MULTIPLICITY OF POSITIVE SOLUTIONS TO BOUNDARY BLOW UP ELLIPTIC PROBLEMS WITH SIGN-CHANGING WEIGHTS

JORGE GARCÍA-MELIÁN

ABSTRACT. In this paper we consider the elliptic boundary blow-up problem

$$\begin{cases} \Delta u = (a_+(x) - \varepsilon a_-(x))u^p & \text{in } \Omega \\ u = \infty & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, a_+, a_- are positive continuous functions supported in disjoint subdomains Ω_+, Ω_- of Ω, respectively, $p > 1$ and $\varepsilon > 0$ is a parameter. We show that there exists $\varepsilon^* > 0$ such that no positive solutions exist when $\varepsilon > \varepsilon^*$, while a minimal positive solution exists for every $\varepsilon \in (0, \varepsilon^*)$. Under the additional hypotheses that $\overline{\Omega}_+$ and $\overline{\Omega}_-$ intersect along a smooth $(N-1)$-dimensional manifold Γ and a_+, a_- have a convenient decay near Γ, we show that a second positive solution exists for every $\varepsilon \in (0, \varepsilon^*)$ if $p < N^* = (N+2)/(N-2)$. Our proofs are mainly based on continuation methods.

Dedicated to the memory of Fuensanta Andreu

1. Introduction and results

The objective of this paper is the study of the following elliptic boundary blow-up problem

$$\begin{cases} \Delta u = a(x)u^p & \text{in } \Omega \\ u = \infty & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $p > 1$ and $a(x)$ is a continuous function in $\overline{\Omega}$. A solution to (1.1) is a function $u \in C^1(\Omega)$ verifying the equation in the weak sense, and the boundary condition is meant as $u(x) \to \infty$ as $d(x) := \text{dist}(x, \partial\Omega) \to 0+$.

Problem (1.1) has been extensively studied in the past years. It seems to have been first considered in [41] (where $a \equiv 1$ and $p = \dfrac{N+2}{N-2}$) and later in [35], [7], [8], [51] and [46] (in all of them $a(x) \geq a_0 > 0$ in Ω). The adaptation of (1.1) to the $p-$Laplacian setting, that is, $\Delta_p u = a(x)u^q$ in Ω, $q > p-1$, $a \geq a_0 > 0$, was analyzed in [19], [20]. Some years later the problem was reconsidered to deal with weights which are positive in Ω, but can vanish on $\partial\Omega$ (cf. [24], [31], [17], [18], [16], [42], [43], [21], [29]) or even be singular there (see [52], [14], [15]). Weights which can also vanish in Ω have been considered in [40].

A characteristic feature of problem (1.1) when $a(x)$ is nonnegative is that *uniqueness* of positive solutions is expected. It usually follows by means of an estimate of all possible solutions near the boundary under the form $u \sim Ad^{-\alpha}$ as $d \to 0+$, where α and A are explicitly given in terms of p and a.

Our main interest in the present paper is to deal with problem (1.1) when the weight $a(x)$ is allowed to *change sign.* Of course, in this case uniqueness is not expected to hold anymore, but even the construction of a single solution does not seem to follow as in the usual case where a is nonnegative. For some previous results in this direction (in a one-dimensional setting) we refer to [44].

It turns out that the negative part of a cannot be too large for positive solutions to (1.1) to exist. This is the contents of our first result, which gives necessary conditions for existence of positive solutions.

Theorem 1. *Assume $a \in C(\overline{\Omega})$ and $p > 1$. Then*

(i) *If $a(x_0) < 0$ for some $x_0 \in \partial\Omega$, then problem* (1.1) *does not have positive solutions.*

(ii) *If $a > 0$ on $\partial\Omega$ and* (1.1) *admits a positive solution then*

$$\int_\Omega a(x) > 0.$$

To gain insight into the influence of the negative part of $a(x)$ on the existence of positive solutions to (1.1), we will consider the parameter-dependent problem:

$$\begin{cases} \Delta u = (a_+(x) - \varepsilon a_-(x)) u^p & \text{in } \Omega \\ u = \infty & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

where a_+, a_- are continuous nonnegative functions with disjoint supports Ω_+, Ω_- and $\varepsilon > 0$ will be regarded as a parameter. Observe in passing that when $\varepsilon \le 0$ the weight in (1.2) becomes nonnegative, hence there exists a unique positive solution. Thus from now on we are only considering positive values of ε.

As a consequence of Theorem 1 (ii) we obtain that solutions may only exist when ε is not too large. As a matter of fact, we will show that when ε is small, there are at least *two* positive solutions. Notice that multiplicity results for boundary blow-up problems are not frequent in the literature: we quote [1], where a problem with three different positive solutions was studied, [22], [23], [26] for a bistable nonlinearity and [2], [45], where sign-changing solutions were obtained aside the positive ones.

Let us state our first existence result. We will always assume that the functions a_+, a_-, are continuous in $\overline{\Omega}$, and $a_+ > 0$ in Ω_+, $a_- > 0$ in Ω_- where Ω_+, Ω_- are subdomains with $\Omega_+ \cap \Omega_- = \emptyset$. Observe that Theorem 1 forces us to assume $\Omega_- \cap \partial\Omega = \emptyset$. We will assume throughout that $a_+ > 0$ on $\partial\Omega$, that is, $\Omega_- \subset\subset \Omega$. For simplicity, we usually write $a_\varepsilon(x) = a_+(x) - \varepsilon a_-(x)$.

We say that a positive solution u is stable if the first eigenvalue λ_1 of the linearized problem

$$\Delta\phi = p(a_+(x) - \varepsilon a_-(x)) u^{p-1}\phi - \lambda\phi \quad \text{in } \Omega \tag{1.3}$$

verifies $\lambda_1 > 0$. The solutions to (1.3) are sought in a Banach space X_α which contains all positive solutions to (1.2), and will be defined in Section 3. It will be seen in Section 4 that (1.3) actually has a first eigenvalue which enjoys the usual properties of simplicity and isolation. It is worth mentioning that equation (1.3) is a problem with no boundary condition.

Theorem 2. *Assume $\Omega_- \subset\subset \Omega$. Then there exists $\varepsilon^* > 0$ such that problem* (1.2) *has no positive solutions when $\varepsilon > \varepsilon^*$, while it has a minimal positive solution u_ε*

for every $\varepsilon \in (0, \varepsilon^*)$. Moreover, u_ε is increasing in ε and stable, and it is the only stable positive solution to problem (1.2) *when $\varepsilon > 0$.*

To obtain a second positive solution to (1.2) for $\varepsilon \in (0, \varepsilon^*)$, we need to make some additional assumptions on the exponent p and the weights a_+, a_-. We point out that these assumptions are only needed to obtain a priori bounds of solutions in Ω_-, and can be replaced by some slightly more general ones (cf. for instance [25]). We first assume that p is subcritical, that is,

$$p < N^* = \frac{N+2}{N-2}. \tag{1.4}$$

As for the weight a_ε, we suppose that $\Gamma = \overline{\Omega}_+ \cap \overline{\Omega}_-$ is an $(N-1)$-dimensional smooth manifold and there exist $\gamma_1, \gamma_2 > 0$ such that

$$\lim_{\substack{x \to x_0 \\ x \in \Omega_+}} \frac{a_+(x)}{d_\Gamma(x)^{\gamma_1}} = A(x_0), \qquad \lim_{\substack{x \to x_0 \\ x \in \Omega_-}} \frac{a_-(x)}{d_\Gamma(x)^{\gamma_2}} = B(x_0),$$

for every $x_0 \in \Gamma$, where A, B are positive continuous functions in a neighborhood of Γ and $d_\Gamma(x) = \operatorname{dist}(x, \Gamma)$. We will refer to this hypothesis as (H).

Our multiplicity result is the following:

Theorem 3. *In addition to the hypotheses of Theorem 2, assume that* (1.4) *and hypothesis* (H) *hold. Then there exists a positive solution to* (1.2) *for $\varepsilon = \varepsilon^*$, and for every $\varepsilon \in (0, \varepsilon^*)$, there exists a second positive solution v_ε. Moreover,*

$$\lim_{\varepsilon \to 0+} \sup_{\overline{\Omega}_-} v_\varepsilon = \infty. \tag{1.5}$$

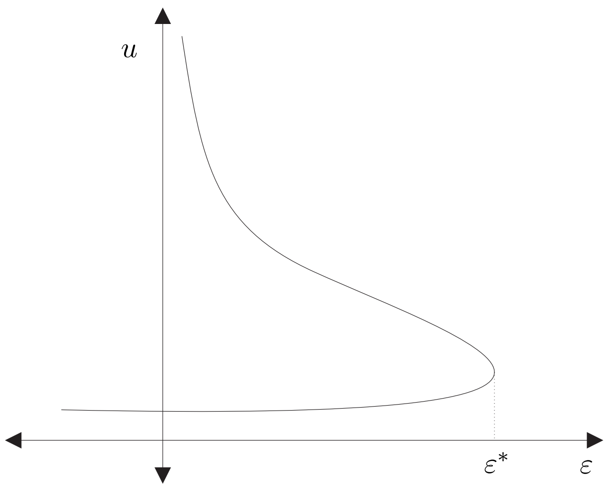

FIGURE 1. Bifurcation diagram for positive solutions to (1.2).

Let us briefly comment on our methods of proof. We will mainly use bifurcation techniques, in the spirit of the classical survey [5]. For this sake, we need to work in suitable ordered Banach spaces in which all possible positive solutions to (1.2) lie (these spaces were introduced in [28] in the context of boundary blow-up solutions). We would like to stress that continuation methods are seldom used when dealing with boundary blow-up problems. In this regard, the present work is a continuation of [28], where they were used in this context.

