma u$ for some positive $\gamma > 0$ and so $\gamma^{p-1} f = f$. Since $f \neq 0$ then $\gamma = 1$ and the uniqueness is shown.

On the other hand, the maximum principle implies that $u = 0$ is the only solution when $f = 0$, so the proof of Theorem 2 is concluded.

Remark. A direct proof of the equivalence "i) $\Leftrightarrow$ iii)" was given in Theorem 5 in [12], for the case $a(x) = a = $ constant. The proof given here of this fact is an immediate adaptation of that result.

Proof of Theorem 1.

Let $\phi \in W_0^{1,p}(\Omega)$, $\mathcal{L}_p\phi \in L^\infty(\Omega)$, a positive supersolution to (1.1). By arguing as in the proof of the step iv) $\Rightarrow$ iii) in Theorem 2, it is shown that $\lambda_{1,p}(a) \geq 0$. Let $u \in W_0^{1,p}(\Omega)$ satisfy,

$$\begin{cases} -\Delta_p u + a(x)\varphi_p(u) = f(x) & \text{in} \quad \Omega \\ u \geq 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where, say $f \in W^{-1,p'}(\Omega)$ and $\langle f, v\rangle \geq 0$ for each nonnegative $v \in W_0^{1,p}(\Omega)$. Then,

$$\int_\Omega \left\{|\nabla u|^{p-2}\nabla u \nabla v + a\varphi_p(u)v\right\} \ dx \geq 0,$$

for each nonnegative $v \in W_0^{1,p}(\Omega)$. Since $u^- \in W_0^{1,p}(\Omega)$ then,

$$\lambda_{1,p}(a) \int_\Omega |u^-|^p \ dx \leq \int_\Omega \{|\nabla u^-|^p + a|u^-|^p\} \ dx \leq 0.$$

If $\lambda_{1,p}(a) > 0$ then, necessarily $u^- = 0$ in Ω. Thus, $u \geq 0$. If, in addition, $f \in L^\infty(\Omega)$ then Lemma 1 and the SMP yield either $u(x) > 0$ in Ω or $u = 0$ in Ω (actually, in order to get the required regularity in u, namely, $u \in C^1(\Omega)$, to apply the strong maximum principle in [24], the condition $f \in L^\infty(\Omega)$ can be relaxed to $f \in L^q_{\text{loc}}(\Omega)$, with $q > p'n$ as in [4]). If, on the contrary, $\lambda_{1,p}(a) = 0$ by assertion iv) in Theorem 2 we must have,

$$-\Delta_p \phi + a\varphi_p(\phi) = 0 \qquad \text{in } \Omega,$$

that is, ϕ is an eigenfunction associated to $\lambda_{1,p}(a) = 0$. Since

$$\int_\Omega \{|\nabla u^-|^p + a|u^-|^p\} \; dx = 0,$$

then either $u^- = 0$ in Ω or $u = u^-$ is a negative eigenfunction associated to $\lambda_{1,p}(a) = 0$. Thus, Lemma 4 gives $u = c\,\phi$ for some negative c. This concludes the proof of theorem 1.

The proofs of Theorems 3, 4 and 5. Let us begin with the proof of Theorem 5. We can rule out the trivial case $u_2 = 0$ in Ω. By Lemma 2, $u_i \in C^{1+\beta}(\overline{\Omega})$, $i = 1, 2$. The SMP ([24]) then ensures $u_2(x) > 0$ in Ω together with $\frac{\partial u_2}{\partial \nu} < 0$ at that part of $\partial\Omega$ where u_2 vanish. Thus, a $c > 1$ exists so that $u_1 < cu_2$ in Ω. Set $g = \mathcal{L}_p u_2$, $g_1 = u_2|_{\partial\Omega}$ and consider the problem,

$$\begin{cases} -\Delta_p v + a\varphi_p(v) = g & \text{in} \quad \Omega \\ \qquad v = g_1 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.1}$$

It is readily seen that $v_- = u_1$ and $v_+ = cu_2$ are sub and supersolutions, respectively, of (3.1). Thus the method of sub and supersolutions (cf. [10], Theorem 4.14, p. 272) yields the existence of a solution $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ to (3.1), $u_1 \leq v \leq c_2 u_2$. Since $\lambda_{1,p}(a) > 0$, v must be nonnegative. Now we claim that (3.1) has a unique nonnegative solution in $W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Therefore $v = u_2$ and $u_1 \leq u_2$. Thus, the proof of Theorem 5 is concluded.

