FUJITA EXPONENTS FOR EVOLUTION PROBLEMS WITH NONLOCAL DIFFUSION

JORGE GARCÍA-MELIÁN AND FERNANDO QUIRÓS

Abstract. We prove the existence of a critical exponent of Fujita type for the nonlocal diffusion problem

$$\begin{cases} u_t(x,t) = J * u(x,t) - u(x,t) + u^p(x,t), & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where J is a compactly supported nonnegative function with unit integral and $p > 1$. Our main result establishes that the Fujita exponent p_F coincides with the classical one when the diffusion is given by the Laplacian. We also deal with reaction terms of nonlocal nature as $J * u^p$ or $(J * u)^p$.

1. Introduction

In the classical paper [26], H. Fujita analyzed the question of global existence in time for positive solutions to certain semilinear parabolic initial value problems posed in $\mathbb{R}^N$. More precisely, he proved that if u_0 is a nonnegative, nontrivial, bounded smooth function in $\mathbb{R}^N$, then the problem

$$\begin{cases} u_t = \Delta u + u^p & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

admits no global solutions when $1 < p < 1 + 2/N$, while it has both bounded global solutions and solutions which blow up in finite time if $p > 1 + 2/N$. When $p = 1 + 2/N$, it was shown later in [30] (for $N = 1, 2$), [35] and [5], that all nontrivial solutions to (1.1) blow up in finite time. The number $1 + 2/N$ is nowadays usually known as the *Fujita exponent* for problem (1.1).

These results have been generalized to deal with some more general situations, where the Laplacian is replaced by a different diffusion operator and/or the reaction term u^p by a nonlinearity $f(u)$. We refer the interested reader to the surveys [36] and [22].

In the present paper we intend to investigate the existence of a Fujita exponent for initial value problems of nonlocal nature. This question has already been addressed when the diffusion operator is the fractional Laplacian in [42] (see also [8] for a probabilistic approach) and when the reaction term is of integral type in [27]. Here we will devote our study to a nonlocal diffusion operator different from the fractional Laplacian, with reaction terms which are both local and nonlocal.

We will firstly be concerned with problem (1.1) when the Laplacian is substituted by a term involving a convolution, namely

$$\begin{cases} u_t = J * u - u + u^p & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.2}$$

where $*$ stands for the usual convolution, with a kernel J that is assumed to be a nonnegative continuous function with unit integral. Evolution problems with this type of diffusion have been widely considered in the literature, since they can be used to model the dispersal of a species by taking into account long-range effects, [25]. In fact, the linear homogeneous problem

$$\begin{cases} u_t = J * u - u & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.3}$$

has been the subject of many recent works, not only in $\mathbb{R}^N$ but also when it is posed in a smooth bounded domain Ω and it is complemented by a suitable Dirichlet or Neumann type boundary condition. We quote for instance [11], [16], [17], [31], [32] and [33]. Nonlinear problems related to (1.3) have also been largely studied: see [7], [9], [12], [20], [21], [41] for the question of traveling waves, [18], [19], [29] for logistic type nonlinearities and [1], [2], [3], [4], [6], [10], [13], [14], [15], for more general nonlocal diffusion operators.

As an outstanding difference with respect to the fractional Laplacian case, it was assumed in most of the previous works that J has compact support. This makes problem (1.2) resemble problem (1.1) a little bit. Indeed, we will prove that the Fujita exponent for (1.2) is given by

$$p_F = 1 + \frac{2}{N}, \tag{1.4}$$

which coincides with the one for problem (1.1).

A word on the notion of solution: by a local solution to (1.2) we mean a function $u \in C^1((0,T), L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)) \cap C([0,T), L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$ for some $T > 0$, which verifies the equation a. e. in $\mathbb{R}^N \times (0,T)$ and takes the initial condition in the usual way. We remark that problem (1.2) is well posed in $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ (see precise statements in Section 3).

Let us next state our results for problem (1.2). Throughout the paper we will always assume that $J \in C(\mathbb{R}^N)$ is compactly supported, radially symmetric, radially decreasing and has unit integral. With no loss of generality we take the support of J to be the unit ball.

Theorem 1. *Let p_F be given by* (1.4).

(a) Assume $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is nonnegative and nontrivial and $1 < p \le p_F$. Then the solution to the Cauchy problem (1.2) *blows up in finite time.*

(b) If $p > p_F$, then there exist nonnegative initial data $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that problem (1.2) *admits global solutions, which are in addition bounded in $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and converge uniformly to zero as $t \to \infty$.*

Remarks 1. (a) As for problem (1.1), there exist positive solutions to (1.2) which blow up in finite time even when $p > p_F$. This is shown as in part (a) of Theorem 1 (see Section 4).

(b) The solutions constructed in part (b) of Theorem 1 decay to zero at the same rate as the solutions to the homogeneous problem (1.3), that is, there exists a positive constant C such that

$$|u(\cdot,t)|_{L^\infty} \le C(1+t)^{-N/2}$$

for all $t > 0$.

We would like to mention that, though the semigroup associated to problem (1.3) shares its asymptotic behavior with that of the heat equation, many features of the former are different from that of the latter. For instance, the nonlocal equation has no regularizing effect, and this translates into a fundamental solution which is highly singular and contains a Dirac delta (see [11]). Therefore, many of the available proofs for problem (1.1) can not work here; indeed, many of them rely even in the explicit form of the heat kernel. Some other proofs make use of self-similar solutions for problem (1.3), which in the present situation can not exist, since the scaling invariance in the operator is lost due to the convolution. Thus, our proofs will be based in different arguments.

