BOUNDARY BLOW-UP SOLUTIONS TO ELLIPTIC SYSTEMS OF COMPETITIVE TYPE *

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Abstract

We consider the elliptic system $\Delta u = u^p v^q$, $\Delta v = u^r v^s$ in Ω, where $p, s > 1$, $q, r > 0$, and $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, subject to different types of Dirichlet boundary conditions: (F) $u = \lambda$, $v = \mu$, (I) $u = v = +\infty$ and (SF) $u = +\infty$, $v = \mu$ on $\partial\Omega$, where $\lambda, \mu > 0$. Under several hypotheses on the parameters p, q, r, s, we show existence and nonexistence of positive solutions, uniqueness and nonuniqueness. We further provide with the exact asymptotic behaviour of the solutions and their normal derivatives near $\partial\Omega$. Some more general related problems are also studied.

1. Introduction

We are concerned in this paper with the following system of semilinear elliptic partial differential equations:

$$\begin{cases} \Delta u = u^p v^q \\ \Delta v = u^r v^s \end{cases} \quad \text{in } \Omega \tag{P}$$

where $p, s > 1$, $q, r > 0$ are real numbers and $\Omega \subset \mathbb{R}^N$ is a bounded domain of class $C^{2,\eta}$ for some η, $0 < \eta < 1$. This system will be studied under three different types of Dirichlet boundary conditions: both components (u, v) bounded on $\partial\Omega$ (finite case), one of them bounded while the other blows up (semifinite case), or both components blowing up simultaneously (infinite case). More precisely,

$$\begin{cases} u = \lambda \\ v = \mu \end{cases} \quad \text{on } \partial\Omega \tag{F}$$

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$$\begin{cases} u = +\infty \\ v = +\infty \end{cases} \quad \text{on } \partial\Omega \tag{I}$$

or

$$\begin{cases} u = +\infty \\ v = \mu \end{cases} \quad \text{on } \partial\Omega \tag{SF}$$

where $\lambda, \mu > 0$. The condition $u = +\infty$ on $\partial\Omega$ is to be understood as $u(x) \to +\infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$. We remark that more general continuous positive functions $\lambda(x)$, $\mu(x)$ can be prescribed on $\partial\Omega$, but we prefer to consider only the constant case for simplicity. Also, the condition $u = \lambda$, $v = +\infty$ on $\partial\Omega$ can be considered in (SF), with obvious modifications of all the results below.

There is a huge amount of literature dealing with single equations with infinite boundary conditions (see for instance [1], [2], [3], [4], [5], [6], [7], [9], [10], [13], [14], [15], [19], [20], [21], [22], [23], [24], [26], [27], [28], [30] and [31]), but very little has been said for the moment for elliptic systems. We quote [8] for predator-prey Lotka–Volterra systems, [11], [12], [16] and [25] for competitive type systems and [17] for cooperative systems. However, we remark that boundary blow-up solutions (sometimes called "large") are explicitly treated only in [16] and [17]. Also, this seems to be the first work regarding the boundary conditions (SF).

Precisely [16] is the starting point for our research. Problem (P) is studied there under the simpler form

$$\begin{cases} \Delta u = v^q \\ \Delta v = u^r \end{cases} \quad \text{in } \Omega$$

for $q, r > 0$, and Ω an interval of $\mathbb{R}$. It is shown that there are infinitely many positive solutions under the boundary conditions (I), that is, $u = v = +\infty$ on $\partial\Omega$, provided that $qr > 1$, and no solutions if $qr \leq 1$. Also, solutions with the boundary condition (SF) can not exist. We mention in passing that a complete classification of all positive solutions (symmetric and nonsymmetric) is performed there, and their boundary behaviour is completely elucidated.

We are showing in the present paper that the more general system (P) can behave in a drastically different way, depending on the sign of $\zeta = (p-1)(s-1) - qr$. If we write the system as $\Delta u = (u^{p-1}v^q)u$, $\Delta v = (u^r v^{s-1})v$, it becomes clear that ζ somehow measures the coupling between the two equations. Thus we will divide our study of system (P) in three cases: "subcritical" when $\zeta > 0$, "critical" if $\zeta = 0$ and "supercritical" for $\zeta < 0$. We will be only concerned with the subcritical and critical cases, delaying a deep study of the supercritical case for a future work.

As an important remark, notice that the cases $q = 0$ and/or $r = 0$ are trivial, since the equations in (P) become uncoupled, and we can deduce all properties of solutions from the results in [7]. Thus we can always assume $q, r > 0$.

The subcritical case is the easiest to handle. We find that problem (P) has a unique positive solution with each of the boundary conditions (F), (I) and (SF). Our proof of existence relies on the method of sub and supersolutions, with some special care with the semifinite conditions (SF). Notice that the system is of competitive type (cf. [29]), and thus the definition of sub and supersolutions is not the standard one (see the Appendix). The uniqueness is achieved by means of estimates near the boundary for all possible positive solutions. Since we follow the approach in [7], some global estimates are also needed, at least for solutions (u, v) verifying $u = v = +\infty$ on $\partial\Omega$. We obtain this global bounds (cf. (3.2) in Section 3.2) through a new interesting iterative procedure, which is based on a deep knowledge of the solutions for a single equation. We mention by the way that the asymptotic result stated in Lemma 7 seems to be completely new. The iterative method for obtaining global bounds turns also to be valid for determining nonexistence of solutions in some regimes of the parameters, both in the subcritical and critical cases.

The critical case, on the contrary is very subtle. While the proof of uniqueness rests unchanged, some facts like existence of solutions for (SF) and boundary behaviour for solutions with (I) seem to be hard to deal with. We answer some of these questions, but we can only solve the problem completely under radial symmetry.

Before stating our results, some remarks on regularity of solutions are in order. Observe that since we are dealing with positive solutions, standard regularity theory (cf. [18]) gives that $u, v \in C^\infty(\Omega)$ in all cases. With respect to regularity up to the boundary, for the conditions (F), we further have $u, v \in C^{2,\eta}(\overline{\Omega})$, while for (SF) we can only assert $v \in C(\overline{\Omega})$. Indeed, it will be shown as a byproduct of our estimates that if $p-1 < 2r$, then $v \notin C^1(\overline{\Omega})$ (cf. Theorem 2).

We now come to our results. We begin by considering the so-called subcritical case $(p-1)(s-1) > qr$. It turns out that in this case, system (P) behaves like a single equation, because the coupling between the two equations is not too strong. With regard to existence, nonexistence and uniqueness of solutions, we have:

Theorem 1. (Subcritical case) *Assume $(p-1)(s-1) > qr$. Then:*

(i) Problem (P) admits a unique positive solution with the boundary conditions (F).

(ii) Problem (P) admits a positive solution with the boundary conditions (I) if and only if $r < p-1$, $q < s-1$. This solution is unique.

(iii) Problem (P) admits a positive solution with the boundary conditions (SF) if and only if $r < p-1$. This solution is unique.

The uniqueness of solutions in each case is a consequence of Lemma 10 in Section 3.3, once we know all possible positive solutions have the same boundary behaviour. We pay attention to this boundary behaviour independently, since it is interesting in its own right.