Let us prove the claim. If $v_i \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, $i = 1, 2$ are two nonnegative solutions to (3.1) then Lemma 2 and the strong maximum principle imply that $\frac{v_1}{v_2}, \frac{v_2}{v_1} \in L^\infty(\Omega)$. Since $v_1 = v_2$ on $\partial\Omega$, a careful checking of the proof of Lemma 4 (see Proposition 1 in [2]) reveals that its conclusions hold even if u and v do not vanish but coincide on $\partial\Omega$. Thus $I(v_1, v_2) \geq 0$ in Ω. The same argument as in the step iii) $\Rightarrow$ v) of Theorem 2 gives then $v_1 = v_2$ and the claim is proved.

As for the proof of Theorem 4 it suffices with showing that iv)' implies $\lambda_{1,p}(a) > 0$. In fact, if $\lambda_{1,p}(a) \leq 0$ and ϕ_0 is a positive associated eigenfunction then $v_- = \epsilon\phi_0$ is, for each $\epsilon > 0$ a positive subsolution to (3.1) where now we set $g = \mathcal{L}_p \phi$ and $g_1 = \phi|_{\partial\Omega}$. Thus, the comparison result in Theorem 5 gives $\epsilon\phi_0 \leq \phi$ for each $\epsilon > 0$, which is not possible. Therefore $\lambda_{1,p}(a) > 0$.

In the case of Theorem 3 we have as a consequence of Theorem 4 that $\lambda_{1,p}(a) > 0$ if ϕ is a strict supersolution, or $\lambda_{1,p}(a) = 0$ otherwise. To achieve the desired result it suffices to apply Theorem 2 in the first case, and Theorem 1 with a positive eigenfunction ϕ_0 associated to $\lambda_{1,p}(a) = 0$ as a supersolution, in the second.

4. A nonresonant sublinear problem

The following result describes general conditions in order to solve "sublinear" type problems,

$$\begin{cases} -\Delta_p u + f(x, u) = g(x) & x \in \Omega \\ \qquad u = 0 & x \in \partial\Omega, \end{cases} \tag{4.1}$$

under a suitable asymptotic behaviour of $\frac{f(x,u)}{\varphi_p(u)}$ when $u \to \pm\infty$.

Theorem 6 *Assume that* $\Omega \in \mathbf{R^n}$ *is a bounded domain of* $\mathbf{R^n}$ *and let* $f : \Omega \times \mathbf{R} \to \mathbf{R}$ *be a Carathéodory function, which is sublinear in the following sense,*

$$|f(x,u)| \leq C_0(|u|^{p-1} + 1), \tag{4.2}$$

for some $C_0 > 0$ *and* $u \in \mathbf{R}$. *Define* $a_{\pm\infty}(x) = \liminf_{u\to\pm\infty} \frac{f(x,u)}{\varphi_p(u)} \in L^\infty(\Omega)$ *and suppose in addition that the following inequalities hold,*

$$\lambda_{1,p}(a_{-\infty}) > 0, \qquad \lambda_{1,p}(a_{+\infty}) > 0. \tag{4.3}$$

Then, for each $g \in W^{-1,p'}(\Omega)$ *problem (4.1) admits at least a weak solution* $u \in W_0^{1,p}(\Omega)$.

Remark. Theorem 6 is a slightly stronger version of a part of Theorem 2.5 in [6]. There, the analog operator $-\hat{\Delta}_p u = -\sum_{i=1}^n \partial_i \left(|\partial_i u|^{p-2}\partial_i u\right)$ is considered instead of $-\Delta_p$, its first Dirichlet eigenvalued being designated by λ_1 ($\lambda_1 > 0$). It is assumed that $|f(x,u)| \leq m(x) + c|u|^{p-1}$ with $m \in L^{p'}(\Omega)$ together with,

$$\liminf_{u\to\pm\infty} \frac{f(x,u)}{\varphi_p(u)} \geq \delta - \lambda_1, \tag{4.2$'$}$$

for some $\delta > 0$. It is easily seen that (4.2)$'$ implies (4.2), while (4.2) is slightly more general than (4.2)$'$.