Two important previous results are needed for our proofs to work: the first one is the precise behavior of the principal eigenvalue of the operator $J * u - u$ in a ball of radius R as $R \to \infty$, which has been obtained in [28]. The second one is the asymptotic behavior of the solutions to (1.3) for large values of t, which coincides with that for the heat equation, and was considered in [11]. For completeness, they will be recalled in Sections 2 and 3, respectively.

We will next consider a modification of problem (1.2). Notice that if it is thought of as describing the evolution of a species with nonlocal diffusion effects, then it makes sense to admit that the reaction effects can also be nonlocal. Thus, it is natural to consider a version on (1.2) in which the local term u^p is replaced by a nonlocal one which also has a power growth. There are, of course, many ways of doing this. However, we will restrict ourselves to two possible "nonlocal reaction" versions of (1.2), namely

$$u_t = J * u - u + J * u^p \quad \text{in } \mathbb{R}^N \times (0,\infty), \tag{1.5}$$

and

$$u_t = J * u - u + (J * u)^p \quad \text{in } \mathbb{R}^N \times (0,\infty), \tag{1.6}$$

where $p > 1$.

At first glance, it could seem that the presence of the nonlocal reaction terms in problems (1.5) and (1.6) –which are in some sense stronger than the local one– could force the solutions to blow-up more easily, but we will prove that in the present case the Fujita exponent is exactly the same as for (1.2), given by (1.4).

Theorem 2. *Under the same hypotheses on J, Theorem 1 also holds for problems* (1.5) *and* (1.6).

Remark 2. A slight generalization of problems (1.5) and (1.6) can be treated by means of the same methods. For instance, the convolution in the right hand side can be made against a different kernel G, as long as it verifies $G \geq cJ$ for some positive constant c. Notice that this condition is met in particular when $G > 0$ in $\mathbb{R}^N$ or more generally when the support of G contains the unit ball.

The rest of the paper is organized as follows: in Section 2 we gather some preliminaries on the eigenvalue problem associated to the operator $J * u - u$. Section 3 is dedicated to recall the asymptotic behavior of solutions to (1.3), and to establish the basic questions of local existence and comparison of nonnegative solutions for problems (1.2), (1.5) and (1.6). Finally, the proofs of Theorems 1 and 2 are carried out in Section 4.

2. Preliminaries on the eigenvalue problem

This section is dedicated to collect some properties of the principal eigenvalue –that is, the eigenvalue associated to a positive eigenfunction– of the problem

$$\begin{cases} J * u - u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.1}$$

where J is as in the introduction and Ω is a smooth bounded domain of $\mathbb{R}^N$. The most important property we will use in Section 4 is the limit behavior of the principal eigenvalue in the dilated domain $\gamma\Omega$ when $\gamma \to \infty$. This problem was dealt with in [28], where some other features were also considered.

The next result is part of Theorems 2.1 and 1.5 in [28], where we refer to for proofs.

Theorem 3. *Problem* (2.1) *admits an eigenvalue $\lambda_1(\Omega)$ associated to a positive eigenfunction $\phi \in C(\overline{\Omega})$. Moreover, it is simple and unique, and it verifies $0 < \lambda_1(\Omega) < 1$. Furthermore, $\lambda_1(\Omega)$ can be variationally characterized as*

$$\lambda_1(\Omega) = 1 - \sup_{\substack{u\in L^2(\Omega)\\ u\neq 0}} \frac{\displaystyle\int_\Omega\int_\Omega J(x-y)u(x)u(y)\,dy\,dx}{\displaystyle\int_\Omega u^2(x)\,dx}. \tag{2.2}$$

Finally, if we denote $\Omega_\gamma = \gamma\Omega$ for positive γ, then $\gamma^2\lambda_1(\Omega_\gamma) \to A(J)\sigma_1(\Omega)$ as $\gamma \to \infty$, where

$$A(J) = \frac{1}{2N}\int_{\mathbb{R}^N} J(z)|z|^2 dz, \tag{2.3}$$

and $\sigma_1(\Omega)$ is the principal eigenvalue of $-\Delta$ in Ω with homogeneous Dirichlet boundary conditions. In addition, if ϕ_γ stands for the positive eigenfunction associated to $\lambda_1(\Omega_\gamma)$ with the normalization $|\phi_\gamma|_{L^1(\Omega_\gamma)} = 1$ and we set $\psi_\gamma(x) = \gamma^N\phi_\gamma(\gamma x)$, then $\psi_\gamma \to \Phi$ in $L^2(\Omega)$ as $\gamma \to \infty$, where Φ is the positive eigenfunction of $-\Delta$ with $|\Phi|_{L^1(\Omega)} = 1$.

The radial symmetry imposed on J implies that problem (2.1) in a ball B_R is invariant with respect to rotations about the origin. Thus the simplicity of the principal eigenvalue shows that all eigenfunctions are radial. However, for its use in the proofs of Section 4, we need a stronger property of the eigenfunctions: they are radially decreasing. We remark though, that the radial symmetry of the eigenfunction does not simplify the equation in (2.1) (as usually happens with elliptic pde's with radial symmetry), and thus this property is not expected to be obtained this way. Therefore, we proceed differently.