Theorem 2. (Boundary behaviour in the subcritical case) *Assume $(p-1)(s-1) > qr$. Then*

(i) If (u,v) denotes the unique solution to problem (P) under the boundary conditions (I) then

$$\begin{aligned}
\lim_{x\to x_0} d(x)^\alpha u(x) &= \left(\frac{(\alpha(\alpha+1))^{s-1}}{(\beta(\beta+1))^q}\right)^{\frac{1}{(p-1)(s-1)-qr}} \\
\lim_{x\to x_0} d(x)^\beta v(x) &= \left(\frac{(\beta(\beta+1))^{p-1}}{(\alpha(\alpha+1))^r}\right)^{\frac{1}{(p-1)(s-1)-qr}} \\
\lim_{x\to x_0} d(x)^{\alpha+1} \nabla u(x)\nu(x_0) &= \alpha\left(\frac{(\alpha(\alpha+1))^{s-1}}{(\beta(\beta+1))^q}\right)^{\frac{1}{(p-1)(s-1)-qr}} \\
\lim_{x\to x_0} d(x)^{\beta+1} \nabla v(x)\nu(x_0) &= \beta\left(\frac{(\beta(\beta+1))^{p-1}}{(\alpha(\alpha+1))^r}\right)^{\frac{1}{(p-1)(s-1)-qr}}
\end{aligned} \tag{1.1}$$

for every $x_0 \in \partial\Omega$, where $\nu(x_0)$ stands for the exterior unit normal to $\partial\Omega$ at x_0 and

$$\alpha = \frac{2(s-1-q)}{(p-1)(s-1)-qr}, \quad \beta = \frac{2(p-1-r)}{(p-1)(s-1)-qr}.$$

(ii) If now (u,v) denotes the solution to (P) with the boundary conditions (SF) then

$$\begin{aligned} &\lim_{x\to x_0} d(x)^{\theta} u(x) = \left(\frac{\theta(\theta+1)}{\mu^q}\right)^{\frac{1}{p-1}} \\ &\lim_{x\to x_0} d(x)^{\theta+1} \nabla u(x) \nu(x_0) = \theta\left(\frac{\theta(\theta+1)}{\mu^q}\right)^{\frac{1}{p-1}}, \end{aligned} \tag{1.2}$$

where $\theta = \frac{2}{p-1}$. If moreover $p-1 < 2r$, then

$$\begin{aligned} &\lim_{x\to x_0} d(x)^{-\tau}(\mu - v(x)) = \frac{\mu^s}{\tau(1-\tau)}\left(\frac{\theta(\theta+1)}{\mu^q}\right)^{\frac{r}{p-1}} \\ &\lim_{x\to x_0} d(x)^{1-\tau} \nabla v(x) \nu(x_0) = \frac{\mu^s}{(1-\tau)}\left(\frac{\theta(\theta+1)}{\mu^q}\right)^{\frac{r}{p-1}} \end{aligned} \tag{1.3}$$

where $\tau = 2 - \theta r$. In particular, v is never C^1 up to $\partial\Omega$.

We now turn to the critical case $(p-1)(s-1) = qr$. With essentially the same ideas as in Theorem 1, we can completely answer the questions of nonexistence, uniqueness and multiplicity of positive solutions. However, the existence seems to be a very subtle question, at least for the boundary conditions (SF).

Theorem 3. (Critical case) *Assume $(p-1)(s-1) = qr$. Then:*

(i) Problem (P) admits a unique positive solution with the boundary conditions (F).

(ii) Problem (P) admits a positive solution with the boundary conditions (I) if and only if $r = p-1$, $q = s-1$. Moreover, if (u,v) is a solution, then $(tu, t^{-\delta}v)$ is also a solution for every $t > 0$, where $\delta = (p-1)/(s-1)$, and hence there are infinitely many positive solutions.

(iii) Problem (P) can only admit a positive solution with the boundary conditions (SF) if $r < p-1$. The solution is unique if it exists. Moreover, if (u,v) is a solution, then $(tu, t^{-\delta}v)$ is also a solution for every $t > 0$, and thus solutions with different boundary datum μ are all obtained from one of them (say with $\mu = 1$).

Notice that Theorem 3 (iii) does not assert that problem (P) always has a positive solution (u,v) with $u = +\infty$, $v = \mu$ on $\partial\Omega$. Also, it would be interesting to know whether for the boundary conditions (I), all possible solutions are of the form $(tu, t^{-\delta}v)$, $t > 0$, for a given one (u,v). Both questions seem to be hard to prove in general domains Ω. However, they are completely solved in the following Theorem in the radial case $\Omega = B$, a ball of $\mathbb{R}^N$.

Theorem 4. *Assume Ω is a ball B, and $(p-1)(s-1) = qr$. Then:*

(i) If $r = p-1$ *and* $q = s-1$*, and* (u, v) *is a radial positive solution with* $u = v = +\infty$ *on* $\partial\Omega$*, then* $u = t^{\frac{s-1}{p+s-2}} U$*,* $v = t^{-\frac{p-1}{p+s-2}} U$*, for some* $t > 0$*, where* U *is the unique solution to*

$$\begin{cases} \Delta u = u^{p+s-1} & in \quad B \\ u = +\infty & on \quad \partial B. \end{cases}$$

Hence we also have the estimates near the boundary:

$$\begin{aligned} &\lim_{x \to x_0} d(x)^{\omega} u(x) = (t^{s-1}\omega(\omega+1))^{\frac{1}{p+s-2}} \\ &\lim_{x \to x_0} d(x)^{\omega+1} \nabla u(x) \nu(x_0) = \omega(t^{s-1}\omega(\omega+1))^{\frac{1}{p+s-2}} \\ &\lim_{x \to x_0} d(x)^{\omega} v(x) = (t^{-(p-1)}\omega(\omega+1))^{\frac{1}{p+s-2}} \\ &\lim_{x \to x_0} d(x)^{\omega+1} \nabla v(x) \nu(x_0) = \omega(t^{-(p-1)}\omega(\omega+1))^{\frac{1}{p+s-2}} \ , \end{aligned} \tag{1.4}$$

where $\omega = \frac{2}{p+s-2}$.

(ii) If $r < p-1$*, then problem (P) has a unique positive solution* (u, v) *with the boundary conditions (SF), which verifies (1.2) and (1.3).*

Remark 1. Part (i) of Theorem 4 is a consequence of Theorem 5 in [6], which determines that if u, v are radial positive solutions to the linear equations $\Delta u = a(r)u$, $\Delta v = b(r)v$, respectively, in a ball B of radius R, such that the weights $a(r)$, $b(r)$ are close near ∂B in the sense that

$$\int_0^R (R-r)|a(r) - b(r)| dr < +\infty \ ,$$

then there exists a constant $\kappa > 0$ such that

$$\lim_{r \to R} \frac{u(r)}{v(r)} = \kappa.$$

We believe that this result ceases to be true in general smooth bounded domains Ω (where κ needs not be constant). Thus it seems a hard task to extend Theorem 4 (i) to those domains.

Finally, we consider again the question of existence of positive solutions to (P) with the boundary conditions (SF) in general domains. We prove that there actually exists a solution provided that the domain Ω is “sufficiently small”. We denote $\rho\Omega = \{x \in \mathbb{R}^N : x/\rho \in \Omega\}$. Then:

Theorem 5. *Assume* $(p-1)(s-1) = qr$ *and* $r < p-1$*. Then there exists* $\rho_0 > 0$*, depending on* p, q, r, s *and* Ω*, such that if* $0 < \rho < \rho_0$*, problem (P) admits a unique positive solution* (u, v) *with the boundary conditions (SF) in* $\rho\Omega$*, which verifies (1.2) and (1.3).*

The techniques used to prove the previous theorems apply to a wider class of elliptic systems, for instance involving (possibly unbounded) weights or with more general nonlinearities in the right-hand side, which essentially behave like powers. They will be treated in Section 5.