Problem (1.2) is equivalent to a fixed point equation for a compact, positive operator. But the first problem is encountered when dealing with the linearization of this equation around a positive solution, since it corresponds to an eigenvalue problem for a positive operator which is not strongly positive. This entails that the strengthened version of the famous theorem of Krein and Rutman (cf. [39]) cannot be used to obtain a simple principal eigenvalue. Moreover, the functional associated to this eigenvalue for the adjoint operator need not be strictly positive, and this could cause some trouble in the local bifurcation results we intend to use. Thus our fist concern will be to obtain a property which enables us to ensure the positivity of this functional.

Another difficulty arises when trying to analyze the bifurcation diagram around a degenerate positive solution, that is, one which has zero as principal eigenvalue. An essential feature is that the bifurcation diagram "bends to the left" at every such solution. This has already been obtained in previous works when dealing with a homogeneous Dirichlet problem (see [36], [37], [34]). Specifically, in [34] a Picone type identity was used for this aim, but we have to remark that this idea is out of use here due to the singularity of the solutions near the boundary. Thus we proceed differently, taking advantage of the strict positivity of the eigenfunction of the adjoint operator associated to the principal eigenvalue.

The paper is organized as follows: in Section 2 we recall some basic facts about ordered Banach spaces, including Krein-Rutman's theorem, and obtain some insight into the failure of strict positivity of the principal eigenvector of the adjoint operator. Section 3 is dedicated to introduce the spaces where we will be working and in Section 4 we solve some linear problems in these spaces. Finally, Section 5 deals with the proof of Theorem 1, while Sections 6 and 7 are concerned with the construction of the minimal solution to (1.2) and the second solution, respectively.

2. On the strict positivity of eigenvectors

This section is dedicated to recall some definitions from the theory of ordered Banach spaces, and to obtain a property on the strict positivity of eigenvectors of adjoint operators, related to Krein-Rutman's theorem, which will be adequate for our purposes.

Let X be a Banach space. A closed subset P is called an order cone if $\lambda x,\ x+y \in P$ whenever $\lambda > 0$ and $x, y \in P$, and $P \cap (-P) = \{0\}$. A cone is said to be generating if $X = P - P$, and total if $X = \overline{P - P}$. It is easily seen that cones with nonempty interior are generating, hence total. The *dual* cone P^* is defined as the set of those functionals $f \in X^*$ such that $\langle f, x\rangle \geq 0$ for every $x \in P$. When P is generating, P^* is a cone, hence X^* is an ordered Banach space.

For a given cone in X, we introduce an order relation by $x \leq y$ if $y - x \in P$. We also write $x < y$ if $y - x \in P \setminus \{0\}$, and in case P has nonempty interior, $x \ll y$ if $y - x \in \operatorname{int} P$.

One of the most interesting concepts in the context of ordered Banach spaces is that of a positive operator: if X, Y are ordered Banach spaces with the cones P_X, P_Y, a linear operator $T : X \to Y$ is positive if $T(P_X) \subset P_Y$. When $T(P_X \setminus \{0\}) \subset P_Y \setminus \{0\}$, we say that T is strictly positive, and in case P_Y has nonempty interior and $T(P_X \setminus \{0\}) \subset \text{int } P_Y$, T is called strongly positive. In particular, a functional $f : X \to \mathbb{R}$ is called strictly positive provided that $\langle f, x\rangle > 0$ for every $x \in P_X \setminus \{0\}$.

Compact, positive operators in ordered Banach spaces have the following remarkable spectral property, which is a generalization of Perron-Frobenius theorem for positive matrices. We denote by $r(T)$ the spectral radius of T, defined as

$$r(T) = \lim_{k\to\infty} \|T^k\|^{\frac{1}{k}}.$$

Then:

Theorem 4 (Krein-Rutman, Theorem 6.1 in [39])**.** *Let X be an ordered Banach space with a total positive cone P. If the linear operator $T : X \to X$ is compact and positive, with $r(T) > 0$, then $r(T)$ is an eigenvalue of T and T^* with eigenvectors in P and P^*, respectively.*

Remark 1. One of the disadvantages when using Krein-Rutman theorem in the present form is the following: even when the positive cone P is total, if F is a closed subspace of X then the cone $P \cap F$ need not be total. This inconvenient somehow limits the applicability of Theorem 4 to an operator T which leaves F invariant (see the proof of Theorem 5 below). Although some extensions of this result have been obtained, they mainly try to remove compactness of the operator (see [12], [13], [47], [48]), but at the best of our knowledge no generalization is made where the totality of the cone is dropped.

When P has nonempty interior and T is strongly positive, more can be said about the eigenvalue $r(T)$: it is simple and the only one with an eigenvector ϕ in the interior of the cone. Moreover, the eigenvector Φ of T^* associated to $r(T)$ is strictly positive (cf. Theorem 6.3 in [39]). But if T is not strongly positive, the simplicity of $r(T)$ and the strict positivity of Φ are not clear.

Assuming that the simplicity of $r(T)$ can be obtained by other means, our main objective in this section is to give a condition which ensures the strict positivity of Φ even when T is not strongly positive.

Theorem 5. *Let X be a Banach space with a total positive cone P and $T : X \to X$ a compact, positive linear operator with $r(T) > 0$. Assume $\phi \in P$ is the unique normalized eigenvector of T in P (associated to $r(T)$) and let Φ be a positive eigenvector of T^* associated to $r(T)$. Further assume that Φ is not strictly positive, and that $r(T_{|E}) > 0$, where $E = \overline{P_F - P_F}$, $P_F = P \cap F$, $F = \{x \in X : \langle \Phi, x\rangle = 0\}$. Then $\langle \Phi, \phi\rangle = 0$.*

Proof. Since Φ is not strictly positive, there exists $x \in P \setminus \{0\}$ such that $\langle \Phi, x\rangle = 0$, so that $P_F \neq \{0\}$. Observe also that $E = \overline{P_F - P_F}$, is an ordered Banach space with the positive cone P_F, which is total.

Moreover, since T leaves F invariant it also follows that $T(P_F) \subset P_F$, hence $T(E) \subset E$. The operator T being compact in E, and using the assumption $r(T_{|E}) > 0$, we obtain from Theorem 4 that $r(T)$ is an eigenvalue of T with an eigenvector in P_F. By our assumption this eigenvector has to be a positive multiple of ϕ, so that $\phi \in F$, that is, $\langle \Phi, \phi\rangle = 0$, as we wanted to prove. □

Remark 2. When X if of finite dimension, the condition $r(T_{|E}) > 0$ is not needed. The following proof is inspired in Theorem 2.5 of [38]: let $B^+ = B_1 \cap P_F$, where B_1

is the unit ball. Let $x_0 \in P_F \setminus \{0\}$, and for small ε define $A_\varepsilon : B^+ \to B^+$ by

$$A_\varepsilon x = \frac{Tx + \varepsilon x_0}{\|Tx + \varepsilon x_0\|}.$$

Observe that A_ε is well defined because $Tx + \varepsilon x_0 \geq \varepsilon x_0 > 0$. By Brouwer's fixed point theorem, there exists $x_\varepsilon \in B^+$ such that $A_\varepsilon x_\varepsilon = x_\varepsilon$. It follows that $\|x_\varepsilon\| = 1$ and

$$Tx_\varepsilon + \varepsilon x_0 = \lambda_\varepsilon x_\varepsilon$$

for some $\lambda_\varepsilon > 0$. If $\varepsilon_n \to 0$ is an arbitrary sequence, passing to subsequences we may assume $\lambda_n \to \lambda_0 \geq 0$, $x_n \to \tilde{x}$, $\|\tilde{x}\| = 1$. Thus $T\tilde{x} = \lambda_0 \tilde{x}$, and we have $\lambda_0 > 0$ since 0 is not an eigenvalue associated to a positive eigenvector. We conclude as before that $\tilde{x}$ is a multiple of ϕ.

It is possible to construct operators T defined in finite dimensional spaces which are positive but not strongly positive and have eigenvectors in the interior or on the boundary of the cone. In the latter case, the positive eigenvector Φ associated to T^* may be strictly positive. If this is not the case, some examples show that if T has more than one positive eigenvector, then there can exist eigenvectors ϕ such that $\langle \Phi, \phi \rangle > 0$.

Examples We assume $X = \mathbb{R}^N$ and identify in what follows linear operators with matrices with respect to a fixed basis.

(a) Let T be given by the matrix

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then it is easily seen that $\phi = (0,0,1)^T$ is an eigenvector associated to $r(T) = 1$ which is on the boundary of the cone P. But $\Phi = (2,1,1)^T$, which is strictly positive.

(b) When T has several eigenvalues with positive eigenvectors, the result in Theorem 5 may be false. One simple example is given by the matrix

$$T = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

We clearly have $r(T) = 2$, $\phi = (1,0)^T$ and $\Phi = (1,0)^T$. Then $\phi \in \partial P$, Φ is not strictly positive and yet $\langle \Phi, \phi \rangle = 1$.

(c) In spite of T not being strongly positive, we can still have $\phi \in \text{int } P$. For instance

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

is not strongly positive, but $r(T) = 2$ is an eigenvalue with a strongly positive eigenvector $\phi = (1,1)^T$.

3. Functional setting

In this section, we recall the definition of the functional spaces we will work in, and briefly analyze the regularity of some operators defined between them.