Theorem 4. *Let B_R be the ball of radius R of $\mathbb{R}^N$ centered at the origin and ϕ a positive eigenfunction associated to $\lambda_1(B_R)$. Then ϕ is radially symmetric and radially nonincreasing.*

Proof. The proof is based on the properties of symmetric decreasing rearrangements (we refer to Chapter 3 in [37] for these properties). Let $\phi \in C(\overline{B_R})$ be a positive eigenfunction, and denote by ϕ^* its symmetric decreasing rearrangement. Since $|\phi^*|_{L^2(B_R)} = |\phi|_{L^2(B_R)}$ and, according to Riesz inequality

$$\begin{aligned}\int_{B_R}\int_{B_R} J(x-y)\phi(x)\phi(y)\,dy\,dx &\le \int_{B_R}\int_{B_R} J^*(x-y)\phi^*(x)\phi^*(y)\,dy\,dx\\ &= \int_{B_R}\int_{B_R} J(x-y)\phi^*(x)\phi^*(y)\,dy\,dx,\end{aligned}$$

we have, thanks to the variational characterization (2.2), that

$$1-\lambda_1(B_R) = \frac{\displaystyle\int_{B_R}\int_{B_R} J(x-y)\phi^*(x)\phi^*(y)\,dy\,dx}{\displaystyle\int_{B_R}\phi^{*2}(x)\,dx},$$

whence ϕ^* is an eigenfunction. The simplicity of $\lambda_1(B_R)$ implies $\phi = \phi^*$, and the theorem follows. □

3. The Cauchy problem

Let us begin by observing that the operator $Au = J * u - u$ is bounded in $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Thus, it generates a uniformly continuous semigroup of bounded operators. Hence the Cauchy problem (1.3) has a unique solution for every $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, which is given by $e^{At}u_0$, [24]. It turns out that, under appropriate conditions on the initial datum u_0, the asymptotic behavior of this solution as $t \to \infty$ is exactly the same as the behavior of the solution of the heat equation with the same initial data. This phenomenon was analyzed in [11], from which we quote the following important result. It is worth noticing that the dependence on the initial datum of the constant in inequality (3.1) below is not explicitly stated in Theorem 1 there. However, it can be checked by means of a careful analysis of its proof.

Theorem 5. *Let $u_0 \in L^1(\mathbb{R}^N)$ be such that its Fourier transform $\hat{u}_0$ also belongs to $L^1(\mathbb{R}^N)$. Then there exists a positive constant C such that the solution u to (1.3) verifies*

$$|u(\cdot,t)|_{L^\infty} \le C(|u_0|_{L^1} + |\hat{u}_0|_{L^1})(1+t)^{-\frac{N}{2}} \quad \text{for all } t \ge 0. \tag{3.1}$$

Next, let us state a result which implies that problem (1.2) is well-posed in $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Most part of the proof is standard, based on contraction and continuation arguments, so we simply omit it (for a related situation see Theorem 1.1 in [39]).

Theorem 6. *Let $p > 1$ and $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $u_0 \ge 0$. Then problem (1.2) admits a unique solution u which is defined in a maximal interval of time $[0,T)$. Moreover, if $T < \infty$, then $|u(\cdot,t)|_{L^\infty}$ blows up as $t \to T-$.*

Proof. We only show that last assertion, the other ones being standard. Notice that when $T < \infty$ then the norm $|u(\cdot,t)|_{L^1} + |u(\cdot,t)|_{L^\infty}$ must blow up as $t \to T-$. Thus, assume that $|u(\cdot,t)|_{L^\infty}$ remains bounded as $t \to T-$. If we integrate the equation in $\mathbb{R}^N$:

$$\left(\int_{\mathbb{R}^N} u\right)_t = \int_{\mathbb{R}^N} u_t = \int_{\mathbb{R}^N} u^p \le |u(\cdot,t)|_{L^\infty}^{p-1} \int_{\mathbb{R}^N} u,$$

and it follows that $|u(\cdot,t)|_{L^1}$ is bounded as $t \to T-$. Thus $|u(\cdot,t)|_{L^1} + |u(\cdot,t)|_{L^\infty}$ can not blow up, a contradiction. □

We finally consider a comparison principle for positive solutions to (1.2) which will be a fundamental tool in order to prove existence of global solutions (we refer to [23] and Appendix F in [40] for related comparison theorems for nonlocal problems).

Theorem 7. *Let $u, v \in C^1((0,T), L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)) \cap C([0,T), L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$ be nonnegative functions verifying*

$$u_t \le J * u - u + u^p, \quad v_t \ge J * v - v + v^p \quad \text{in } \mathbb{R}^N \times (0,T)$$

for some $T > 0$, with $u(\cdot,0) \le v(\cdot,0)$ in $\mathbb{R}^N$. Then $u \le v$ in $\mathbb{R}^N \times [0,T)$.

Proof. Let $w = u - v$ and choose $\tilde{T} < T$. In $\mathbb{R}^N \times (0,\tilde{T})$ we have

$$w_t \le J * w - w + a(x,t)w \tag{3.2}$$

for some function $a(x,t)$ which is bounded in $\mathbb{R}^N \times [0,\tilde{T}]$. We now notice that, by interpolation, we have $w(\cdot,t) \in L^2(\mathbb{R}^N)$ for all $t \in [0,\tilde{T}]$. Thus, if we multiply (3.2) by $w^+ = \max\{w,0\}$, integrate in $\mathbb{R}^N$ and use Fubini's theorem, we have

$$\left(\int_{\mathbb{R}^N} w^+(x,t)^2 dx\right)_t \le C \int_{\mathbb{R}^N} w^+(x,t)^2 dx. \tag{3.3}$$

Since $w^+(\cdot,0)=0$, we arrive to $w^+ \equiv 0$ in $\mathbb{R}^N \times [0,\tilde{T}]$. Noticing that $\tilde{T} < T$ is arbitrary, we finally have $u \le v$ in $\mathbb{R}^N \times [0,T)$. □

Remarks 3. (a) With a quite similar argument it is shown that the comparison principle also holds for the homogeneous problem (1.3). In particular, if $u_0 \ge 0$, then $e^{At}u_0 \ge 0$.