The paper is organized as follows: Section 2 provides us with some preliminary results regarding a single equation, which will be useful in the other sections. Section 3 is devoted to the subcritical case, treating separately in Sections 3.1, 3.2, 3.3 and 3.4 the existence, global estimates, boundary estimates with uniqueness and nonexistence, respectively. The critical case will be covered in Section 4, while in §5 we quote without proof some results on existence, uniqueness and boundary behaviour of solutions to some more general systems related to (P). Finally, in the Appendix, the method of sub and supersolutions for system (P) will be revisited under the three types of boundary conditions, paying special attention to conditions (SF).

2. Preliminaries

In this section we collect some results concerning the solutions to the problem

$$\begin{cases} \Delta u = d(x)^{-\gamma} u^p & \text{in} \quad \Omega, \\ u = +\infty & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

that will be used in the next sections. Here $d(x)$ stands for the distance of a point $x \in \Omega$ to the boundary $\partial\Omega$. Since Ω is C^2, it is well known (cf. Lemma 14.16 in [18]) that $d(x)$ is C^2 in a neighbourhood of $\partial\Omega$. Redefining $d(x)$ outside this neighbourhood if necessary, we can always assume that $d(x) \in C^2(\overline{\Omega})$.

The following Lemma, part of Theorem 1 in [7], contains the basic features of problem (2.1) (see also Theorem 7 in [7]).

Lemma 6. *Assume $p > 1$ and $\gamma < 2$. Then problem (2.1) has a unique positive solution, which will be denoted by $U_{p,\gamma}$. This solution is obtained as the limit as $n \to +\infty$ of the solutions U_n to the problem*

$$\begin{cases} \Delta u = d(x)^{-\gamma} u^p & \text{in} \quad \Omega, \\ u = n & \text{on} \quad \partial\Omega. \end{cases}$$

Moreover,

$$\lim_{x \to x_0} d(x)^{\alpha} U_{p,\gamma}(x) = (\alpha(\alpha+1))^{\frac{1}{p-1}}$$
$$\lim_{x \to x_0} d(x)^{\alpha+1} \nabla U_{p,\gamma}(x) \nu(x_0) = \alpha(\alpha(\alpha+1))^{\frac{1}{p-1}}$$

for every $x_0 \in \partial\Omega$, where $\alpha = \frac{2-\gamma}{p-1}$ and $\nu(x_0)$ stands for the exterior unit normal to $\partial\Omega$ at x_0.

In particular the quantities

$$A_{p,\gamma} = \sup_{x \in \Omega} d(x)^{\alpha} U_{p,\gamma}(x), \qquad B_{p,\gamma} = \inf_{x \in \Omega} d(x)^{\alpha} U_{p,\gamma}(x) \tag{2.2}$$

are finite and positive. For the purposes of global estimates in Section 3, we need to analyze them in more detail for varying γ.

Lemma 7. *The quantities $A_{p,\gamma}$ and $B_{p,\gamma}$ are bounded and bounded away from zero when γ is bounded and bounded away from 2. Also,*

$$\lim_{\gamma\to 2-} A_{p,\gamma} = \lim_{\gamma\to 2-} B_{p,\gamma} = 0 \ .$$

In particular, $U_{p,\gamma} \to 0$ uniformly on compact subsets of Ω when $\gamma \to 2-$.

The first part of Lemma 7 is a consequence of the following "comparison principle", which will also be used in Section 3. Its proof follows from the method of sub and supersolutions (cf. Lemma 4 in [15] and Lemma 1 in [14]) and the uniqueness of solutions to (2.1) given by Lemma 6.

Lemma 8. *Let $u \in C^2(\Omega)$ verify $\Delta u \le Cd(x)^{-\gamma}u^p$ in Ω for some positive constant C, and $u = +\infty$ on $\partial\Omega$. Then $u(x) \ge C^{-\frac{1}{p-1}}U_{p,\gamma}(x)$. Similarly, if $\Delta u \ge Cd(x)^{-\gamma}u^p$ in Ω, then $u(x) \le C^{-\frac{1}{p-1}}U_{p,\gamma}(x)$, regardless the value of u on the boundary.*

Proof of Lemma 7. We claim that $(\sup_\Omega d(x))^{-\frac{\gamma}{p-1}}A_{p,\gamma}$ is a decreasing function of γ for $\gamma < 2$. We just observe that if $\sigma > \gamma$, then $U_{p,\gamma}$ verifies

$$\Delta U_{p,\gamma} = d^{-\gamma}U_{p,\gamma}^p = d^{\sigma-\gamma}d^{-\sigma}U_{p,\gamma}^p \le (\sup_\Omega d(x))^{\sigma-\gamma}d^{-\sigma}U_{p,\gamma}^p.$$

Hence, using Lemma 8 we get $U_{p,\gamma} \ge ((\sup_\Omega d(x))^{\frac{\gamma-\sigma}{p-1}}U_{p,\sigma}$, and the claim follows. In particular, if $\sigma_1 < \gamma < \sigma_2 < 2$, then

$$(\sup_\Omega d(x))^{\frac{\gamma-\sigma_2}{p-1}}A_{p,\sigma_2} \le A_{p,\gamma} \le (\sup_\Omega d(x))^{\frac{\gamma-\sigma_1}{p-1}}A_{p,\sigma_1}.$$

A similar calculation for $B_{p,\gamma}$ shows the first part of the Lemma.

To prove the second part, it suffices to see that $\lim_{\gamma\to 2-} A_{p,\gamma} = 0$. Define $V_\gamma = d(x)^\alpha U_{p,\gamma}$, where $\alpha = (2-\gamma)/(p-1) \to 0$ as $\gamma \to 2-$. Then V_γ is continuous on $\overline{\Omega}$, and we can assert that there exists $x_\gamma \in \overline{\Omega}$ such that $V_\gamma(x_\gamma) = A_{p,\gamma}$. There are two possibilities: if $x_\gamma \in \partial\Omega$, then $A_{p,\gamma} = (\alpha(\alpha+1))^{\frac{1}{p-1}}$, according to Lemma 6. If, on the contrary, $x_\gamma \in \Omega$, then $\nabla V_\gamma(x_\gamma) = 0$, $\Delta V_\gamma(x_\gamma) \le 0$. It is easy to see that $V = V_\gamma$ satisfies the equation

$$d^2\Delta V - 2\alpha d\nabla d\nabla V + \alpha(\alpha+1)|\nabla d|^2 V - \alpha d\Delta d V = V^p \ .$$

Setting $x = x_\gamma$, we arrive at $A_{p,\gamma}^{p-1} \le \alpha(\alpha+1)|\nabla d(x_\gamma)|^2 - \alpha d(x_\gamma)\Delta d(x_\gamma) = O(\alpha)$ as $\alpha \to 0+$. In either case we obtain that $A_{p,\gamma} \to 0$ as $\gamma \to 2-$. □

We close this section by stating and proving an extension of Lemma 8 to the case where Ω is a half-space $D = \{x \in \mathbb{R}^N : \ x_1 > 0\}$ (for a point $x \in \mathbb{R}^N$ we write $x = (x_1, x')$, with $x' \in \mathbb{R}^{N-1}$). This result will be useful when dealing with the boundary estimates for solutions that satisfy conditions (I).