Proof of Theorem 6. For $g \in W^{-1,p'}(\Omega)$ consider the standard functional,

$$J(u) = \int_\Omega \left\{\frac{1}{p}|\nabla u|^{p-2} + F(x,u)\right\}\, dx - \langle g, u\rangle\,,$$

with $u \in W_0^{1,p}(\Omega)$, where $F(x,u) = \int_0^u F(x,s)\, ds$. It is well-known that J is sequentially weakly lower semicontinuous. We are now proving, following the spirit of the proof of Theorem 2 in [7], that J is also coercive in $W_0^{1,p}(\Omega)$, i. e., that $J(u_n) \to \infty$ provided $|u_n|_{1,p} \to \infty$. In fact, assume by contradiction that $J(u_n) \leq C_1$ for each n. We obtain then,

$$\begin{aligned} \tfrac{1}{p}|u_n|_{1,p}^p - \|g\||u_n|_{1,p} &\leq \tfrac{C_0}{p}|u_n|_p^p + C_0|\Omega|^{\frac{1}{p'}}|u_n|_p + C_1 \\ &\leq C_2(|u_n|_p^p + 1), \end{aligned} \tag{4.3}$$

for certain $C_2 > 0$, $|u_n|_p$ standing for the L^p norm of u. Thus (4.3) implies that $t_n = |u_n|_p \to \infty$ and setting $v_n = \frac{u_n}{t_n}$ we arrive at

$$\frac{1}{p}|v_n|_{1,p}^p - \frac{\|g\|}{t_n^{p-1}}|v_n|_{1,p} \leq C_2(|v_n|_p^p + \frac{1}{t_n^p}),$$

which implies that v_n is bounded in $W_0^{1,p}(\Omega)$. Passing through a subsequence we get that $v_n \rightharpoonup v \in W_0^{1,p}(\Omega)$ weakly in $W_0^{1,p}(\Omega)$, being such a convergence strong in $L^p(\Omega)$, $|v|_p = 1$.

On the other hand, $J(u_n) \leq C_1$ implies,

$$\frac{1}{p}|v_n|_{1,p}^p \leq \int_\Omega -\frac{F(x,t_nv_n)}{t_n^p}\, dx + \frac{1}{t_n^{p-1}}\langle g, v_n\rangle + \frac{C_1}{t_n^p},$$

and taking $\limsup$ we arrive at,

$$\frac{1}{p}|v|_{1,p}^p \leq \limsup \int_\Omega -\frac{F(x,t_nv_n)}{t_n^p}\, dx. \tag{4.4}$$

To estimate the integral in (4.4) it is convenient to write,

$$\int_\Omega \frac{F(x,t_n v_n)}{t_n^p}\,dx = \int_{\{v>0\}} \frac{F(x,t_n v_n)}{t_n^p}\,dx + \int_{\{v<0\}} \frac{F(x,t_n v_n)}{t_n^p}\,dx.$$

Observe now that $\liminf_{u\to\pm\infty} \frac{F(x,u)}{|u|^p} = \frac{1}{p}a_{\pm\infty}(x)$ a. e. in Ω. Thus, modulus a subsequence we obtain,

$$\liminf \frac{F(x,t_n v_n(x))}{t_n^p} \geq \frac{1}{p}a_{\pm\infty}(x)|v^\pm(x)|^p,$$

a. e. in Ω. By using Fatou's lemma in (4.4) we arrive at

$$\limsup \int_\Omega -\frac{F(x,t_n v_n)}{t_n^p}\,dx \leq \int_\Omega -\frac{a_\infty}{p}|v^+|^p\,dx + \int_\Omega -\frac{a_{-\infty}}{p}|v^-|^p\,dx.$$

Therefore,

$$\int_\Omega |\nabla u|^p\,dx \leq \int_\Omega -a_\infty|v^+|^p\,dx + \int_\Omega -a_{-\infty}|v^-|^p\,dx. \tag{4.5}$$

However, (4.5) implies,

$$\lambda_{1,p}(a_{-\infty})\int_\Omega |v^-|^p\,dx + \lambda_{1,p}(a_\infty)\int_\Omega |v^+|^p\,dx \leq 0,$$

what, due to (4.2), entails $v^- = v^+ = 0$. But this contradicts the fact $|v|_p = 1$. In conclusion, J must be coercive and Theorem 6 is proved.