(b) Similar conclusions hold for problems (1.5) and (1.6). The proofs of the last statement in Theorem 6 and of Theorem 7 are only minor modifications of the ones just given. For instance in the proof of the comparison principle for problem (1.6) we would have instead of (3.3):

$$\left(\int_{\mathbb{R}^N} w^+(x,t)^2 dx\right)_t \le C \int_{\mathbb{R}^N} w^+(x,t)(J * w^+)(x,t) dx.$$

Using Hölder's inequality in the last integral and then Jensen's inequality and Fubini's theorem, we would obtain

$$\begin{aligned}\int_{\mathbb{R}^N} w^+(x,t)(J * w^+)(x,t) dx &\le \left(\int_{\mathbb{R}^N} w^+(x,t)^2 dx\right)^{1/2} \left(\int_{\mathbb{R}^N} (J * w^+(x,t))^2 dx\right)^{1/2} \\ &\le \left(\int_{\mathbb{R}^N} w^+(x,t)^2 dx\right)^{1/2} \left(\int_{\mathbb{R}^N} J * (w^+)^2(x,t) dx\right)^{1/2} \\ &= \int_{\mathbb{R}^N} w^+(x,t)^2 dx,\end{aligned}$$

and the proof concludes as before.

4. Fujita exponents

This section is dedicated to the proof of Theorems 1 and 2. We will give in full detail that of Theorem 1. But before proceeding further, we give an $L^p(\mathbb{R}^N \times [0,\infty))$ bound for global solutions to (1.2) and (1.5).

Lemma 8. *Let u be a nonnegative global solution to problem* (1.2) *or* (1.5)*, and assume that there is a constant C such that $|u(\cdot,t)|_{L^1} \le C$ for all $t \ge 0$. Then*

$$\int_0^\infty \int_{\mathbb{R}^N} u(x,t)^p \, dx \, dt < \infty.$$

Proof. Assume first that u is a solution to (1.5). Integrate the equation in (1.5) in $\mathbb{R}^N$ and apply Fubini's theorem to get

$$\left(\int_{\mathbb{R}^N} u\right)_t = \int_{\mathbb{R}^N} J * u^p = \int_{\mathbb{R}^N} u^p. \tag{4.1}$$

Integrating in $(0,t)$ for arbitrary t we have

$$\int_{\mathbb{R}^N} u(\cdot,t) - \int_{\mathbb{R}^N} u_0 = \int_0^t \int_{R^N} u^p.$$

Thanks to the assumption of boundedness in $L^1(\mathbb{R}^N)$, we may let $t \to \infty$ to get the lemma proved for solutions to (1.5). The proof for solutions to (1.2) is the same, except that the middle integral in (4.1) does not appear. $\square$

Proof of Theorem 1. (a) This proof is inspired in an argument of Kaplan, [34]. Choose $R > 0$ and let ϕ_R be the positive eigenfunction of (2.1) associated to $\lambda_1(B_R)$ normalized by $|\phi_R|_{L^1(B_R)} = 1$. Multiplying the equation (1.2) by ϕ_R and integrating in B_R we have

$$\left(\int_{B_R} u\phi_R\right)_t = \int_{B_R} u_t\phi_R = \int_{B_R} (J*u-u)\phi_R + \int_{B_R} u^p\phi_R. \tag{4.2}$$

For the term with the convolution in the right-hand side of (4.2) we obtain, by means of Fubini's theorem, and using the symmetry of J:

$$\begin{aligned}\int_{B_R}(J*u-u)\phi_R &= \int_{B_R}\left(\int_{\mathbb{R}^N} J(x-y)u(y)dy - u(x)\right)\phi_R(x)dx \\ &= \int_{\mathbb{R}^N}\left(\int_{B_R} J(x-y)\phi_R(x)dx\right)u(y)dy - \int_{B_R} u(x)\phi_R(x)dx \\ &\geq \int_{B_R}\left(\int_{B_R} J(x-y)\phi_R(x)dx\right)u(y)dy - \int_{B_R} u(x)\phi_R(x)dx \\ &= -\lambda_1(B_R)\int_{B_R} u(x)\phi_R(x)dx,\end{aligned}$$

while for the second term in the right-hand side of (4.2) we may apply Jensen's inequality to arrive at

$$\left(\int_{B_R} u\phi_R\right)_t \geq -\lambda_1(B_R)\int_{B_R} u\phi_R + \left(\int_{B_R} u\phi_R\right)^p.$$

It follows that $\int_{B_R} u\phi_R$ blows up at a finite time (and hence blow up occurs in L^1_{loc} and in L^∞) provided that

$$\int_{B_R} u(x,t_0)\phi_R(x)dx \geq (\lambda_1(B_R))^{\frac{1}{p-1}} \tag{4.3}$$

at some $t_0 \geq 0$. Let us prove that this condition always holds for $t_0 = 0$ and every initial datum $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ provided $1 < p < p_F$.