Let Ω be a bounded C^2 domain of $\mathbb{R}^N$. We denote by $d(x)$ the distance function from a point x to $\partial\Omega$. Observe that our assumptions imply that d is C^2 in a neighborhood of $\partial\Omega$, where it verifies $|\nabla d| = 1$ (cf. [33]).

For fixed $\alpha > 0$ we introduce the spaces

$$X_\alpha = \{u \in C^1(\Omega) \cap H^2_{\rm loc}(\Omega) : \ \|u\|_{X_\alpha} < \infty\},$$
$$Y_\alpha = \{u \in L^\infty_{\rm loc}(\Omega) : \ \|u\|_{Y_\alpha} < \infty\},$$

where

$$\|u\|_{X_\alpha} = \sup_\Omega d(x)^\alpha |u(x)| + \max_{1\le i\le N} \sup d(x)^{\alpha+1} |\partial_i u(x)| + \sup_\Omega d(x)^{\alpha+2} |\Delta u(x)|,$$
$$\|u\|_{Y_\alpha} = \sup_\Omega d(x)^\alpha |u(x)|.$$

All supremums are understood to be essential here and in what follows.

It was shown in [28] that X_α, Y_α are Banach spaces when equipped with the respective norms $\|\cdot\|_{X_\alpha}$ and $\|\cdot\|_{Y_\alpha}$. We next require a further property of these spaces.

Lemma 6. *The inclusion $X_\alpha \hookrightarrow Y_\beta$ is compact when $\alpha < \beta$.*

Proof. It is clear that $X_\alpha \subset Y_\alpha \subset Y_\beta$ when $\alpha < \beta$. Now let $\{u_n\}_{n=1}^\infty \subset X_\alpha$ be a sequence with $\|u_n\|_{X_\alpha} \le C$ for some positive constant C. If $\Omega' \subset\subset \Omega$, we have

$$\left.\begin{array}{rcl} |u_n(x)| & \le & \dfrac{C}{(\inf_{\Omega'} d)^\alpha} \\ |\partial_i u_n(x)| & \le & \dfrac{C}{(\inf_{\Omega'} d)^{\alpha+1}} \end{array}\right. \qquad x \in \Omega'.$$

We deduce that $\{u_n\}_{n=1}^\infty$ is equicontinuous and uniformly bounded in Ω'. Since Ω' is arbitrary, by means of Ascoli-Arzela's theorem and a diagonal procedure, we obtain a function $u \in C(\Omega)$ such that (by passing to a subsequence) $u_n \to u$ uniformly on compact subsets of Ω. Since $|u_n(x)| \le Cd(x)^{-\alpha}$ in Ω, we also have $|u(x)| \le Cd(x)^{-\alpha}$ in Ω, so that $u_n \in Y_\alpha \subset Y_\beta$.

We finally have to show that $u_n \to u$ in Y_β. Take $\varepsilon > 0$ and let $\delta > 0$ be small. If $x \in \Omega$ is such that $d(x) < \delta$:

$$d(x)^\beta |u_n(x) - u(x)| \le d(x)^{\beta-\alpha} d(x)^\alpha (|u_n(x)| + |u(x)|) \le 2C\delta^{\beta-\alpha} \le \varepsilon$$

if δ is chosen small enough. On the other hand, $u_n \to u$ uniformly in $\Omega_\delta = \{x \in \Omega : d(x) \ge \delta\}$, so that if n is large enough:

$$d(x)^\beta |u_n(x) - u(x)| \le (\sup_\Omega d)^\beta |u_n(x) - u(x)| \le \varepsilon$$

for $x \in \Omega_\delta$. Thus $\sup_\Omega d(x)^\beta |u_n(x) - u(x)| \le \varepsilon$ if n is large enough, and this means $u_n \to u$ in Y_β, as was to be proved. □

Remark 3. It is worth mentioning that the inclusion $X_\alpha \hookrightarrow Y_\alpha$ is *not* compact. To see this, let $\{f_n\}_{n=1}^\infty \subset C(\partial\Omega)$ be a bounded sequence which does not admit uniformly convergent subsequences. Let v_n be the unique solution to the Dirichlet problem

$$\left\{\begin{array}{ll} \Delta v = 0 & \text{in } \Omega \\ v = f_n & \text{on } \partial\Omega. \end{array}\right.$$

It follows by standard properties of harmonic functions that $|v_n(x)| \le C$ and $d(x)|\partial_i v_n(x)| \le C$ for $x \in \Omega$ and $i = 1, \dots, N$, where C is a positive constant (cf. Chapter 2 in [33]). Define

$$u_n(x) = d(x)^{-\alpha} v_n(x) \quad x \in \Omega.$$

Then clearly $u_n \in C^2(\Omega)$ and it is not hard to check that $\|u_n\|_{X_\alpha} \le C$, that is, $\{u_n\}_{n=1}^\infty$ is bounded in X_α. If we had that $u_n \to u$ in Y_α for some subsequence –still denoted by $\{u_n\}_{n=1}^\infty$ for the sake of simplicity– we would obtain that v_n converges uniformly in $\overline{\Omega}$, and therefore $\{f_n\}_{n=1}^\infty$ converges on $\partial\Omega$, which is impossible. Hence no subsequence of $\{u_n\}_{n=1}^\infty$ can converge in Y_α and the inclusion $X_\alpha \hookrightarrow Y_\alpha$ is not compact.

It is important to stress that the spaces X_α, Y_α are ordered Banach spaces with the cones P_{X_α}, P_{Y_α} of nonnegative functions. These cones have nonempty interior, given by

$$\text{int } P_{X_\alpha} = \{u \in X_\alpha : \ \inf_\Omega d(x)^\alpha u(x) > 0\}$$
$$\text{int } P_{Y_\alpha} = \{u \in Y_\alpha : \ \inf_\Omega d(x)^\alpha u(x) > 0\},$$

and consequently are generating. Observe that all nonnegative functions which are continuous up to $\partial\Omega$ belong to the boundary of the cone, but solutions to (1.2) lie in the interior of both cones.

To conclude this section, let us consider some operators defined in X_α with values in Y_β for positive α and β. The first important example is the Laplacian, from X_α to $Y_{\alpha+2}$, since by definition $\Delta u \in Y_{\alpha+2}$ for every $u \in X_\alpha$ and $\|\Delta u\|_{Y_{\alpha+2}} \le \|u\|_{X_\alpha}$. Next, let us consider conditions on a function f for its Nemytskii operator to be well-defined and smooth. The next result is a slight generalization of Theorem 4 in [28], and its proof will be omitted.

Theorem 7. *Assume $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous. Let $p > 1$, $\alpha, \beta > 0$ and $\gamma \le \beta - \alpha p$. Then:*

(a) *If $|f(x,u)| \le Cd(x)^{-\gamma}(1+|u|)^p$ for some C, then $F : Y_\alpha \to Y_\beta$ is well-defined.*
(b) *If f verifies (a), the derivative of f with respect to u exists and $|f'(x,u)| \le Cd(x)^{-\gamma}(1+|u|)^{p-1}$, then F is locally Lipschitz-continuous.*
(c) *If f verifies (a), the second derivative of f with respect to u exists and $|f''(x,u)| \le Cd(x)^{-\gamma}(1+|u|)^{p-2}$, then F is Fréchet differentiable at any u which satisfies $\inf_\Omega d(x)^\alpha |u(x)| > 0$, and*

$$F'(u)\varphi = f'(x,u)\varphi.$$

In addition, F is C^1 in a neighborhood of u.

(d) *If f verifies (a), the third derivative of f with respect to u exists and satisfies $|f'''(x,u)| \le Cd(x)^{-\gamma}(1+|u|)^{p-3}$, then F is C^2 in a neighborhood of any u with $\inf_\Omega d(x)^\alpha |u(x)| > 0$, and*

$$F''(u)[\varphi,\psi] = \frac{1}{2} f''(x,u)\varphi\psi.$$

We single out an important consequence of the previous theorem which will be used in the forthcoming sections.

Corollary 8. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ verify (a) and (d) in Theorem 7 with $\gamma \le 2 - \alpha(p-1)$. Then the operator $H : X_\alpha \to Y_{\alpha+2}$ given by*

$$H(u) = \Delta u + f(x,u)$$

is C^2 in a neighborhood of any function $u \in X_\alpha$ which verifies $u \in \operatorname{int} P_{X_\alpha}$. Moreover:

$$\begin{aligned} H'(u)\varphi &= \Delta\varphi + f'(x,u)\varphi \\ H''(u)[\varphi,\psi] &= \frac{1}{2} f''(x,u)\varphi\psi, \end{aligned}$$

for every $\varphi, \psi \in X_\alpha$.

4. Solvability of linear problems

This section is devoted to solve some linear problems related to (1.2). Their main feature is the absence of boundary conditions, and the unique solvability is guaranteed by the singularity of the coefficients involved. We mention in passing that related results for the *Dirichlet* problem have been obtained for instance in [3] (see also [49], [50] and references therein).

For a fixed function $b \in C(\Omega)$, we want to solve

$$\Delta u = b(x)u - f \qquad \text{in } \Omega \tag{4.1}$$

for $f \in Y_{\alpha+2}$, $u \in X_\alpha$, and fixed $\alpha > 0$. As usual, the solvability of (4.1) depends on the homogeneous problem

$$\Delta u = b(x)u \qquad \text{in } \Omega. \tag{4.2}$$

For a given $f \in Y_{\alpha+2}$, it is required that the solution u belongs to X_α. This entails that the singularity of b has to be somehow related to α. The main hypothesis we will impose is:

$$\liminf_{d\to 0} d(x)^2 b(x) > \alpha(\alpha+1). \tag{4.3}$$

This condition is met for instance when $b \geq pu^{p-1}$ and u is a solution to (1.2).