5. Some remarks on weak comparison

One of the less restrictive weak comparison principles in use for the p-Laplacian operator is the following result due to Tolksdorf (cf. Lemma 3.1 in [23]).

Theorem (Tolksdorf)*Let Ω be a bounded domain in $\mathbf{R^n}$ and let $f:\Omega\times\mathbf{R}\to\mathbf{R}$ be a Carathéodory function such that $\frac{\partial f}{\partial u}(x,u)$ exists and is nonnegative for each u, a. e. in Ω. If $u,v \in W^{1,p}(\Omega)$ satisfy,*

$$\begin{cases} -\Delta_p u + f(x,u) \leq -\Delta_p v + f(x,v) & \text{in} \quad \Omega \\ u \leq v & \text{on} \quad \partial\Omega, \end{cases} \tag{5.1}$$

then

$$u \leq v \qquad \text{in} \quad \Omega. \tag{5.2}$$

The main consequence of the fact that an operator $-\Delta_p + f(x,\cdot)$ satisfy the WCP is that the problem $-\Delta_p u + f(x,u) = 0$ in Ω, $u = 0$ on $\partial\Omega$, admits at most a weak solution $u \in W_0^{1,p}(\Omega)$. Our next result states that under suitable growth conditions on $f(x,u)$, the condition $\frac{\partial f}{\partial u} \geq 0$ can not be relaxed in Tolksdorf's result. The proof is inspired in the counterexample to uniqueness for the one-dimensional operator $-(|u'|^{p-2}u')' - \lambda\varphi_p(u)$, $0 < \lambda < \lambda_{1,p}(0)$, $p > 2$, given in [9]. Thus the condition $p > 2$ will be imposed in the following statements.

Theorem 7 *Let Ω be a smooth enough bounded domain in $\mathbf{R^n}$ and let $f:\Omega\times\mathbf{R}\to\mathbf{R}$ be a Carathéodory function such that $\frac{\partial f}{\partial u}(x,u)$ exists for each u, a. e. in Ω. Let us assume that,* *i)* $|f(x,u)| \leq C_0(|u|^{p-1}+1)$ *for some* $C_0 > 0$ *and* $u \in \mathbf{R}$.

ii) $\lambda_{1,p}(a_{-\infty}) > 0$ *and* $\lambda_{1,p}(a_{\infty}) > 0$, *where* $a_{\pm\infty} = \liminf_{u\to\pm\infty} \frac{f(x,u)}{\varphi_p(u)}$.

iii) There exists $u_0 \in \mathbf{R}$ *such that* $\frac{\partial f}{\partial u}(x,u_0) < 0$ *on some set with positive Lebesgue's measure, together with* $\frac{\partial f}{\partial u}(\cdot,u_0) \in L^1_{\text{loc}}(\Omega)$.

Then, there exists $g \in C(\Omega) \cap L^\infty(\Omega)$ *such that the problem,*

$$\begin{cases} -\Delta_p u + f(x,u) = g & in \quad \Omega \\ \qquad u = 0 & on \quad \partial\Omega, \end{cases} \tag{5.3}$$

admits at least two weak solutions. In particular, the weak comparison principle for $-\Delta_p + f(x,\cdot)$ *fails.*

As an immediate consequence of Theorem 7 we obtain the following characterization of the weak comparison principle for the operator $\mathcal{L}_p = -\Delta_p + a\varphi_p$, $a \in L^\infty(\Omega)$.

Corollary 8 *A necessary and sufficient condition for* $-\Delta_p + a(x)\varphi_p$, $a \in L^\infty(\Omega)$, *to satisfy the weak comparison principle is that* $a = a(x)$ *be nonnegative in* Ω. *In that case,* $\mathcal{L}_p^{-1} : W^{-1,p'}(\Omega) \to L^p(\Omega)$ *is well defined, compact and order preserving.*

Proof of Theorem 7. We are firstly constructing a suitable $v \in C^2(\overline{\Omega})$ so that $v = 0$ on $\partial\Omega$ with $-\Delta_p v + f(x,v) \in C(\Omega) \cap L^\infty(\Omega)$. Thus, by setting $g := -\Delta_p v + f(x,v)$ we get that v is a weak solution of (5.3). It will be seen later that the choice of such a function g entails the existence of a second solution to (5.3).