Lemma 9. *Assume $u \in C^2(D)$ verifies $\Delta u \ge Cx_1^{-\gamma}u^p$ (resp. $\Delta u \le Cx_1^{-\gamma}u^p$) in D, together with $u \le Kx_1^{-\alpha}$ (resp. $u \ge Kx_1^{-\alpha}$), where K, C are positive constants and $\alpha = (2-\gamma)/(p-1)$. Then*

$$u \le \left(\frac{\alpha(\alpha+1)}{C}\right)^{\frac{1}{p-1}} x_1^{-\alpha} \quad \left(\textit{resp. } u \ge \left(\frac{\alpha(\alpha+1)}{C}\right)^{\frac{1}{p-1}} x_1^{-\alpha}\right) \quad \textit{in } D.$$

Proof. The proof is an adaptation of the arguments used in [20] and [7]. We only show the first case, the other being treated similarly.

Assume there exist $x^0 \in D$ and $k > 1$ such that $u(x^0) > kA(x_1^0)^{-\alpha}$, where x_1^0 is the first component of x^0 and $A = (\alpha(\alpha+1)/C)^{\frac{1}{p-1}}$. Set

$$D_0 = \{u > kAx_1^{-\alpha}\} \cap B_r(x^0)$$

with $r = d(x^0)/2$. Then $\Delta(u - kAx_1^{-\alpha}) > kA\alpha(\alpha+1)(k^{p-1}-1)x_1^{-\alpha-2}$ in D_0. Since $x_1 \le 3r/2$ in D_0, if we define $w(x) = (kA\alpha(\alpha+1)(k^{p-1}-1)2^{\alpha+2}r^{-\alpha-2})/(2N3^{\alpha+2})(r^2 - |x-x^0|^2)$, then $\Delta(u - kAx_1^{-\alpha} + w) > 0$ in D_0. The maximum principle implies the existence of a point $x^1 \in \partial D_0$ such that

$$u(x^0) - kA(x_1^0)^{-\alpha} + w(x^0) < u(x^1) - kA(x_1^1)^{-\alpha} + w(x^1).$$

If $x^1 \in B_r(x^0)$, it follows from this inequality that $w(x^0) < w(x^1)$, which is impossible. Thus, $x^1 \in \partial B_r(x^0)$, and we arrive at

$$w(x^0) < u(x^1) - kA(x_1^1)^{-\alpha}.$$

Now taking into account that $x_1^1 \ge r/2$ and the definition of w, we deduce

$$u(x^1) > \left(1 + \frac{2\alpha(\alpha+1)(k^{p-1}-1)}{N3^{\alpha+2}}\right) kA(x_1^1)^{-\alpha}.$$

Proceeding inductively we find a sequence of points $x^n \in D$ such that

$$u(x^n) > \left(1 + \frac{2\alpha(\alpha+1)(k^{p-1}-1)}{N3^{\alpha+2}}\right)^n kA(x_1^n)^{-\alpha},$$

which contradicts the inequality $u \le Kx_1^{-\alpha}$. This proves the Lemma. □

3. The subcritical case $(p-1)(s-1) > qr$

This section is devoted to the proof of Theorems 1 and 2. To clarify the exposition, we split the proof into four different sections, considering in each of them the three types of boundary conditions.

3.1. Existence of solutions

We are proving in what follows that problem (P) admits a solution with each of the boundary conditions (F), (SF) and (I), provided that the conditions in Theorem 1 hold. We will use the method of sub and supersolutions, as stated in the Appendix (Theorems A.1, A.2 and A.3). We recall that a pair $(\underline{u}, \underline{v})$ is a subsolution provided $\Delta\underline{u} \ge \underline{u}^p\underline{v}^q$, $\Delta\underline{v} \le \underline{u}^r\underline{v}^s$ in Ω, and a supersolution $(\overline{u}, \overline{v})$ is defined by reversing the above inequalities.

The finite case, that is the boundary conditions (F) is very easy to handle. Indeed, we can take $(\underline{u}, \underline{v}) = (0, M)$ and $(\overline{u}, \overline{v}) = (M, 0)$ as sub and supersolution, respectively, for large positive M, and Theorem A.1 guarantees the existence of a positive solution.

Next consider the boundary conditions (I). We look for a subsolution of the form $(\underline{u},\underline{v}) = (\varepsilon U_{p,\gamma},\varepsilon^{-\delta}U_{s,\sigma})$, where the functions $U_{p,\gamma}$, $U_{s,\sigma}$ are as introduced in Section 2, $\varepsilon>0$ is small and $\gamma,\sigma,\delta>0$ are to be chosen. Then $(\underline{u},\underline{v})$ is a subsolution provided

$$\varepsilon^{p-\delta q-1}d^{\gamma}U^{q}_{s,\sigma}\le 1, \qquad \varepsilon^{r-\delta s+\delta}d^{\sigma}U^{r}_{p,\gamma}\ge 1. \tag{3.1}$$

If we select $\gamma,\sigma<2$ so that

$$\frac{\gamma}{q}=\frac{2-\sigma}{s-1}, \qquad \frac{\sigma}{r}=\frac{2-\gamma}{p-1}$$

which is always possible since $r<p-1$, $q<s-1$, inequality (3.1) will hold for ε small if $p-\delta q-1>0$, $r-\delta s+\delta<0$. Thus, fixing $r/(s-1)<\delta<(p-1)/q$, we obtain our subsolution. We leave to the reader to check that $(\overline{u},\overline{v})=(MU_{p,\gamma},M^{-\delta}U_{s,\sigma})$ is a supersolution when $M>0$ is large enough. Since the sub and supersolution are ordered, that is $\underline{u}\le\overline{u}$, $\underline{v}\ge\overline{v}$, Theorem A.2 in the Appendix implies the existence of a positive solution (u,v) to (P), verifying $u=v=+\infty$ on $\partial\Omega$.

We now turn to the boundary conditions (SF). By Lemma 6, we know that the problem

$$\begin{cases} \Delta u = d(x)^{-\gamma}u^p & \text{in} \quad \Omega,\\ u=1 & \text{on} \quad \partial\Omega,\end{cases}$$

has a unique positive solution which will be denoted by $V_{p,\gamma}$, provided that $p>1$, $\gamma<2$. With the choice $\sigma=2r/(p-1)$, it is not hard to prove that $(\varepsilon U_{p,0},\varepsilon^{-\delta}V_{s,\sigma})$ is a subsolution for small positive ε and $(MU_{p,0},M^{-\delta}V_{s,\sigma})$ is a supersolution for large positive M. Thus, since $U_{p,0}\le A_{p,0}d^{-\frac{2}{p-1}}$ and $r<p-1$, Theorem A.3 in the Appendix implies the existence of a positive solution (u,v) to (P) such that $u=+\infty$, $v=\mu$ on $\partial\Omega$.