The first important result of the section is the following:

Theorem 9. *Assume $b \in C(\Omega)$ is nonnegative and verifies (4.3). Then for every $f \in Y_{\alpha+2}$ there exists a unique solution to (4.1) in X_α. Moreover, there exists a positive constant C independent of f and u such that*

$$\|u\|_{X_\alpha} \leq C\|f\|_{Y_{\alpha+2}}.$$

Finally, if $f \geq 0$ then $u \geq 0$.

Before coming to the proof of Theorem 9, we need a couple of results. The most important ingredient is the existence of a strict supersolution to (4.2) with a convenient growth near $\partial\Omega$. A function $\overline{u} \in C^2_{\text{loc}}(\Omega)$ will be called a strict supersolution if $\Delta\overline{u} \leq \gamma b(x)\overline{u}$ for some $\gamma \in (0,1)$. The necessary growth condition is

$$C_1 d(x)^{-\alpha} \leq \overline{u}(x) \leq C_2 d(x)^{-\alpha} \qquad \text{in } \Omega \tag{4.4}$$

for some positive constants C_1, C_2.

Lemma 10. *Let $b \in C(\Omega)$ be nonnegative and verify (4.3). Then there exists a strict supersolution to problem (4.2) verifying (4.4).*

Proof. Let w be the solution to $-\Delta w = 1$ in Ω, $w_{|\partial\Omega} = 0$. For a positive B, define

$$\overline{u} = w^{-\alpha} + Bw.$$

Then, by Hopf's principle, $\overline{u}$ verifies (4.4). Let us check that it is a strict supersolution to (4.2). Since $\Delta\overline{u} = \alpha(\alpha+1)w^{-\alpha-2}|\nabla w|^2 + \alpha w^{-\alpha-1} - B$, $\overline{u}$ will be a strict supersolution provided that

$$\alpha(\alpha+1)|\nabla w|^2 + \alpha w - Bw^{\alpha+2} \le \gamma w^2 b(1 + Bw^{\alpha+1}) \tag{4.5}$$

in Ω, where $\gamma \in (0,1)$ is to be chosen.

Now observe that when $0 < d < \delta$, for small δ, (4.5) is implied by

$$\alpha(\alpha+1)|\nabla w|^2 + \alpha w < \gamma w^2 b. \tag{4.6}$$

Taking into account that $w/d = |\nabla w|$ at $d = 0$, according to l'Hôpital rule and Hopf's principle, we can obtain (4.6) by our assumption on b if γ is close to 1 and δ is chosen small enough, independent of B.

When $d \ge \delta$, (4.5) holds by simply taking B large enough. This concludes the proof. □

The next lemma determines the boundary behavior of the solutions to (4.1) in some special cases. It will be important in the proof of uniqueness in Theorem 9.

Lemma 11. *Let $b \in C(\Omega)$ be nonnegative and verify* (4.3). *Assume $u \in X_\alpha$ is a solution to* (4.1) *for some $f \in Y_{\alpha+2}$ with*

$$\lim_{d\to 0} d(x)^{\alpha+2} f(x) = 0. \tag{4.7}$$

Then

$$\lim_{d\to 0} d(x)^{\alpha} u(x) = 0. \tag{4.8}$$

Proof. This proof is is similar to that of Theorem 3 in [30]. According to Lemma 10, there exists a strict supersolution $\overline{u}$ to (4.2) verifying (4.4). Observe also that (4.3) and (4.7) imply $f/(b\overline{u}) \to 0$ as $d \to 0$.

Since $u \in X_\alpha$,

$$\theta = \limsup_{d\to 0} \frac{u(x)}{\overline{u}(x)}$$

is finite. We claim that $\theta = 0$. Indeed, assume $\theta > 0$. Choose $\varepsilon > 0$. Then there exists $\delta > 0$ such that $u(x) \le (\theta+\varepsilon)\overline{u}(x)$ if $d(x) < \delta$, and $x_0 \in \Omega$ with $d(x_0) < 2\delta/3$ and $u(x_0) \ge (\theta-\varepsilon)\overline{u}(x_0)$. By diminishing δ if necessary we can also achieve $f/(b\overline{u}) \le (\theta-\varepsilon)(1-\gamma)/2$ when $d(x) < \delta$.

Let $r = d(x_0)/2$ and $\Omega_0 = \{x \in \Omega : \ u(x) > (\theta-\varepsilon)\overline{u}(x)\} \cap B_r(x_0)$. Notice that $d(x) < \delta$ for every $x \in \Omega_0$. Hence in Ω_0

$$\begin{aligned}\Delta(u - (\theta-\varepsilon)\overline{u}) &\ge b(x)u - \gamma(\theta-\varepsilon)b(x)\overline{u} - f(x)\\ &\ge (1-\gamma)(\theta-\varepsilon)b(x)\overline{u} - f(x)\\ &\ge \frac{(1-\gamma)}{2}(\theta-\varepsilon)b(x)\overline{u}.\end{aligned}$$

By (4.4) and (4.3), there exists a positive constant such that $\Delta(u-(\theta-\varepsilon)\overline{u}) \ge C(\theta-\varepsilon)d(x)^{-\alpha-2} \ge C(\theta-\varepsilon)r^{-\alpha-2}$ in Ω_0 (from now on we will use the same letter C to denote different positive constants, not necessarily the same everywhere). Let w be the unique solution to $-\Delta w = 1$ in $B_r(x_0)$ with $w = 0$ on $\partial B_r(x_0)$. Since

$\Delta(u-(\theta-\varepsilon)\overline{u}+C(\theta-\varepsilon)r^{-\alpha-2}w)\geq 0$ in Ω_0, we may use the maximum principle to find $x_1\in\partial\Omega_0$ such that

$$(u-(\theta-\varepsilon)\overline{u}+C(\theta-\varepsilon)r^{-\alpha-2}w)(x_0)<(u-(\theta-\varepsilon)\overline{u}+C(\theta-\varepsilon)r^{-\alpha-2}w)(x_1).$$

If $u(x_1)=(\theta-\varepsilon)\overline{u}(x_1)$, we obtain from this inequality that $w(x_0)<w(x_1)$, which is impossible since w is radially decreasing. Thus $x_1\in\partial B_r(x_0)$, and taking into account that $w(x_0)=Cr^2$, we have

$$C(\theta-\varepsilon)r^{-\alpha}<u(x_1)-(\theta-\varepsilon)\overline{u}(x_1).$$

Also, (4.4) gives $\overline{u}(x_1)\leq Cr^{-\alpha}$, so that $u(x_1)\geq(1+C)(\theta-\varepsilon)\overline{u}(x_1)$. Since $d(x_1)<\delta$, we also have $u(x_1)\leq(\theta+\varepsilon)\overline{u}(x_1)$, hence

$$(1+C)(\theta-\varepsilon)\leq\theta+\varepsilon$$

and letting $\varepsilon\to 0$ we arrive at a contradiction. This contradiction means $\theta=0$, so that

$$\limsup_{d\to 0}\frac{u(x)}{\overline{u}(x)}=0.$$

Since $-u$ is solution to (4.1) with f replaced by $-f$, the previous proof gives

$$\lim_{d\to 0}\frac{u(x)}{\overline{u}(x)}=0.$$

and (4.8) follows from (4.4). The proof is finished. □

Next, let us proceed to prove the existence and uniqueness of solutions to (4.1).

Proof of Theorem 9. Assume first that the homogeneous problem (4.2) admits a solution $u\in X_\alpha$. Let $v=u/\overline{u}$, where $\overline{u}$ is the supersolution given in Lemma 10. From Lemma 11, we have $v=0$ on $\partial\Omega$. Moreover,

$$\overline{u}\Delta v+2\nabla\overline{u}\nabla v+(\Delta\overline{u}-b(x)\overline{u})v=0\quad\text{in }\Omega.$$

Since $\Delta\overline{u}-b(x)\overline{u}\leq 0$, we may apply the maximum principle (Corollary 3.2 in [33]) to obtain $v=0$, that is, $u=0$.

Next let $f\in Y_{\alpha+2}$, so that $|f|\leq\|f\|_{Y_{\alpha+2}}d^{-\alpha-2}$ in Ω. Let us check that $\bar{v}=A(\overline{u}+w)$ is a supersolution to (4.1) if A is large enough, where w is as in the proof of Lemma 10. It suffices to have

$$A\Delta\overline{u}-A\leq b(x)A\overline{u}+Ab(x)w-\|f\|_{Y_{\alpha+2}}d^{-\alpha-2}, \tag{4.9}$$

which in turn is implied by $\|f\|_{Y_{\alpha+2}}\leq A(1-\gamma)d^2bd^\alpha\overline{u}+Ad^{\alpha+2}$. If $\delta>0$ is small then $d^2b(x)\geq C$ for some positive constant C when $d(x)<\delta$, so that (4.9) holds if A is large enough, depending on $\|f\|_{Y_{\alpha+2}}$. For $d(x)\geq\delta$, (4.9) holds when A is taken large. It is clear that the election $A=C\|f\|_{Y_{\alpha+2}}$ would suffice.