Set $\psi_R(x) = R^N\phi_R(Rx)$ for $x \in B := B_1$. Then condition (4.3) with $t_0 = 0$ is equivalent to

$$\int_{B_R} u_0(x)\psi_R\left(\frac{x}{R}\right)dx \geq R^{N-\frac{2}{p-1}}(R^2\lambda_1(B_R))^{\frac{1}{p-1}}. \tag{4.4}$$

Now choose $R_0 > 0$ and observe that Theorem 4 implies that ψ_R is radially decreasing. Hence, for $R > R_0$ we find that

$$\int_{B_R} u_0(x)\psi_R\left(\frac{x}{R}\right)dx \geq \int_{B_{R_0}} u_0(x)\psi_R\left(\frac{x}{R_0}\right)dx,$$

so that, letting first $R \to \infty$, using Theorem 3, and then letting $R_0 \to \infty$, we arrive at

$$\liminf_{R\to\infty}\int_{B_R} u_0(x)\psi_R\left(\frac{x}{R}\right)dx \geq \Phi(0)\int_{\mathbb{R}^N} u_0(x)dx > 0.$$

On the other hand, if $p < p_F$, and using again Theorem 3, we arrive at

$$\lim_{R\to\infty} R^{N-\frac{2}{p-1}}(R^2\lambda_1(B_R))^{\frac{1}{p-1}} = 0.$$

Thus (4.4) always holds for large R, and hence blow-up occurs at finite time for every nonnegative nontrivial initial condition $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Remark 4. Observe that the above reasoning fails when $p = p_F$. Nevertheless, it shows that global solutions u to (1.2) are uniformly bounded in $L^1(\mathbb{R}^N)$. Indeed, they cannot verify (4.4) and, thus, for fixed R_0 and $R > R_0$ we have

$$\int_{B_{R_0}} u(x,t)\psi_R\left(\frac{x}{R_0}\right)dx \le \int_{B_R} u(x,t)\psi_R\left(\frac{x}{R}\right)dx \le (R^2\lambda_1(B_R))^{\frac{1}{p-1}}$$

for every $t \ge 0$. Letting $R\to\infty$ and then $R_0\to\infty$ and using Theorem 3 once again, we have

$$\int_{\mathbb{R}^N} u(x,t)dx \le \frac{(A(J)\sigma_1(\Omega))^{\frac{1}{p-1}}}{\Phi(0)}, \tag{4.5}$$

which provides with the desired L^1 bound.

To conclude the proof of (a), we finally show that nonnegative global solutions to (1.2) (aside the trivial one) do not exist when $p = p_F$. We use a modification of arguments in [38]. For this aim we choose functions $\xi \in C_0^\infty(B)$, $\psi \in C^\infty(-1,1)$ verifying $0 \le \xi, \psi \le 1$, $\xi = 1$ in $B_{1/2}$, $\psi = 1$ in $[0,1/2)$, and set

$$\xi_R(x) = \xi\left(\frac{x}{R}\right), \quad \psi_R(t) = \psi\left(\frac{t-t_0}{R^2}\right),$$

where $t_0 > 0$ is arbitrary, but fixed. Multiplying the equation in (1.2) by $\xi_R(x)\psi_R(t)$ and integrating in $[t_0,\infty)$ we get

$$\begin{aligned}\int_{t_0}^\infty\int_{\mathbb{R}^N} u_t(x,t)\xi_R(x)\psi_R(t)dxdt = &\int_{t_0}^\infty\int_{\mathbb{R}^N}(J*u-u)(x,t)\xi_R(x)\psi_R(t)dxdt\\ &+\int_{t_0}^\infty\int_{\mathbb{R}^N} u(x,t)^p\xi_R(x)\psi_R(t)dxdt.\end{aligned}\tag{4.6}$$

Integrating by parts with respect to time in the integral in the left-hand side of (4.6) we obtain

$$\int_{t_0}^\infty\int_{\mathbb{R}^N} u_t(x,t)\xi_R(x)\psi_R(t)dxdt \le -\int_{t_0}^\infty\int_{\mathbb{R}^N} u(x,t)\xi_R(x)\psi_R'(t)dxdt,$$

so that, applying Fubini's theorem in the first integral in the right-hand side of (4.6), we arrive at

$$\begin{aligned}\int_{t_0}^\infty\int_{\mathbb{R}^N} u(x,t)^p\xi_R(x)\psi_R(t)dxdt \le &-\int_{t_0}^\infty\int_{\mathbb{R}^N} u(x,t)\xi_R(x)\psi_R'(t)dxdt\\ &-\int_{t_0}^\infty\int_{\mathbb{R}^N}(J*\xi_R-\xi_R)(x,t)u(x,t)\psi_R(t)dxdt.\end{aligned}\tag{4.7}$$

We notice next that

$$|\psi_R'(t)| = \left|\frac{1}{R^2}\psi'\left(\frac{t-t_0}{R^2}\right)\right| \le \frac{C}{R^2}\chi_{\{t_0+\frac{1}{2}R^2,t_0+R^2\}}, \qquad t \ge t_0, \tag{4.8}$$

where we denote by χ_A the characteristic function of a set A (here and in what follows, the letter C will denote a positive constant, not necessarily the same everywhere). Moreover,

$$(J*\xi_R-\xi_R)(x)=\int_{\mathbb{R}^N}J(z)\left(\xi\left(\frac{x+z}{R}\right)-\xi\left(\frac{x}{R}\right)\right)dz, \tag{4.9}$$

and thanks to Taylor's theorem, since ξ is smooth,

$$\xi\left(\frac{x+z}{R}\right)-\xi\left(\frac{x}{R}\right)=\frac{1}{R}\sum_{i=1}^{N}z_i\frac{\partial\xi}{\partial x_i}\left(\frac{x}{R}\right)+\frac{1}{2R^2}\sum_{i,j=1}^{N}z_iz_j\frac{\partial^2\xi}{\partial x_ix_j}\left(\frac{x}{R}\right)+O\left(\frac{1}{R^3}\right),$$