3.2. Global estimates for solutions

We are showing here that positive solutions (u,v) to (P) which satisfy the boundary conditions (I), i.e. $u=v=+\infty$ on $\partial\Omega$, also verify

$$Ad(x)^{-\alpha}\le u\le Bd(x)^{-\alpha}, \qquad Ad(x)^{-\beta}\le v\le Bd(x)^{-\beta}, \tag{3.2}$$

for some positive constants A,B where α and β are as given in Theorem 2 (i). For this aim, we are using a new iteration method, which will prove to be valid also for nonexistence results in §§3.4 and 4.1.

Let $a_0=\inf v>0$. Then, since $\Delta u\ge a_0^q u^p$ in Ω, Lemma 8 implies $u\le a_0^{-\frac{q}{p-1}}U_{p,0}$, and thus $u\le a_0^{-\frac{q}{p-1}}A_{p,0}d^{-\alpha_0}$, where $\alpha_0=2/(p-1)$. Inserting this into the second equation in (P) we have

$$\Delta v\le a_0^{-\frac{qr}{p-1}}A^r_{p,0}d^{-\alpha_0 r}v^s, \qquad \text{in } \Omega,$$

and Lemma 8 again gives

$$v\ge \left(a_0^{-\frac{qr}{p-1}}A^r_{p,0}\right)^{-\frac{1}{s-1}}B_{s,\alpha_0 r}d^{-\beta_0}, \qquad \text{in } \Omega,$$

where $\beta_0 = (2-\alpha_0 r)/(s-1)$. Proceeding inductively, we obtain

$$\begin{aligned} u &\le a_n^{-\frac{q}{p-1}} A_{p,\beta_{n-1}q} d^{-\alpha_n} \\ v &\ge a_{n+1} d^{-\beta_n} \end{aligned} \qquad \text{in } \Omega, \tag{3.3}$$

where

$$\begin{aligned} \alpha_n &= \frac{2-\beta_{n-1}q}{p-1} \\ \beta_n &= \frac{2-\alpha_n r}{s-1} \\ a_{n+1} &= a_n^{\frac{qr}{(p-1)(s-1)}} A_{p,\beta_{n-1}q}^{-\frac{r}{s-1}} B_{s,\alpha_n r}\ . \end{aligned} \tag{3.4}$$

Let us see that all these quantities converge as $n \to +\infty$. It is a straightforward calculation to check that

$$\beta_n = \frac{2(p-1-r)}{(p-1)(s-1)} + \frac{qr}{(p-1)(s-1)}\,\beta_{n-1}$$

and that $\beta_1 > \beta_0$, $\beta_n \le \beta$. In particular, we deduce that β_n converges to $\beta = 2(p-1-r)/((p-1)(s-1)-qr)$ as $n \to +\infty$. As a consequence, also $\alpha_n \to \alpha = 2(s-1-q)/((p-1)(s-1)-qr)$.

Since in this case the numbers $\beta_{n-1}q$ and $\alpha_n r$ will be bounded and bounded away from 2, Lemma 7 and the third equation in (3.4) imply the existence of a constant $K>0$ such that $a_{n+1} \ge K a_n^\delta$, where $\delta = \frac{qr}{(p-1)(s-1)} < 1$. Iterating this inequality we arrive at

$$a_{n+1} \ge a_0^{\delta^{n+1}} K^{\delta^n + \delta^{n-1} + \ldots + \delta + 1}\ .$$

Passing to the limit we deduce $\liminf_{n\to+\infty} a_{n+1} \ge K^{\frac{1}{1-\delta}} > 0$. We now come back to (3.3) and let $n \to +\infty$ to obtain $u \le Bd^{-\alpha}$, $v \ge Ad^{-\beta}$ in Ω, for some positive constants A, B. The symmetric argument proves the reversed inequalities and thus (3.2) is established. □

3.3. Boundary estimates. Uniqueness

In this section, we are obtaining the boundary estimates (1.1), (1.2) and (1.3) stated in Theorem 2. The approach used is the same as in [7].

Proof of (1.1). Let (u,v) be a positive solution to (P) with $u = v = +\infty$. Take $x_0 \in \partial\Omega$ and $\{x_n\} \subset \Omega$ such that $x_n \to x_0$. Choose an open neighbourhood $\mathcal{U}$ of x_0 so that $\partial\Omega$ admits $C^{2,\eta}$ local coordinates $\xi : \mathcal{U} \to \mathbb{R}^N$, and $x \in \mathcal{U} \cap \Omega$ if and only if $\xi_1(x) > 0$ ($\xi = (\xi_1, \xi_2, \ldots, \xi_N)$). Also, assume $\xi(x_0) = 0$. Denoting $u(x) = \bar u(\xi(x))$, $v(x) = \bar v(\xi(x))$ we see that $(\bar u, \bar v)$ verifies the system

$$\begin{cases} \displaystyle\sum_{i,j=1}^N a_{ij}(\xi) \frac{\partial^2 \bar u}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^N b_i(\xi) \frac{\partial \bar u}{\partial \xi_i} = \bar u^p \bar v^q \\ \displaystyle\sum_{i,j=1}^N a_{ij}(\xi) \frac{\partial^2 \bar v}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^N b_i(\xi) \frac{\partial \bar v}{\partial \xi_i} = \bar u^r \bar v^s \end{cases}$$

in $\xi(\mathcal{U} \cap \Omega)$, where a_{ij}, b_i are C^η, and $a_{ij}(0) = \delta_{ij}$.

Let t_n be the projections of $\xi(x_n)$ onto $\xi(\mathcal{U} \cap \partial\Omega)$, and define

$$u_n(y) = d_n^\alpha\ \bar{u}(t_n + d_n y), \qquad v_n(y) = d_n^\beta\ \bar{v}(t_n + d_n y)\ ,$$

where $d_n = d(\xi(x_n))$, and α, β are as given in Theorem 2 (i). Then the functions (u_n, v_n) verify

$$\begin{cases} \displaystyle\sum_{i,j=1}^{N} a_{ij}(t_n + d_n y) \frac{\partial^2 u_n}{\partial\xi_i \partial\xi_j} + d_n \sum_{i=1}^{N} b_i(t_n + d_n y) \frac{\partial u_n}{\partial \xi_i} = u_n^p v_n^q \\ \displaystyle\sum_{i,j=1}^{N} a_{ij}(t_n + d_n y) \frac{\partial^2 v_n}{\partial\xi_i \partial\xi_j} + d_n \sum_{i=1}^{N} b_i(t_n + d_n y) \frac{\partial v_n}{\partial \xi_i} = u_n^r v_n^s\ . \end{cases}$$

We are now making use of the global estimates (3.2). They imply that $Ay_1^{-\alpha} \le u_n(y) \le By_1^{-\alpha}$, $Ay_1^{-\beta} \le v_n(y) \le By_1^{-\beta}$. In particular, we have estimates for u_n and v_n in compact subdomains of $D := \{y \in \mathbb{R}^N :\ y_1 > 0\}$. Then it is standard to conclude that (for a subsequence) $u_n \to u_0$, $v_n \to v_0$ in $C^2_{\rm loc}(D)$, where (u_0, v_0) is a positive solution to

$$\begin{cases} \Delta u_0 = u_0^p v_0^q \\ \Delta v_0 = u_0^r v_0^s \end{cases} \qquad \text{in } D, \tag{3.5}$$

which verifies $Ay_1^{-\alpha} \le u_0(y) \le By_1^{-\alpha}$, $Ay_1^{-\beta} \le v_0(y) \le By_1^{-\beta}$ in D.