It is plain that $-\bar{v}$ is then a subsolution. Next, let $\{\Omega_k\}_{k=1}^\infty$ a sequence of smooth domains verifying $\Omega_k\subset\Omega_{k+1}\subset\subset\Omega$, and consider the Dirichlet problem

$$\begin{cases}\Delta v=b(x)v-f & \text{in }\Omega_k\\ v=0 & \text{on }\partial\Omega_k.\end{cases} \tag{4.10}$$

Since $b,f\in C(\overline{\Omega}_k)$ and $b\geq 0$, there exists a unique (strong) solution v_k to (4.10), which verifies $v_k\in H^{2,q}(\Omega_k)$ for every $q>1$. By uniqueness, since $\bar{v}>0$ on $\partial\Omega_k$, we have $-\bar{v}\leq v_k\leq\bar{v}$ in Ω_k. This entails that the sequence $\{v_k\}_{k=1}^\infty$ is bounded in $C^{1,\eta}_{\text{loc}}(\Omega)$ for every $\eta\in(0,1)$. Hence, by Ascoli-Arzelá theorem and a diagonal procedure we obtain that, passing to a subsequence, $v_k \to u$ in $C^1(\Omega)$. Passing to the limit in the weak formulation of (4.10) we obtain that u is a weak solution to (4.1), which in addition verifies $-\bar{v} \le u \le \bar{v}$.

By standard regularity, we have $u \in C^1(\Omega) \cap H^2_{\text{loc}}(\Omega)$, and of course $|u| \le C\|f\|_{Y_{\alpha+2}} d^{-\alpha}$ in Ω. It follows that $|\Delta u| \le C\|f\|_{Y_{\alpha+2}} d^{-\alpha-2}$ in Ω. Using Lemma 12 in [28], we also have $|\partial_i u| \le C\|f\|_{Y_{\alpha+2}} d^{-\alpha-1}$ in Ω for $i = 1, \ldots, N$. That is, $\|u\|_{X_\alpha} \le C\|f\|_{Y_\alpha}$.

Finally, notice that when $f \ge 0$ we can take $\underline{v} = 0$ as a subsolution to (4.10) and obtain that $v_k \ge 0$ for every k. Hence $u \ge 0$. The proof is concluded. □

Observe that Theorem 9 allows us to define a bounded, linear positive operator $T : Y_{\alpha+2} \to X_\alpha$ by $Tf = u$. Since the inclusion $X_\alpha \to Y_{\alpha+2}$ is compact by Lemma 6, the restriction of T to X_α will be compact. We will also consider in some places $T : Y_{\alpha+2} \to Y_{\alpha+2}$, which is compact and positive as well.

However T is not strongly positive in $Y_{\alpha+2}$, since $T(P_{Y_{\alpha+2}}) \subset P_{X_\alpha} \subset \partial P_{Y_{\alpha+2}}$, nor in X_α due to Lemma 11. This means that the improved version of Krein-Rutman's theorem (Theorem 6.3 in [39]) can not be used to obtain the existence of an eigenvalue associated to a positive eigenfunction. Thus we will make use of Theorem 4, and for this purpose we need to know that $r(T) > 0$. Since we will need to invoke at some moment Theorem 5, we require a slightly more general result.

Lemma 12. *Let $F \subset Y_{\alpha+2}$ be a closed subspace such that $T(F) \subset F$ and $F \cap P_{Y_{\alpha+2}} \ne \{0\}$. Then*

$$r(T_{|F}) > 0.$$

In particular, $r(T) > 0$.

Proof. Let $P_F = F \cap P_{Y_{\alpha+2}}$. Then F is an ordered Banach space with P_F as a positive cone, and $T : F \to F$ is compact and positive. Assume $r(T_{|F}) = 0$. Take an arbitrary $\lambda > 0$, and fix $f \in P_F \setminus \{0\}$. By Theorem 2.16 in [38], there exists a unique solution $u \in P_F$ to

$$\frac{1}{\lambda} u = Tu + \frac{1}{\lambda} Tf.$$

It follows that $u \in X_\alpha$ and it verifies

$$\Delta u = bu - \lambda u - f \qquad \text{in } \Omega. \tag{4.11}$$

Let $\Omega' \subset\subset \Omega$ be arbitrary and φ a positive eigenfunction associated to the principal eigenvalue λ_1 of the problem

$$\begin{cases} \Delta\varphi = b\varphi - \lambda_1\varphi & \text{in } \Omega' \\ \varphi = 0 & \text{on } \partial\Omega'. \end{cases}$$

Multiplying (4.11) by φ, integrating in Ω' and integrating by parts we arrive at

$$-\lambda_1 \int_{\Omega'} \varphi u = -\lambda \int_{\Omega'} \varphi u - \int_{\Omega'} f\varphi \le -\lambda \int_{\Omega'} \varphi u,$$

which yields $\lambda \le \lambda_1$, which is not possible, since $\lambda > 0$ was chosen arbitrarily. Thus $r(T_{|F}) > 0$. □

To close this section, we consider the eigenvalue problem related to (4.1), that is,

$$\begin{cases} \Delta u = b(x)u - \lambda u & \text{in } \Omega \\ u \in X_\alpha \end{cases} \tag{4.12}$$

and look for an eigenvalue associated to a positive eigenfunction. We remark that although the existence of this eigenvalue will be a consequence of Theorem 4, its simplicity and uniqueness follow by entirely different arguments, based on the maximum principle.

Theorem 13. *Let $b \in C(\Omega)$ verify (4.3). Then the eigenvalue problem (4.12) admits a least eigenvalue $\lambda_1(b)$, which is the only one with a positive eigenfunction ϕ. Moreover, $\lambda_1(b)$ is simple and $\phi > 0$ in Ω but*

$$\phi \in \partial P_{X_\alpha}.$$

Proof. Assume initially that $b \geq 0$. Observe that (4.12) is equivalent to $u = \lambda T u$, $u \in Y_{\alpha+2}$. Since $T : Y_{\alpha+2} \to Y_{\alpha+2}$ is compact, positive and $r(T) > 0$ thanks to Lemma 12, we may apply Theorem 4 to obtain the existence of an eigenvector $\phi \in P_{Y_{\alpha+2}}$ associated to $r(T)$. Actually, since $T(Y_{\alpha+2}) \subset X_\alpha$, we have $\phi \in P_{X_\alpha}$. Then $\lambda_1 = \lambda_1(b) := 1/r(T)$ will be an eigenvalue of (4.12), and it is the least one. By the strong maximum principle we obtain that $\phi > 0$ in Ω, but Lemma 11 gives $\phi \in \partial P_{X_\alpha}$.

Let us see that λ_1 is the only eigenvalue associated to a positive eigenfunction, and that it is simple (see the proof of Theorem 2.2 in [11]). Let ψ be a positive eigenfunction associated to an eigenvalue $\mu \geq \lambda_1$. Using again the strong maximum principle, $\psi > 0$ in Ω. Let $\overline{u}$ be the supersolution to (4.2) given by Lemma 10 (recall that we are assuming $b \geq 0$). Define

$$\phi_1 = \frac{\phi}{\overline{u}}, \qquad \psi_1 = \frac{\psi}{\overline{u}}.$$

Then ϕ_1 verifies $\overline{u}\Delta\phi_1 + 2\nabla\overline{u}\nabla\phi_1 + (\Delta\overline{u} - b(x)\overline{u} + \lambda_1\overline{u})\phi_1 = 0$, and a similar equation for ψ_1, with λ_1 replaced by μ. For $t > 0$, the function $z_t = t\phi_1 - \psi_1$ verifies

$$\overline{u}\Delta z_t + 2\nabla\overline{u}\nabla z_t + c(x) z_t \geq 0 \qquad \text{in } \Omega,$$

where

$$c(x) = \Delta\overline{u} - b(x)\overline{u} + \mu\overline{u}.$$

Notice that, since $\overline{u}$ is a strict supersolution, we obtain $c(x) \leq ((\gamma - 1)b(x) + \mu)\overline{u}$, hence $c(x) < 0$ in $\Omega \setminus \Omega_\delta$ for some small $\delta > 0$, where $\Omega_\delta = \{x \in \Omega : \ d(x) > \delta\}$.

We next observe that $z_t < 0$ in Ω_δ if t is small enough, while $z_t > 0$ when t is large. Then

$$\tau = \sup\{t > 0 : \ z_t < 0 \text{ in } \Omega_\delta\}$$

is finite, and we have $z_\tau \leq 0$ in Ω_δ. Since, according to Lemma 11, $z_t = 0$ on $\partial\Omega$, and $c \leq 0$ in $\Omega \setminus \Omega_\delta$, we may apply again the maximum principle to obtain that $z_\tau \leq 0$ throughout Ω.

By the strong maximum principle, either $z_\tau < 0$ in Ω or $z_\tau \equiv 0$ in Ω. But the first possibility would imply $z_{\tau+\varepsilon} < 0$ in Ω_δ, contradicting the maximality of τ, so the second possibility holds. Thus $\psi_1 = \tau\phi_1$, that is, $\psi = \tau\phi$.

Of course, this rules out the possibility $\mu > \lambda_1$, and we have proved that λ_1 is the only eigenvalue of (4.12) with a positive eigenfunction. When $\mu = \lambda_1$, the above proof shows that all eigenfunctions are multiples of ϕ, that is, λ_1 is simple.

To deal with the general case where b is not necessarily positive, we just observe that problem (4.12) is equivalent to $\Delta u = (b(x) + M)u - (\lambda + M)u$, and $M > 0$ can be selected so that $b(x) + M > 0$ in Ω. The proof is finished. □

Remark 4. A minor modification of the proof of simplicity in Theorem 13 also yields the following: if $\psi \in X_\alpha$ is positive and verifies

$$\Delta\psi \le b(x)\psi \tag{4.13}$$

and $\lim_{d\to 0} d(x)^\alpha \psi(x) = 0$, then $\lambda_1(b) > 0$ unless equality holds in (4.13), in which case $\lambda_1(b) = 0$. If the inequality in (4.13) is reversed we obtain $\lambda_1(b) < 0$ unless equality holds.