where $O(1/R^3)$ is a function which vanishes in $|x|<R/2$ and in $|x|>R$. Now observe that, thanks to the symmetry of J we obtain

$$\int_{\mathbb{R}^N}J(z)z_idz=0,\qquad\int_{\mathbb{R}^N}J(z)z_iz_j=2A(J)\delta_{ij}$$

where $A(J)$ is as in (2.3). Thus, from (4.9), we have

$$|(J*\xi_R-\xi_R)(x)|=\left|\frac{A(J)}{R^2}\Delta\xi\left(\frac{x}{R}\right)+O\left(\frac{1}{R^3}\right)\right|\le\frac{C}{R^2}\chi_{\{\frac12R<|x|<R\}}. \tag{4.10}$$

Plugging (4.8) and (4.10) into (4.7) we obtain the inequality

$$\int_{t_0}^{\infty}\int_{\mathbb{R}^N}u(x,t)^p\xi_R(x)\psi_R(t)dxdt\le\frac{C}{R^2}\left(\int_{t_0+\frac12R^2}^{t_0+R^2}\int_{|x|<R}u(x,t)dxdt+\int_{t_0}^{t_0+R^2}\int_{\frac12R<|x|<R}u(x,t)dxdt\right).$$

Making use of Hölder's inequality and taking into account that $p=p_F$, we obtain

$$\int_{t_0}^{\infty}\int_{\mathbb{R}^N}u(x,t)^p\xi_R(x)\psi_R(t)dxdt\le C\left[\left(\int_{t_0+\frac12R^2}^{t_0+R^2}\int_{|x|<R}u(x,t)^pdxdt\right)^{1/p}+\left(\int_{t_0}^{t_0+R^2}\int_{\frac12R<|x|<R}u(x,t)^pdxdt\right)^{1/p}\right]. \tag{4.11}$$

It now follows from Remark 4 and Lemma 8, after letting $R\to\infty$, that

$$\int_{t_0}^{\infty}\int_{\mathbb{R}^N}u(x,t)^pdxdt=0,$$

and because t_0 was arbitrary we arrive at $u\equiv0$ in $\mathbb{R}^N\times(0,\infty)$. This shows that no nontrivial nonnegative global solutions exist in this case.

(b) We will show the existence of positive global solutions to (1.2) for some initial datum $u_0\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ by constructing a supersolution in the spirit of [43].

Choose $u_0\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ such that its Fourier transform $\hat{u}_0$ belongs to $L^1(\mathbb{R}^N)$, and let $\delta=|u_0|_{L^1}+|\hat{u}_0|_{L^1}$. The supersolution will take the form

$$\bar{u}(x,t)=h(t)v(x,t) \tag{4.12}$$

where $v = e^{At}u_0$ is the unique solution to (1.3), as in Section 3, and h is to be chosen. It is not hard to check that $\bar{u}$ will be a supersolution provided that

$$\frac{h'(t)}{h(t)^p} \geq |v(\cdot,t)|_\infty^{p-1}. \tag{4.13}$$

According to (3.1) in Theorem 5, for (4.13) to hold it is sufficient that

$$\frac{h'(t)}{h(t)^p} \geq C\delta(1+t)^{-\frac{N(p-1)}{2}}.$$

Thus a good choice for the function h is

$$h(t) = \left(1 - \frac{C\delta(p-1)}{\alpha}\left(1-(1+t)^{-\alpha}\right)\right)^{-\frac{1}{p-1}},$$

where $\alpha = N(p-1)/2 - 1 > 0$, provided that δ is small enough. Since $\bar{u}(\cdot,0) = u_0$, Theorem 7 shows that the solution to (1.2) is global and bounded. □

We observe that, since the proof of Theorem 2 is similar to that of Theorem 1, we will not give the full details of the former, and will only sketch the main differences with respect to the latter.

Proof of Theorem 2. Let us start by noticing that the proof of part (a) remains valid, if we take into account that, thanks to Jensen's inequality applied twice and Fubini's theorem

$$\begin{aligned}\int_{B_R}(J*u^p)\phi_R \geq \int_{B_R}(J*u)^p\phi_R &\geq \left(\int_{B_R}(J*u)\phi_R\right)^p \\ &= \left(\int_{B_R}(J*\phi_R)u\right)^p = (1-\lambda_1(B_R))^p\left(\int_{B_R}\phi_R u\right)^p,\end{aligned}$$

so that

$$\left(\int_{B_R}u\phi_R\right)_t \geq -\lambda_1(B_R)\int_{B_R}u\phi_R + (1-\lambda_1(B_R))^p\left(\int_{B_R}u\phi_R\right)^p.$$

If $1<p<p_F$, the rest of the proof goes as before.

Now consider the case $p=p_F$. It is not hard to check that, if u is a global solution to (1.6), then $v = J*u$ is a global solution to (1.5) and $|u(\cdot,t)|_{L^1} = |v(\cdot,t)|_{L^1}$ for all $t>0$. Thus it suffices to show that problem (1.5) has no global solutions.