Claim: $u_0 = C_1 y_1^{-\alpha}$, $v_0 = C_2 y_1^{-\beta}$, where

$$C_1 = \left(\frac{(\alpha(\alpha+1))^{s-1}}{(\beta(\beta+1))^q} \right)^{\frac{1}{(p-1)(s-1)-qr}}, \qquad C_2 = \left(\frac{(\beta(\beta+1))^{p-1}}{(\alpha(\alpha+1))^r} \right)^{\frac{1}{(p-1)(s-1)-qr}}. \tag{3.6}$$

Let us show the claim. Notice that $\Delta u_0 \ge A^q y_1^{-\beta q} u_0^p$ in Ω, and thus Lemma 9 implies $u_0 \le B_1 y_1^{-\alpha}$ in Ω, where

$$B_1 = \left(\frac{\alpha(\alpha+1)}{A^q} \right)^{\frac{1}{p-1}}.$$

Similarly, since $\Delta v_0 \le B_1^r y_1^{-\alpha r} v_0^s$ in Ω, Lemma 9 again gives $v_0 \ge A_1 y_1^{-\beta}$ in Ω, for

$$A_1 = \left(\frac{\beta(\beta+1)}{B_1^r} \right)^{\frac{1}{s-1}}.$$

Iterating this procedure, we obtain that $u_0 \le B_n y_1^{-\alpha}$, $v_0 \ge A_n y_1^{-\beta}$ in Ω, A_n, B_n being given by

$$B_{n+1} = \left(\frac{\alpha(\alpha+1)}{A_n^q} \right)^{\frac{1}{p-1}}, \qquad A_{n+1} = \left(\frac{\beta(\beta+1)}{B_{n+1}^r} \right)^{\frac{1}{s-1}}.$$

It is not hard to see that if A is small enough (which can always be assumed), the sequence $\{A_n\}$ will be increasing and bounded from above, hence convergent. This also entails the convergence of $\{B_n\}$. A little algebra then gives that $A_n \to C_2$ and $B_n \to C_1$, where C_1 and C_2 are as in (3.6). Thus $u_0 \le C_1 y_1^{-\alpha}$ and $v_0 \ge C_2 y_1^{-\beta}$. The symmetric argument provides with the reversed inequality, and the claim is proved.

To summarize, we have shown that $u_n \to C_1 y_1^{-\alpha}$ and $v_n \to C_2 y_1^{-\beta}$ in $C^2_{\rm loc}(D)$. The limits in (1.1) are then obtained by setting $y = e_1$ in the above convergence, and recalling that $\xi(x_n) = t_n + d_n e_1$. □

Proof of (1.2) and (1.3). We now turn our attention once more to the boundary conditions (SF). Let (u, v) be the positive solution to (P). Since $v = \mu$ on $\partial\Omega$, and u is the solution to

$$\begin{cases} \Delta u = v^q u^p & \text{in} \quad \Omega \\ u = +\infty & \text{on} \quad \partial\Omega, \end{cases}$$

then Theorem 1 in [7] with $\gamma = 0$ and $C_0 \equiv \mu^q$ shows the validity of the limits in (1.2). To prove the corresponding limits for v, we notice that by the maximum principle $v \le \mu$, so that $w = \mu - v \ge 0$ verifies

$$\begin{cases} \Delta w = -u^r(\mu - w)^s & \text{in} \quad \Omega \\ w = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Thus, when $p - 1 < 2r$, Lemma A.4 in the Appendix implies the existence of a positive constant K such that $w \le K d^\tau$, where $\tau = 2 - \theta r > 0$ and $\theta = 2/(p-1)$.

Take an arbitrary point $x_0 \in \partial\Omega$ and a sequence $x_n \to x_0$. We straighten the boundary as in the previous case with a local change of coordinates ξ, and introduce the functions

$$u_n(y) = d_n^{\theta}\, \bar{u}(t_n + d_n y), \qquad w_n(y) = d_n^{-\tau} \bar{w}(t_n + d_n y)\ ,$$

where $d_n = d(\xi(x_n))$, t_n are the projections of $\xi(x_n)$ onto $\xi(\mathcal{U} \cap \partial\Omega)$ and $u(x) = \bar{u}(\xi(x))$, $w(x) = \bar{w}(\xi(x))$. Since $A y_1^{-\theta} \le u_n(y) \le B y_1^{-\theta}$ and $0 \le w_n \le B y_1^{\tau}$, we obtain that (passing to a subsequence) $u_n \to u_0$, $w_n \to w_0$ in $C^2_{\rm loc}(D)$ where

$$\Delta u_0 = \mu^q u_0^p, \qquad \Delta w_0 = -u_0^r \mu^s, \qquad \text{in } D$$

and $A y_1^{-\theta} \le u_0 \le B y_1^{-\theta}$, $0 \le w_0 \le B y_1^{\tau}$ in D. Lemma 9 readily implies that

$$u_0 = \left(\frac{\theta(\theta+1)}{\mu^q} \right)^{\frac{1}{p-1}} y_1^{-\theta}\ .$$

Thus $\Delta w_0 = -\mu^s C_0^r y_1^{-\theta r}$, where $C_0 = (\theta(\theta+1)/\mu^q)^{\frac{1}{p-1}}$. Let

$$z = w_0 - \frac{\mu^s C_0^r}{\tau(1-\tau)} y_1^{\tau}.$$

Then $\Delta z = 0$ in D and $|z| \le K y_1^{\tau}$, for some positive constant K. Observe that $z = 0$ on $\partial\Omega$, and thus we can extend z by reflection to a harmonic function in $\mathbb{R}^N$ such that $|z| \le K |y_1|^{\tau}$. Since $\tau < 1$, the interior derivatives for harmonic functions (cf. Theorem 2.10 in [18]) imply that z is constant, and hence $z \equiv 0$. In conclusion, $w_0 = \mu^s C_0^r / (\tau(1-\tau)) y_1^{\tau}$, and the proof of (1.3) finishes as before, taking $y = e_1$. □

Remarks 2. a) It is clear that the proof of (1.2) and (1.3) continues to be valid in the critical case $(p-1)(s-1) = qr$.

b) The proof can also be extended to a more general boundary condition $v = \mu(x)$ on $\partial\Omega$, where $\mu(x)$ is a continuous positive function (see Remark 5 (a) in the Appendix).

A very important consequence of estimates (1.1), and (1.2) is that any two pairs of solutions to (P) (u_1, v_1), (u_2, v_2) (under any of the boundary conditions we are considering) "agree" on the boundary, that is

$$\lim_{x \to x_0} \frac{u_1(x)}{u_2(x)} = \lim_{x \to x_0} \frac{v_1(x)}{v_2(x)} = 1$$

for every $x_0 \in \partial\Omega$ (for simplicity, we write in what follows $u_1/u_2 = v_1/v_2 = 1$ on $\partial\Omega$). Once we have this information, the uniqueness is obtained in all cases through the following Lemma.