5. Proof of Theorem 1

This brief section is dedicated to prove the necessary conditions gathered in Theorem 1. We have decided to separate it from the rest since their proofs are of a different nature.

Proof of Theorem 1. (i) (See the proof of Theorem 1 in [32].) Let $x_0 \in \partial\Omega$ be such that $a(x_0) < 0$. Choose an open neighborhood $\mathcal{U}$ of x_0 such that $a < 0$ in $D := \Omega\cap\mathcal{U}$, and a function $\psi \in C_0^\infty(\mathcal{U})$ such that $0 \le \psi \le 1$ and $\psi \equiv 1$ in a neighborhood of x_0. Consider the problem

$$\begin{cases} \Delta v = 0 & \text{in } D \\ v = \psi & \text{on } \partial D, \end{cases}$$

which has a unique positive solution v.

Now assume that there exists a positive solution to (1.1). Then for every $n \in \mathbb{N}$ we have $u \ge n\psi$ on ∂D and $\Delta u = a(x)u^p \le 0$ in D, so that, by comparison, we have $u \ge nv$ in D. We reach a contradiction when $n \to \infty$.

(ii) Since $a > 0$, we have that

$$\begin{aligned} &u(x) \sim A(x_0)d(x)^{-\alpha} \\ &\nabla u(x)\nu(x_0) \sim \alpha A(x_0)d(x)^{-\alpha-1} \end{aligned} \qquad \text{as } x \to x_0 \in \partial\Omega, \tag{5.1}$$

where ν is the outward unit normal to $\partial\Omega$, $\alpha = 2/(p-1)$ and $A(x_0) = (\alpha(\alpha+1)/a(x_0))^{\frac{1}{p-1}}$ (cf. for instance [15] or [9]). It also follows from the analysis in [15] that

$$|\nabla u(x)| \sim \alpha A(x_0)d(x)^{-\alpha-1} \tag{5.2}$$

when $x \to x_0$.

Assume u is a positive solution to (1.1). For small $\delta > 0$, let $\Omega_\delta = \{x \in \Omega : d(x) > \delta\}$. Then

$$\int_{\Omega_\delta} a(x) = \int_{\Omega_\delta} \frac{\Delta u}{u^p} = p\int_{\Omega_\delta} \frac{|\nabla u|^2}{u^{p+1}} + \int_{\partial\Omega_\delta} \frac{1}{u^p}\frac{\partial u}{\partial\nu}. \tag{5.3}$$

From (5.2) we have

$$\frac{|\nabla u|^2}{u^{p+1}} \le C\frac{d^{-2\alpha-2}}{d^{-\alpha(p+1)}} = C,$$

for some positive constant C, when $d(x)$ is small enough, and

$$\left|\frac{1}{u^p}\frac{\partial u}{\partial\nu}\right| \le C\frac{d^{-\alpha-1}}{d^{-\alpha p}} = C\delta$$

when $d(x) = \delta$. Letting $\delta \to 0$ in (5.3), we obtain

$$\int_\Omega a(x) = p\int_\Omega \frac{|\nabla u|^2}{u^{p+1}} > 0,$$

where the integral in the right-hand side is convergent. This concludes the proof. □

6. The minimal solution

We will concentrate next in proving the existence of an interval $(0, \varepsilon^*)$ such that the problem

$$(P_\varepsilon) \qquad \begin{cases} \Delta u = (a_+(x) - \varepsilon a_-(x))u^p & \text{in } \Omega \\ u = \infty & \text{on } \partial\Omega \end{cases}$$

has no solution for $\varepsilon > \varepsilon^*$, while it has a minimal positive solution if $\varepsilon \in (0, \varepsilon^*)$. Our proofs include continuation methods, as well as the method of sub and supersolutions (developed for boundary blow-up problems in [27]). We will frequently denote for simplicity $a_\varepsilon = a_+ - \varepsilon a_-$.

The first step consists in proving that solutions to (P_ε) exist when $\varepsilon > 0$ is small enough. The idea is to use a continuation argument beginning with the unique solution to (P_0), which will be denoted by U. We will work in the spaces defined in Section 3 with $\alpha = 2/(p-1)$.

Lemma 14. *There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, problem (P_ε) admits a positive solution. Moreover, this solution is unique in a neighborhood of U in X_α.*

Proof. We will use the implicit function theorem. For $\varepsilon \in \mathbb{R}$ and $u \in X_\alpha$, define $H(\varepsilon, u) = \Delta u - (a_+ - \varepsilon a_-)u^p$. Then $H : \mathbb{R} \times X_\alpha \to Y_{\alpha+2}$ is C^1 in a neighborhood of $(0, U)$ according to Corollary 8. Moreover, $H(0, U) = 0$ and

$$H'(0, U)\varphi = \Delta\varphi - pa_+U^{p-1}\varphi$$

for $\varphi \in X_\alpha$. We now notice that $b(x) = pa_+U^{p-1}$ is nonnegative and verifies

$$\lim_{d \to 0} d(x)^2 b(x) = p\alpha(\alpha + 1) > \alpha(\alpha + 1),$$

and by Theorem 9 we deduce that $H'(0, U) : X_\alpha \to Y_{\alpha+2}$ is an isomorphism. Thus we may apply the implicit function theorem to obtain $\delta > 0$ and a C^1 function $u : (-\delta, \delta) \to X_\alpha$ with $u(0) = 0$ such that $H(\varepsilon, u(\varepsilon)) = 0$ for $|\varepsilon| < \delta$ and the unique solutions to $H(\varepsilon, u) = 0$ in a neighborhood of U are of the form $u = u(\varepsilon)$.

The proof is completed by noticing that, since $\|u(\varepsilon) - U\|_{X_\alpha}$ is small enough, we have $u(\varepsilon) \geq cd(x)^{-\alpha}$ for some small positive c, so that $u(\varepsilon) = \infty$ on $\partial\Omega$, and is a solution to (P_ε). □

Once solutions have been constructed for small ε, we will prove that the set of values of ε for which a positive solution to (P_ε) exists is an interval of the form $(0, \varepsilon^*)$, and that a minimal positive solution can always be constructed when ε belongs to this interval.

Lemma 15. *There exists $\varepsilon^* > 0$ such that (P_ε) does not admit positive solutions when $\varepsilon > \varepsilon^*$ and for $\varepsilon \in (0, \varepsilon^*)$, there exists a minimal positive solution u_ε to (P_ε), which is increasing with ε.*

Proof. It follows immediately from Theorem 1 that positive solutions to (P_ε) do not exist when ε is large enough. Define

$$\varepsilon^* = \sup\{\varepsilon > 0 : (P_\varepsilon) \text{ has a positive solution}\}.$$

Then clearly no positive solutions to (P_ε) exist when $\varepsilon > \varepsilon^*$. To prove the existence of solutions when $\varepsilon \in (0, \varepsilon^*)$, we first construct a "universal" subsolution. This is easily achieved by selecting a small enough $\delta > 0$ and setting $\underline{u} = \delta U$. Then

$$\Delta \underline{u} = \delta a_+ U^p \geq a_+ (\delta U)^p \geq (a_+ - \varepsilon a_-)\underline{u}^p$$

in Ω, with $\underline{u} = \infty$ on $\partial\Omega$, so that $\underline{u}$ is a subsolution to (P_ε) for every $\varepsilon > 0$.

Now take $\varepsilon \in (0, \varepsilon^*)$ arbitrary. By the definition of ε^*, problem $(P_{\bar\varepsilon})$ has a positive solution v for some $\bar\varepsilon \in (\varepsilon, \varepsilon^*)$. Since we can always have $v \geq \underline{u}$ by diminishing δ if necessary, we may apply the method of sub and supersolutions to obtain a positive solution u to (P_ε).

Finally, observe that for every solution u to (P_ε) we have $\Delta u \leq a_+ u^p$, and then the uniqueness of solutions of this problem implies $u \geq U \geq \underline{u}$. Since we have a subsolution below all possible solutions, the existence of a minimal positive solution u_ε is obtained in a standard way. The monotonicity of u_ε is also a consequence of the method of sub and supersolutions. This concludes the proof. ☐

Next we deal with the nonsingular character of the minimal solution u_ε. Thus we consider the linearized equation

$$\left\{ \begin{array}{ll} \Delta \phi = p(a_+ - \varepsilon a_-) u_\varepsilon^{p-1} \phi - \lambda \phi & \text{in } \Omega \\ \phi \in X_\alpha. & \end{array} \right. \tag{6.1}$$

Since $\lim_{d \to 0} p d^2 a_\varepsilon u^{p-1} = p\alpha(\alpha+1)$, by Theorem 13 there exists a least eigenvalue $\lambda_1(u_\varepsilon)$ with an associated strictly positive eigenfunction which lies on the boundary of the positive cone of $Y_{\alpha+2}$. As a first property of this eigenvalue we have:

Lemma 16. *Let u_ε be the minimal solution to (P_ε) for $\varepsilon \in (0, \varepsilon^*)$, and $\lambda_1(u_\varepsilon)$ the first eigenvalue of the linearized problem* (6.1)*. Then $\lambda_1(u_\varepsilon) \geq 0$.*

Proof. Assume $\lambda_1(u_\varepsilon) < 0$. Let us choose an associated positive eigenfunction $\phi \in X_\alpha$, normalized as $\sup_\Omega \phi/u_\varepsilon = 1$. We claim that $\overline{u} = u_\varepsilon - \eta\phi$ is a supersolution to (P_ε) if η is small enough. Observe first that $\overline{u} > 0$ in Ω and $\overline{u} = \infty$ on $\partial\Omega$ if η is small. Moreover:

$$\begin{aligned} \Delta(u_\varepsilon - \eta\phi) - a_\varepsilon (u - \eta\phi)^p &= \lambda_1(u_\varepsilon)\eta\phi - a_\varepsilon((u_\varepsilon - \eta\phi)^p - u_\varepsilon^p + p u_\varepsilon^{p-1}\eta\phi) \\ &= \eta\phi \left(\lambda_1(u_\varepsilon) - a_\varepsilon \frac{p(p-1)}{2}\, \xi^{p-2}\eta\phi \right) \\ &\leq \eta\phi \left(\lambda_1(u_\varepsilon) + \varepsilon a_- \frac{p(p-1)}{2}\, \xi^{p-2}\eta\phi \right), \end{aligned}$$

where $u_\varepsilon - \eta\phi \leq \xi \leq u_\varepsilon$. Since a_- is supported in $\Omega_- \subset\subset \Omega$ we can bound the last term by a positive constant C times η. Hence $\Delta(u_\varepsilon - \eta\phi) - a_\varepsilon(u - \eta\phi)^p \leq \eta\phi(\lambda_1(u_\varepsilon) + C\eta) < 0$, if η is sufficiently small.