As in Remark 4, we obtain an L^1 bound for global solutions to (1.5), and then Lemma 8 shows that $u \in L^p(\mathbb{R}^N \times (0,\infty))$. With exactly the same reasoning which led to (4.11) we obtain

$$\begin{aligned}\int_{t_0}^\infty\int_{\mathbb{R}^N} J*u(x,t)^p\xi_R(x)\psi_R(t)\,dxdt \leq\ & C\left[\left(\int_{t_0+\frac{1}{2}R^2}^{t_0+R^2}\int_{|x|<R}u(x,t)^p\,dxdt\right)^{1/p}\right. \\ &\left.+\left(\int_{t_0}^{t_0+R^2}\int_{\frac{1}{2}R<|x|<R}u(x,t)^p\,dxdt\right)^{1/p}\right],\end{aligned}$$

and we can let $R\to\infty$ and then $t_0\to 0$ to have $u\equiv 0$ in $\mathbb{R}^N\times(0,\infty)$.

Part (b) follows by constructing again a supersolution. We recall that if u is a nonnegative global solution to (1.6), then $J*u$ is a nonnegative global solution to (1.5), thus it suffices with showing that problem (1.6) admits global solutions for some choices of initial data. However, the introduction of a nonlocal term as a reaction implies that the choice of the initial datum has to be very particular. We first set

$$u_0(x) = \delta \frac{1}{(1+|x|)^\theta}$$

for small δ and $\theta > N$. Then $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. It is not hard to check that $J * u_0(x) \le 2^\theta u_0(x)$ for all $x \in \mathbb{R}^N$. Moreover, if $v = e^{At}u_0$, that is, the solution to the linear problem (1.3), then we have that $2^\theta v - J * v$ is the solution to the same problem with initial datum $2^\theta u_0 - J * u_0 \ge 0$. Thus, by the comparison principle (see Remarks 3 (a)), $2^\theta v - J * v \ge 0$, i. e. $J * v \le 2^\theta v$ in $\mathbb{R}^N \times (0, \infty)$.

For this particular initial datum, we choose a supersolution in the form (4.12). Then $\bar{u}$ will be a supersolution if

$$h'(t)v(x,t) \ge h(t)^p (J * v(x,t))^p,$$

which will hold when

$$\frac{h'(t)}{h(t)^p} \ge 2^{\theta p} |v(\cdot,t)|_{L^\infty}^{p-1}.$$

Now the rest of the proof goes as in Theorem 1. □

Acknowledgements. Supported by Spanish Ministerio de Ciencia e Innovación and FEDER under grants MTM2008-05824 and MTM2008-06326-C02-01, and by ESF Programme "Global and geometric aspects of nonlinear partial differential equations".

References

[1] F. Andreu, J. M. Mazon, J.D. Rossi, J. Toledo, *The Neumann problem for nonlocal nonlinear diffusion equations*, J. Evol. Eqns. **8** (2008), 189–215.

[2] F. Andreu, J. M. Mazon, J. D. Rossi, J. Toledo, *A nonlocal p-Laplacian evolution equation with Neumann boundary conditions*, J. Math. Pures Appl. **90** (2008), 201–227.

[3] F. Andreu, J. M. Mazon, J. D. Rossi, J. Toledo, *A nonlocal p−Laplacian evolution equation with non homogeneous Dirichlet boundary conditions*, SIAM J. Math. Anal. **40** (2009), 1815–1851.

[4] F. Andreu, J. M. Mazon, J. D. Rossi, J. Toledo, *The limit as $p \to \infty$ in a nonlocal p−Laplacian evolution equation. A nonlocal approximation of a model for sandpiles*, Calc. Var. P. D. E. **35** (2009), 279–316.

[5] D. G. Aronson, H. F. Weinberger, *Multidimensional nonlinear diffusion arising in population genetics*, Adv. Math. **30** (1978), 33–76.

[6] G. Barles, E. Chasseigne, C. Imbert, *On the Dirichlet problem for second-order elliptic integro-differential equations*, Indiana Univ. Math. J. **57** (2008), 213–246.

[7] P. Bates, P. Fife, X. Ren, X. Wang, *Traveling waves in a convolution model for phase transitions*, Arch. Rat. Mech. Anal. **138** (1997), 105–136.

[8] M. Birkner, J. A. López-Mimbela, A. Wakolbinger, *Blow-up of semilinear PDE's at the critical dimension. A probabilistic approach*, Proc. Amer. Math. Soc. **130** (8) (2002), 2431–2442.

[9] J. Carr, A. Chmaj, *Uniqueness of traveling waves for nonlocal monostable equations*, Proc. Amer. Math. Soc. **132** (2004), 2433–2439.

[10] E. Chasseigne, *The Dirichlet problem for some nonlocal diffusion equations*, Diff. Int. Eqns. **20** (2007), 1389– 1404.

[11] E. Chasseigne, M. Chaves, J.D. Rossi, *Asymptotic behavior for nonlocal diffusion equations*, J. Math. Pures Appl. **86** (2006), 271–291.

[12] X. Chen, *Existence, uniqueness and asymptotic stability of traveling waves in nonlocal evolution equations*, Adv. Diff. Eqns. **2** (1997), 125–160.

[13] C. Cortázar, J. Coville, M. Elgueta, S. Martínez, *A non local inhomogeneous dispersal process*, J. Diff. Eqns. **241** (2007), 332–358.

[14] C. Cortázar, M. Elgueta, J. García-Melián, S. Martínez, *Existence and asymptotic behavior of solutions to some inhomogeneous nonlocal diffusion problems*, to appear in SIAM J. Math. Anal.

[15] C. Cortázar, M. Elgueta, J.D. Rossi, *A nonlocal diffusion equation whose solutions develop a free boundary*, Ann. Henri Poincaré **6** (2005), 269–281.