Lemma 10. *Let (u_1, v_1), (u_2, v_2) be positive solutions to (P) such that $u_1/u_2 = v_1/v_2 = 1$ on $\partial\Omega$, and assume $(p-1)(s-1) \geq qr$. Then $u_1 = u_2$ and $v_1 = v_2$.*

Proof. Let $w = u_1/u_2$, and assume $k := \sup_\Omega w > 1$. Then, since $w = 1$ on $\partial\Omega$, there exists $x_0 \in \Omega$ such that $w(x_0) = k$, and hence $\nabla w(x_0) = 0$, $\Delta w(x_0) \leq 0$. In particular, $u_2 \Delta u_1 - u_1 \Delta u_2 \leq 0$ at x_0. This leads to $v_2(x_0) \geq k^{\frac{p-1}{q}} v_1(x_0)$.

We now claim that $v_2 < k^{\frac{r}{s-1}} v_1$ in Ω. Assume on the contrary that $\Omega_0 := \{v_2 > k^{\frac{r}{s-1}} v_1\}$ is nonempty. Notice that $\partial\Omega_0 \subset \Omega$, since $k > 1$ and $v_1/v_2 = 1$ on $\partial\Omega$, thus $v_2 = k^{\frac{r}{s-1}} v_1$ on $\partial\Omega_0$. Then

$$\Delta v_2 = u_2^r v_2^s \geq k^{-r+\frac{rs}{s-1}} u_1^r v_1^s = \Delta(k^{\frac{r}{s-1}} v_1)$$

in Ω_0 and the maximum principle implies $v_2 \leq k^{\frac{r}{s-1}} v_1$ in Ω_0, which is impossible. Hence $v_2 \leq k^{\frac{r}{s-1}} v_1$ in Ω, and by the strong maximum principle it follows that $v_2 < k^{\frac{r}{s-1}} v_1$ in Ω.

Combining the two assertions we have $k^{\frac{p-1}{q}} v_1(x_0) < k^{\frac{r}{s-1}} v_1(x_0)$, that is $k^{\frac{(p-1)(s-1)-qr}{q(s-1)}} < 1$. If $(p-1)(s-1) = qr$, this is a contradiction, while for $(p-1)(s-1) > qr$ we obtain $k < 1$, which is also a contradiction. To summarize, we conclude $k \leq 1$, i.e. $u_1 \leq u_2$. The symmetric argument proves $u_1 = u_2$, and using the equation for u_1 and u_2, we deduce $v_1 = v_2$. The Lemma is proved. □

3.4. Nonexistence

This section is devoted to prove nonexistence of positive solutions (u, v) to (P) both for the boundary conditions (I) and (SF), when the conditions in Theorem 1 (*ii*) and (*iii*) do not hold.

Nonexistence for the boundary conditions (I). We begin assuming that (u, v) is a positive solution to (P) verifying $u = v = +\infty$ on $\partial\Omega$ with $r < p-1$ and $q \geq s-1$, and we will reach a contradiction. Notice that since $(p-1)(s-1) > qr$, both conditions $r \geq p-1$ and $q \geq s-1$ can not hold simultaneously. The remaining case $r \geq p-1$, $q < s-1$ is treated in the same way.

Consider first $q > s-1$. With the same notation as in §3.2, we have $\alpha_n \to \alpha < 0$ and $\beta_n > 0$. Since $\alpha_0 > 0$, we can choose n so that $\alpha_n > 0$, $\alpha_{n+1} < 0$. Thus $\beta_{n-1} q < 2$ and $\beta_n q > 2$. Also, recall from (3.3) that $v \geq a_{n+1} d^{-\beta_n}$ in Ω, and thus

$$\Delta u \geq a_{n+1}^q d^{-\beta_n q} u^p \qquad \text{in } \Omega.$$

According to Theorem 7 in [7], this implies that v is bounded. Actually, the proof is similar to the one given below to prove nonexistence of solutions with the boundary conditions (SF) when $r \geq p-1$, and therefore will not be given.

Now assume $q = s-1$. The iteration argument in §3.2 makes full sense, but $\alpha = 0$. Also, thanks to (3.3) and (3.4), we obtain $v \geq a_{n+1} d^{-\beta_n}$ in Ω, where

$$a_{n+1} = a_n^\delta A_{p,\beta_{n-1}q}^{-\frac{r}{s-1}} B_{s,\alpha_n r},$$

and $\delta = \frac{qr}{(p-1)(s-1)} < 1$. But $\beta_{n-1} q \to 2$ as $n \to +\infty$, and so Lemma 7 implies $A_{p,\beta_{n-1}q} \to 0$, while $B_{s,\alpha_n r}$ is bounded away from zero. In particular, for every $K > 0$ there exists n_0 such that

$$a_{n+1} \geq K a_n^\delta, \quad n \geq n_0.$$

This readily gives $\liminf_{n\to+\infty} a_{n+1} \geq K^{\frac{1}{1-\delta}}$, and since K is arbitrary $\lim_{n\to+\infty} a_{n+1} = +\infty$. But then $v = +\infty$, which is not possible, and no solution exists in this case. □

Nonexistence for the boundary conditions (SF). We will now undertake the nonexistence of positive solutions to (P) with the boundary conditions $u = +\infty$, $v = \mu$ on $\partial\Omega$, and $r \geq p-1$. Notice that u verifies

$$\begin{cases} \Delta u = v^q u^p & \text{in} \quad \Omega \\ u = +\infty & \text{on} \quad \partial\Omega, \end{cases}$$

with v^q bounded and bounded away from zero. Then, it is well-known (cf. [3]) that $u \sim Cd^{-\frac{2}{p-1}}$ as $d \to 0$. We are showing next, following the ideas of Theorem 7 in [7], that this gives rise to a contradiction with $v = \mu$ on $\partial\Omega$ when $r \geq p-1$.

Indeed, fix $x_0 \in \partial\Omega$. In a small neighbourhood of x_0 (relative to Ω) we have

$$v(x) \leq \mu + \varepsilon, \quad u(x)^r \geq K d(x)^{-\gamma} ,$$

for $\varepsilon > 0$ small, $K > 0$ and $\gamma = 2r/(p-1) \geq 2$. Choose an arbitrary sequence $\{x_k\} \subset \Omega$, $x_k \to x_0$. Then $B(x_k, d_k) \subset \Omega$, where $d_k = d(x_k)$. Since $d(x) \leq 2d_k$ in $B(x_k, d_k)$,

$$\begin{cases} \Delta v \geq K(2d_k)^{-\gamma} c^s & \text{in} \quad B(x_k, d_k) \\ v \leq \mu + \varepsilon & \text{on} \quad \partial B(x_k, d_k) , \end{cases}$$

for $c = \inf_\Omega v > 0$. Denoting $v_k(x) = v(x_k + d_k x)$ we find that

$$\begin{cases} \Delta v_k \geq 2^{-\gamma} K d_k^{2-\gamma} c^s & \text{in} \quad B(0,1) \\ v_k \leq \mu + \varepsilon & \text{on} \quad \partial B(0,1) . \end{cases}$$

Let ϕ be the unique solution to $\Delta\phi = 1$ in $B(0,1)$, $\phi = 0$ on $\partial B(0,1)$ (notice that $\phi < 0$ in $B(0,1)$). The maximum principle implies $v_k \leq \mu + \varepsilon + 2^{-\gamma} K d_k^{2-\gamma} c^s \phi$, in $B(0,1)$, and we deduce

$$v(x_k) = v_k(0) \leq \mu + \varepsilon + 2^{-\gamma} K d_k^{2-\gamma} c^s \phi(0) . \tag{3.7}$$