Next recall that δU is a subsolution when $\delta < 1$. By diminishing δ, we can always achieve $\delta U \leq u_\varepsilon - \eta\phi$, so that the method of sub and supersolutions would give a solution u to (P_ε) which verifies $u \leq u_\varepsilon - \eta\phi$, contradicting the minimality of u_ε. Thus $\lambda_1(u_\varepsilon) \geq 0$. ☐

Remark 5. Choose a function $b \in C(\Omega)$ verifying $b(x) > p a_\varepsilon(x) u_\varepsilon(x)^{p-1}$, and notice that this entails $\liminf_{d \to 0} d(x)^2 b(x) > \alpha(\alpha+1)$. Thus problem (P_ε) is equivalent to $u = T(bu - a_\varepsilon u^p)$ in X_α, where T is the operator given by Theorem 9. Denote $S(\varepsilon, u) = T(bu - a_\varepsilon u^p)$, so that $S : \mathbb{R} \times X_\alpha \to X_\alpha$ is compact.

Let us see that $r := r(S'(\varepsilon, u_\varepsilon)) \leq 1$. Indeed, if we assume $r > 1$, since we have that $S'(\varepsilon, u_\varepsilon)v = T(bv - pa_\varepsilon u_\varepsilon^{p-1} v)$ is compact and positive by the choice of b, we may apply Krein-Rutman's theorem to obtain that r is an eigenvalue associated to a positive eigenfunction φ. This means

$$\Delta\varphi = \left(1 - \frac{1}{r}\right) b\varphi + \frac{p}{r} a_\varepsilon u_\varepsilon^{p-1}\varphi > pa_\varepsilon u_\varepsilon^{p-1}\varphi,$$

and by Remark 4 we deduce $\lambda_1(u_\varepsilon) < 0$, which contradicts Lemma 16. Thus $r \leq 1$. Notice also that $r = 1$ if and only if $\lambda_1(u_\varepsilon) = 0$.

When $\lambda_1(u_\varepsilon) > 0$, we can argue as in Lemma 14 to conclude the uniqueness of positive solutions to (P_ε) in a neighborhood of u_ε. In the next result we analyze the situation $\lambda_1(u_\varepsilon) = 0$.

Lemma 17. *Assume $\lambda_1(u_{\varepsilon_0}) = 0$ for some $\varepsilon_0 \in (0, \varepsilon^*)$. Then there exists a neighborhood $\mathcal{U}$ of u_ε such that for $\varepsilon > \varepsilon_0$ there are no solutions to (P_ε) in $\mathcal{U}$, while for $\varepsilon < \varepsilon_0$ there are exactly two solutions to (P_ε) in $\mathcal{U}$.*

Proof. As in the proof of Lemma 14, let $H : \mathbb{R} \times X_\alpha \to Y_{\alpha+2}$ be given by $H(\varepsilon, u) = \Delta u - (a_+ - \varepsilon a_-)u^p$. By Theorem 7, H is C^2 in a neighborhood of $u_0 := u_{\varepsilon_0}$. Observe that $H(\varepsilon_0, u_0) = 0$ and since $\lambda_1(u_\varepsilon) = 0$, the kernel of $H'(\varepsilon_0, u_0)$ is one-dimensional and is spanned by some $\phi \in X_\alpha$, by Theorem 13.

Moreover, the problem $\Delta\varphi = p(a_+ - \varepsilon a_-)u_0^{p-1}\varphi - f$ with $\varphi \in X_\alpha$, $f \in Y_{\alpha+2}$ is equivalent to $\varphi = Tf$, where $T : Y_{\alpha+2} \to Y_{\alpha+2}$. It is plain that

$$\text{Im } H_u(\varepsilon_0, u_0) = \{u \in Y_{\alpha+2} : \ \langle \Phi, u\rangle = 0\},$$

where $\Phi \in Y^*_{\alpha+2}$ is the positive eigenvector of T^* given by Theorem 4.

We claim that Φ is strictly positive. Indeed, we have

$$H_u(\varepsilon_0, u_0)u_0 = \Delta u_0 - p(a_+ - \varepsilon_0 a_-)u_0^p = -(p-1)(a_+ - \varepsilon_0 a_-)u_0^p,$$

so that $\langle \Phi, (a_+ - \varepsilon_0 a_-)u_0^p\rangle = 0$. Next, observe that $a_+ u_0^p + \phi \in \text{int } P_{Y_{\alpha+2}}$, since $a_+ u_0^p + \phi > 0$ in Ω and $\liminf_{d\to 0} d^\alpha (a_+ u_0^p + \phi) > 0$. Also, there exists a positive constant C such that $\varepsilon_0 a_- u_0^p \leq C\phi$ in Ω, because a_- is supported in Ω_- and $\phi > 0$ in Ω. Thus

$$0 < \langle \Phi, a_+ u_0^p + \phi\rangle = \langle \Phi, \varepsilon_0 a_- u_0^p\rangle + \langle \Phi, \phi\rangle \leq (C+1)\langle \Phi, \phi\rangle .$$

We conclude, using Theorem 5 (and also Lemma 12) that Φ is strictly positive.

In particular, $\langle \Phi, a_- u_0^p\rangle > 0$ and we can use Theorem 2.1 in [4] to obtain that there exists $\delta > 0$ and C^2 functions $\varepsilon : (-\delta, \delta) \to \mathbb{R}$, $u : (-\delta, \delta) \to X_\alpha$ such that, in a neighborhood of the point (ε_0, u_0), all solutions to $H(\varepsilon, u) = 0$ are of the form $(\varepsilon(s), u(s))$. Moreover

$$\varepsilon'(0) = 0,$$

and we also make the important claim:

$$\varepsilon''(0) < 0. \tag{6.2}$$

To see this we just differentiate twice in $H(\varepsilon(s), u(s)) = 0$ and set $s = 0$ (we take into account that the parameter s can be selected so that $u'(0) = \phi$):

$$H_\varepsilon(\varepsilon_0, u_0)\varepsilon''(0) + H_{uu}(\varepsilon_0, u_0)\phi^2 + H_u(\varepsilon_0, u_0)u''(0) = 0,$$

which is equivalent to

$$\Delta\psi = p(a_+ - \varepsilon_0 a_-)u_0^{p-1}\psi - \varepsilon''(0)a_- u_0^p + \frac{p(p-1)}{2}u_0^{p-2}(a_+ - \varepsilon_0 a_-)\phi^2, \tag{6.3}$$

where $\psi = u''(0)$.

In order to determine the sign of $\varepsilon''(0)$, we will not apply the Fredholm alternative directly to (6.3). Instead, we set

$$w = \psi + \beta\frac{\phi^2}{u_0},$$

for β to be determined. A straightforward calculation gives

$$\Delta w = \Delta\psi + \beta\left(\frac{2\phi\Delta\phi}{u_0} - \frac{\phi^2\Delta u_0}{u_0^2}\right) + 2\beta u_0\left|\nabla\left(\frac{\phi}{u_0}\right)\right|^2,$$

so that

$$\Delta w - pa_{\varepsilon_0}u_0^{p-1}w = -\varepsilon''(0)a_- u_0^p + (p-1)\left(\beta + \frac{p}{2}\right)a_{\varepsilon_0}u_0^{p-2}\phi^2 + 2\beta u_0\left|\nabla\left(\frac{\phi}{u_0}\right)\right|^2.$$

We now set $\beta = -p/2$, and use the Fredholm alternative to obtain an expression for $\varepsilon''(0)$:

$$\varepsilon''(0) = -p\frac{\left\langle \Phi, u_0\left|\nabla\left(\frac{\phi}{u_0}\right)\right|^2\right\rangle}{\langle \Phi, a_- u_0^p\rangle}. \tag{6.4}$$

Finally, we observe that the function $v = u_0\left|\nabla\left(\frac{\phi}{u_0}\right)\right|^2 \in Y_{\alpha+2}$, since $u_0, \phi \in X_\alpha$ and $u_0 \geq Cd^{-\alpha}$. Since v is clearly nonnegative it belongs to $P_{Y_{\alpha+2}}$, and (6.2) is a consequence of (6.4) and the strict positivity of Φ.