[16] C. Cortázar, M. Elgueta, J.D. Rossi, N. Wolanski, *Boundary fluxes for nonlocal diffusion*, J. Diff. Eqns. **234** (2007), 360–390.

[17] C. Cortázar, M. Elgueta, J.D. Rossi, N. Wolanski, *How to approximate the heat equation with Neumann boundary conditions by nonlocal diffusion problems*, Arch. Rat. Mech. Anal. **187** (2008), 137–156.

[18] J. Coville, J. Dávila, S. Martínez, *Existence and uniqueness of solutions to a nonlocal equation with monostable nonlinearity*, SIAM J. Math. Anal. **39** (2008), 1693–1709.

[19] J. Coville, J. Dávila, S. Martínez, *Non-local anisotropic dispersal with monostable nonlinearity.* J. Diff. Eqns. **244** (2008), no. 12, 3080–3118.

[20] J. Coville, L. Dupaigne, *Propagation speed of travelling fronts in nonlocal reaction diffusion equations*, Nonl. Anal. **60** (2005), 797–819.

[21] J. Coville, L. Dupaigne, *On a nonlocal equation arising in population dynamics*, Proc. Roy. Soc. Edinburgh **137** (2007), 1–29.

[22] K. Deng, H. A. Levine, *The role of critical exponents in blow-up theorems: the sequel*, J. Math. Anal. Appl. **243** (2000), 85–126.

[23] K. Deng, M. K. Kwong, H. A. Levine, *The influence of nonlocal nonlinearities on the long-time behavior of solutions of Burgers' equation*, Quart. Appl. Math. **50** (1992), 173–200.

[24] K. J. Engel, R. Nagel, "One-parameter semigroups for linear evolution equations", Graduate Texts in Mathematics 194, Springer-Verlag, New York, 2000.

[25] P. Fife, *Some nonclassical trends in parabolic and parabolic-like evolutions*, in "Trends in nonlinear analysis", pp. 153–191, Springer-Verlag, Berlin, 2003.

[26] H. Fujita, *On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$*, J. Fac. Sci. Univ. Tokyo Sect. 1 **13** (1966), 109–124.

[27] V. A. Galaktionov, H. A. Levine, *A general approach to critical Fujita exponents in nonlinear parabolic problems*, Nonlinear Anal. **34** (1998), 1005–1027.

[28] J. García-Melián, J. D. Rossi, *On the principal eigenvalue of some nonlocal diffusion operators.* J. Diff. Eqns. **246** (2009), 21–38.

[29] J. García-Melián, J. D. Rossi, *A logistic equation with refuge and nonlocal diffusion*, to appear in Comm. Pure Appl. Anal.

[30] K. Hayakawa, *On nonexistence of global solutions of some semilinear parabolic differential equations*, Proc. Japan Acad. **49** (1973), 503–505.

[31] L. I. Ignat, J. D. Rossi, *Refined asymptotic expansions for nonlocal evolution equations*, J. Evol. Eqns. **8** (2008), 617–629.

[32] L. I. Ignat, J. D. Rossi, *Decay estimates for nonlocal problems via energy methods*, J. Math. Pures Appl. **92** (2009), 163–187.

[33] L. I. Ignat, J. D. Rossi, *Asymptotic expansions for nonlocal diffusion equations in L^q-norms for* $1 \le q \le 2$, preprint.

[34] S. Kaplan, *On the growth of solutions of quasi-linear parabolic equations*, Comm. Pure Appl. Math. **16** (1963), 305–330.

[35] K. Kobayashi, T. Sirao, H. Tanaka, *On the growing up problem for semilinear heat equations*, J. Math. Soc. Japan **29** (1977), 407–424.

[36] H. A. Levine, *The role of critical exponents in blow-up theorems*, SIAM Rev. **32** (1990), 262–288.

[37] E. H. Lieb, M. Loss, "Analysis", Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2001.

[38] È. Mitidieri, S. I. Pokhozhaev, *A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities*, (Russian) Tr. Mat. Inst. Steklova **234** (2001), 1–384; translation in Proc. Steklov Inst. Math. **234** (2001), 1–362.

[39] M. Pérez-Llanos, J. D. Rossi, *Blow-up for a non-local diffusion problem with Neumann boundary conditions and a reaction term*, Nonlinear Anal. TM&A. **70** (2009), 1629–1640.

[40] P. Quittner, P. Souplet, "Superlinear parabolic problems: blow-up, global existence and steady states", Birkhäuser Advanced Texts, Basel, 2007.

[41] K. Schumacher, *Travelling-front solutions for integro-differential equations I*, J. Reine Angew. Math. **316** (1980), 54–70.

[42] S. Sugitani, *On nonexistence of global solutions fo some nonlinear integral equations*, Osaka J. Math. **12** (1975), 45–51.

[43] F. B. Weissler, *Existence and non-existence of global solutions for a semilinear heat equation*, Israel J. Math. **38** (1981), 29–40.

J. García-Melián
Departamento de Análisis Matemático, Universidad de La Laguna.
C/. Astrofísico Francisco Sánchez s/n, 38271 – La Laguna, SPAIN
and
Instituto Universitario de Estudios Avanzados (IUdEA) en Física Atómica, Molecular y Fotónica, Facultad de Física, Universidad de La Laguna
C/. Astrofísico Francisco Sánchez s/n, 38203 – La Laguna, SPAIN
E-mail address: jjgarmel@ull.es

F. Quirós
Departamento de Matemáticas, Universidad Autónoma de Madrid.
Campus de Cantoblanco s/n, 28049 – Madrid, SPAIN
E-mail address: fernando.quiros@uam.es