For $\epsilon > 0$ define $\Omega_\epsilon = \{x \in \Omega | d(x,\partial\Omega) > \epsilon\}$. If ϵ is small enough, a global tangential-normal coordinate system can be defined in $\Omega \setminus \Omega_\epsilon$ so that, for each $x \in \Omega \setminus \Omega_\epsilon$, $x = \phi(s) - \tau\nu(\phi(s))$, where $\phi = \phi(s)$, $s \in \mathbf{R^{n-1}}$ stands for a local parameterization of $\partial\Omega$, ν being the outer unit normal field on $\partial\Omega$. We define $v = u_0$ in Ω_ϵ. To proceed in $\Omega \setminus \Omega_\epsilon$ we consider $\zeta \in C_0^\infty(\mathbf{R})$ such that $\zeta = 0$ in $x \leq 0$, $\zeta = u_0$ in $x \geq \epsilon$, while $\zeta'(x)u_0 > 0$ in $0 < x < \epsilon$. Thus we set $v(x) = \zeta(\tau(x))$ in $\Omega \setminus \Omega_\epsilon$. Then it is easily seen that $|\nabla v| = |\zeta'|$ while $\Delta_p v = |\zeta'|^{p-2}\,(2\zeta" + \zeta'\Delta\tau)$ in $\Omega \setminus \Omega_\epsilon$. Therefore $\Delta_p v \in C(\overline{\Omega})$ and $g \in C(\Omega) \cap L^\infty(\Omega)$.

On the other hand, as it was shown in the proof of Theorem 6, conditions i) and ii) imply that the functional

$$J(u) = \int_\Omega \left\{ \frac{1}{p}|\nabla u|^p + F(x,u) - gu \right\}\, dx,$$

with $F(x,u) = \int_0^u f(x,s)\, ds$, is coercive in $W_0^{1,p}(\Omega)$. Thus, (5.3) admits a solution $u_1 \in W_0^{1,p}(\Omega)$ such that,

$$J(u_1) = \inf_{u\in W_0^{1,p}(\Omega)} J(u).$$

If it is shown now that v does not minimize J then it will be obtained that $u_1 \neq v$ and hence, (5.3) has at least two solutions. In fact, for any $\psi \in C_0^\infty(\Omega_\epsilon)$ we obtain,

$$J(v+t\psi) = \int_{\Omega_\epsilon} \left\{ \frac{1}{p}|t|^p|\nabla\psi|^p + F(x,u_0+t\psi) - g(u_0+t\psi) \right\}\, dx,$$

and so,

$$\frac{d^2}{dt^2}\left(J(v+t\psi)\right)|_{t=0} = \int_{\Omega_\epsilon} \frac{\partial f}{\partial u}(x,u_0)\psi^2\, dx.$$

Define now $A = \{x | \frac{\partial f}{\partial u}(x, u_0) < 0\}$, while $A_{2\epsilon} = A \cap \Omega_{2\epsilon}$, with ϵ so small as meas$(A_{2\epsilon}) > 0$. Let $\chi_{A_{2\epsilon}}$ be the characteristic function of $A_{2\epsilon}$ and designate by ψ_n the $\frac{1}{n}$ order regularization of $\chi_{A_{2\epsilon}}$ (cf. [13]). Then, $\psi_n \in C_0^\infty(\Omega_\epsilon)$ and $\psi_n \to \chi_{A_{2\epsilon}}$ in $L^q(\Omega)$ for every $1 \leq q < \infty$. In conclusion,

$$\lim_{n\to\infty} \int_{\Omega_\epsilon} \frac{\partial f}{\partial u}(x, u_0)\psi_n^2 \, dx = \int_{A_{2\epsilon}} \frac{\partial f}{\partial u}(x, u_0) \, dx < 0.$$

Thus $\int_{\Omega_\epsilon} \frac{\partial f}{\partial u}(x, u_0)\psi_n^2 \, dx < 0$ for n large. Therefore, v can not even locally minimize J, and the proof of Theorem 7 is concluded.

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