It clearly follows from (6.2) that there are no solutions in a neighborhood of u_0 if $\varepsilon > \varepsilon_0$ while there are exactly two when $\varepsilon < \varepsilon_0$. This concludes the proof. $\square$

Completion of the proof of Theorem 2. It only remains to prove that the minimal solution is stable, and it is the only stable solution when $\varepsilon > 0$ (this argument is borrowed from [34]). We first remark that, since u_ε is increasing with ε, then it is bounded as $\varepsilon \downarrow 0$. Thus u_ε is also bounded in X_α, and by compactness for every sequence $\varepsilon_n \downarrow 0$ we can extract a subsequence $\{u_n\}$ which converges in X_α. Thus the limit is a solution to (P_0) and by uniqueness it has to be U. It follows that $u_\varepsilon \to U$ as $\varepsilon \downarrow 0$.

According to Lemma 14, we obtain that u_ε is the only solution in a neighborhood of U when $\varepsilon \in (0, \varepsilon_0)$ for some small $\varepsilon_0 > 0$. Lemma 17 also yields $\lambda_1(u_\varepsilon) > 0$ if $\varepsilon \in (0, \varepsilon_0)$, for if $\lambda_1(u_{\bar\varepsilon}) = 0$ for some $\bar\varepsilon \in (0, \varepsilon_0)$, we would have that no solutions exist if $\varepsilon > \bar\varepsilon$ is close to $\bar\varepsilon$ in a neighborhood of $u_{\bar\varepsilon}$.

Let us show that $\lambda_1(u_\varepsilon) > 0$ for every $\varepsilon \in (0, \varepsilon^*)$. We have just shown that this holds for small positive ε. Arguing as in Lemma 14, we can apply the implicit function theorem and produce the branch of minimal solutions to the right as long as $\lambda_1(u_\varepsilon) > 0$. Assume that $\lambda_1(u_{\varepsilon_0}) = 0$ for some $\varepsilon_0 \in (0, \varepsilon^*)$, and choose the first value with this property.

Observe that, by Lemma 17, there exists $\varepsilon_1 \in (\varepsilon_0, \varepsilon^*)$ such that $\lambda_1(u_{\varepsilon_1}) > 0$. We can apply the implicit function theorem to produce the branch of solutions to the left of ε_1. Let us show that if $u = u(\varepsilon)$ is such solution, then $u(\varepsilon)$ is increasing. Indeed, we have $H_\varepsilon(\varepsilon, u(\varepsilon)) + H_u(\varepsilon, u(\varepsilon))u'(\varepsilon) = 0$, which is equivalent to

$$\Delta u'(\varepsilon) = pa_\varepsilon u(\varepsilon)^{p-1}u'(\varepsilon) - a_- u(\varepsilon)^p. \tag{6.5}$$

Notice that (6.5) is equivalent to $u'(\varepsilon) = S'(u'(\varepsilon)) + T(a_- u(\varepsilon)^p)$, where S is as in Remark 5. Now $\lambda_1(u(\varepsilon)) > 0$ implies $r(S'(u'(\varepsilon))) < 1$, so that, since $T(a_- u(\varepsilon)^p) \geq 0$, by Theorem 2.16 in [38] we obtain $u'(\varepsilon) \geq 0$, and the strong maximum principle yields $u'(\varepsilon) > 0$ in Ω. That is, u is increasing in ε. In particular, $u(\varepsilon)$ is bounded for $\varepsilon \leq \varepsilon_1$.

Now we produce this branch to the left until one of the following two options arise: (a) there exists $\varepsilon \in (0, \varepsilon_1)$ such that $\lambda_1(u(\varepsilon)) = 0$; (b) the branch reaches the value $\varepsilon = 0$. The former case is impossible in view of Lemma 17, while the latter implies that $u(\varepsilon)$ coincides with the branch of minimal solutions u_ε, which is also impossible. This contradiction shows that $\lambda_1(u_\varepsilon) > 0$ for every $\varepsilon \in (0, \varepsilon^*)$.

To conclude the proof we only need to show that the minimal solution is the only stable one for $\varepsilon > 0$. The argument is similar as the one used above: if $\lambda_1(u) > 0$ for some solution u then we can produce the branch to the left until we reach $\varepsilon = 0$, so that this branch has to coincide with that of the minimal solutions, which is impossible. This finishes the proof. □

7. The second solution

In this final section we show the existence of a positive solution to (P_ε) when $\varepsilon = \varepsilon^*$ and of a second positive solution when $\varepsilon \in (0, \varepsilon^*)$. The most important feature is the existence of a priori bounds for all possible positive solutions. This is the contents of the next lemma, where we assume the hypotheses on a quoted in the introduction. We use some results from [25], although slightly different assumptions could be placed on the weights and the a priori bounds in [10] or [6] could be invoked instead.

Lemma 18. *Assume $a_\varepsilon = a_+ - \varepsilon a_-$, where a_+, a_- verify* (H). *Then for every $\varepsilon_0 > 0$, there exists a positive constant $M > 0$ such that*

$$\|u\|_{X_\alpha} \leq M$$

for every positive solution u to (P_ε) with $\varepsilon \geq \varepsilon_0$.

Proof. By our hypotheses, we may apply the results in Section 3 of [25] (particularly Lemma 3.4) and obtain that

$$\sup_{\overline{\Omega}_-} u \leq C$$

for every positive solution u to (P_ε) with $\varepsilon \geq \varepsilon_0$. Next, we consider the problem

$$\begin{cases} \Delta w = a_+ w^p & \text{in } \Omega_+ \\ w = \infty & \text{on } \partial\Omega \\ w = C & \text{on } \partial\Omega_+ \setminus \partial\Omega \end{cases} \tag{7.1}$$

which has a unique positive solution W. If u is a positive solution to (P_ε) with $\varepsilon \geq \varepsilon_0$ we have $\Delta u = a_+ u^p$ in Ω_+, together with $u = \infty$ on $\partial\Omega$ and $u \leq C$ on $\partial\Omega_-$, that is, u is a subsolution to (7.1). It follows by uniqueness that $u \leq W$, so that $u \leq Cd^{-\alpha}$ in Ω for some positive constant C, independent of u.

But then $|\Delta u| \le C d^{-\alpha-2}$ and we may use once again Lemma 12 in [28] to obtain the required bound in X_α. This concludes the proof. $\square$

Proof of Theorem 3. Let us first show that a solution to (P_{ε^*}) exists. For this aim let $\varepsilon_n \uparrow \varepsilon^*$. Since the sequence $\{u_{\varepsilon_n}\}$ is bounded in X_α by Lemma 18, we can extract a subsequence such that $u_{\varepsilon_n} \to u$. It easily follows that u is a solution to (P_{ε^*}), which is strictly positive since u_ε is increasing.

Next let us show that there exists a second solution to (P_ε) if $\varepsilon \in (0, \varepsilon^*)$. Fix $\varepsilon_0 \in (0, \varepsilon^*)$. Since every positive solution to (P_ε) with $\varepsilon \ge \varepsilon_0$ verifies $U \le u \le M d^{-\alpha}$ by Lemma 18, all of them lie in the open set

$$\mathcal{U} = \{u \in X_\alpha : \ \delta d^{-\alpha} < u < M d^{-\alpha}\},$$

where δ is small and M large. Using the homotopy invariance of the Leray-Schauder degree, we have for $\varepsilon \in (\varepsilon_0, \varepsilon^*)$ and $\varepsilon_1 > \varepsilon^*$:

$$\deg(I - S(\varepsilon, \cdot), \mathcal{U}, 0) = \deg(I - S(\varepsilon_1, \cdot), \mathcal{U}, 0) = 0,$$

where S is given in Remark 5, since there are no fixed points of $S(\varepsilon_1, \cdot)$ in $\mathcal{U}$.

On the other hand, we have also seen in Remark 5 that $r(\varepsilon, S'(u_\varepsilon)) < 1$, so that u_ε is an isolated fixed point of $S(\varepsilon, \cdot)$ in X_α and we may apply the classical Leray-Schauder formula to obtain that

$$i(I - S(\varepsilon, \cdot), u_\varepsilon) = 1$$

where i denotes the local index. It follows by the excision property of the degree that $\deg(I - S(\varepsilon, \cdot), \mathcal{U} \setminus B_\delta(u_\varepsilon), 0) = -1$ for small $\delta > 0$, so that there exists another fixed point $v(\varepsilon)$ of $S(\varepsilon, \cdot)$ in $\mathcal{U} \setminus B_\delta(u_\varepsilon)$. Observe that $v(\varepsilon) \in \mathcal{U}$ implies that $v(\varepsilon) = \infty$ on $\partial\Omega$, so that $v(\varepsilon)$ is a solution to our original problem (P_ε).

To conclude the proof, only (1.5) remains to be shown. Assume that for some sequence $\varepsilon_n \downarrow 0$, the second solution $v_n := v_{\varepsilon_n}$ verifies

$$\sup_{\overline{\Omega}_-} v_n \le C.$$

Then, by the proof of Lemma 18, we obtain that $\|v_n\|_{X_\alpha}$ is bounded. Passing to a subsequence, we obtain that $v_n \to v$ in X_α, where v is a solution to (P_0). By uniqueness, we have $v = U$, but then Lemma 14 is contradicted, because the only solution in a neighborhood of U for small ε is the minimal solution u_ε. $\square$

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Departamento de Análisis Matemático, Universidad de La Laguna.
C/. Astrofísico Francisco Sánchez s/n, 38271 – La Laguna, SPAIN
and
Instituto Universitario de Estudios Avanzados (IUdEA) en Física Atomica, Molecular y Fotonica, Facultad de Física, Universidad de La Laguna
C/. Astrofísico Francisco Sánchez s/n, 38203 – La Laguna , SPAIN
E-mail address: jjgarmel@ull.es