If $\gamma > 2$, we obtain $\lim_{k\to+\infty} v(x_k) = -\infty$, which is impossible. If $\gamma = 2$, letting $k \to +\infty$ and then $\varepsilon \to 0$, we have $\mu \leq \mu + 2^{-\gamma} K c^s \phi(0)$ which is also impossible, since $\phi(0) < 0$. □

Remarks 3. a) It is easily checked that the proof of nonexistence of positive solutions to (P) with the boundary conditions (SF) when $r \geq p-1$ does not depend on the sign of $(p-1)(s-1)-qr$.

b) Let (u_μ, v_μ) be the unique solution to (P) with the boundary conditions $u=+\infty$, $v=\mu$ on $\partial\Omega$. When $q<s-1$, it is not hard to show that as $\mu \to +\infty$, $u_\mu \to u$, $v_\mu \to v$, where (u,v) is the unique solution to (P) with $u=v=+\infty$. It would be interesting to ascertain the asymptotic behaviour of (u_μ, v_μ) in the complementary regime $q \geq s-1$. We believe that $u_\mu \to 0$ uniformly on compact subsets of Ω and $v_\mu \to +\infty$ uniformly on $\overline{\Omega}$ as $\mu \to +\infty$. However, it seems difficult to obtain a proof of this fact, although the reason why this should happen is rather clear: there is no solution to (P) with the conditions $u=v=+\infty$ in this case.

Unable to cover the general case, we content ourselves to prove it when Ω is a ball B of radius R (this will be used in Section 4.2). In this case, the solutions are radial, and can be characterized by their minima, which is achieved at $r=0$. Moreover, $u_\mu(0)$ is decreasing while $v_\mu(0)$ is increasing. Assume $v_\mu(0)$ is bounded. Then $u_\mu(0) \to u_0 \geq 0$ and $v_\mu(0) \to v_0 > 0$. By continuity with respect to initial data, it follows that, as $\mu \to +\infty$, $u_\mu \to u$ and $v_\mu \to v$, uniformly on compacts of $[0,R)$, where u and v solve

$$\begin{cases} (r^{N-1}u')' = r^{N-1}u^p v^q & \text{in } (0,R) \\ (r^{N-1}v')' = r^{N-1}u^r v^s & \text{in } (0,R) \\ u(0)=u_0, \ v(0)=v_0 \\ u'(0)=v'(0)=0. \end{cases} \tag{3.8}$$

Also $v(R)=+\infty$, since v_μ is increasing in μ and $v_\mu(R)=\mu$. If $u_0>0$, (u,v) will be a positive solution to (P) with $v(R)=+\infty$, and $u(R)$ either finite or infinite. This is a contradiction with $q \geq s-1$. If $u_0=0$, we just note that $(r^{N-1}u')' = r^{N-1}(u^{p-1}v^q)u$ in $(0,R)$ with $u(0)=u'(0)=0$ implies $u \equiv 0$, and hence $v \equiv v_0$, which is not possible (observe that we can not directly appeal to uniqueness of ode's, since $r<1$ is possible, and the right hand-side in (3.8) would not be locally Lipschitz). In either case we obtain that $v_\mu(0)$ can not be bounded, and hence $v_\mu \to +\infty$ uniformly in $\overline{B}_R$.

Finally, since $\Delta u_\mu \geq (\inf v_\mu)^q u_\mu^p$ in B, Lemma 8 gives $u_\mu \leq (\inf v_\mu)^{-\frac{q}{p-1}} U_{p,0} \to 0$ uniformly on compact subsets of B.

4. THE CRITICAL CASE $(p-1)(s-1)=qr$

We focus our attention now on the critical case $(p-1)(s-1)=qr$. It can be easily checked that the system (P) with the boundary conditions (F) presents exactly the same features as in the subcritical case. Hence we need only consider the other boundary conditions (I) and (SF).

4.1. Boundary conditions (I)

Proof of Theorem 3 (ii). Let us begin by proving that $r=p-1$, $q=s-1$ is necessary for the existence of positive solutions. Assume $r<p-1$, and thus $q>s-1$, and let (u,v) be a positive solution. By means of the iterative procedure in §3.2, we obtain (see (3.3) and (3.4))

$$\begin{aligned} u &\le a_n^{-\frac{q}{p-1}} A_{p,\beta_{n-1}q} d^{-\alpha_n} \\ v &\ge a_{n+1} d^{-\beta_n} \end{aligned}$$

where $\beta_n = 2(p-1-r)/(p-1)(s-1) + \beta_{n-1}$. Hence $\beta_n \to +\infty$ and $\alpha_n \to -\infty$. Let n be the minimum positive integer such that $\beta_n q \ge 2$. We deduce $\Delta u \ge a_{n+1}^q d^{-\beta_n q} u^p$ in Ω, and it follows as in §3.4 that u is bounded. Thus no solution can exist. In the same way we rule out the possibility $r > p-1$, and thus $r = p-1$, $q = s-1$.

To show the existence in this case is simpler. We look for a solution with $u = v$, and we find that u has to satisfy

$$\begin{cases} \Delta u = u^{p+s-1} & \text{in} \quad \Omega \\ u = +\infty & \text{on} \quad \partial\Omega, \end{cases}$$

that is $u = v = U_{p+s-1,0}$, using the notation introduced in Section 2 (notice that $p+s-1 > 1$, since $p > 1$, $s > 1$). Is is easy to show that $(tU_{p+s-1,0}, t^{-\delta} U_{p+s-1,0})$ is a solution for $t > 0$ and $\delta = (p-1)/(s-1)$. □

Proof of Theorem 4 (i). We now consider the radial case $\Omega = B$. Let (u, v) be a radial positive solution to (P) with $u = v = +\infty$ on ∂B. Denoting $a(r) = u^{p-1} v^{s-1}$, we observe that both u and v are solutions to the linear equation $\Delta w = a(r) w$. Hence, Theorem 5 in [6] (see Remark 1) implies the existence of $\kappa > 0$ such that

$$\lim_{r \to R-} \frac{u(r)}{v(r)} = \kappa \ .$$

We claim that $u = \kappa v$. Indeed, let $k > 1$ and assume $B_0 := \{u > k\kappa v\}$ is nonempty. Since $\Delta u = a(r) u > k a(r) \kappa v = \Delta(k\kappa v)$ in B_0 and $u = k\kappa v$ on ∂B_0, the maximum principle implies $u \le k\kappa v$ in B_0, which is impossible. Thus $B_0 = \emptyset$ and $u \le k\kappa v$. Letting $k \to 1$, we obtain $u \le \kappa v$ and a similar argument shows $u = \kappa v$. It is now easy to conclude that $u = \kappa^{\frac{s-1}{p+s-1}} U_{p+s-1,0}$, $v = \kappa^{-\frac{p-1}{p+s-1}} U_{p+s-1,0}$. The boundary behaviour of the solutions, equations (1.4), follows from Lemma 6. □

4.2. Boundary conditions (SF)

Proof of Theorem 3 (iii). As already quoted before (cf. Remark 3 a)) the proof of necessity of the condition $r < p-1$ in §3.4 also holds in the critical case. On the other hand, estimates (1.2) and (1.3) also remain valid (cf. Remark 2). In particular, Lemma 10 provides with uniqueness. These observations prove the Theorem. □

Proof of Theorem 4 (ii). We only need to show existence. Fix $\varepsilon > 0$ such that $q > s + \varepsilon - 1$, and consider the unique solution (u_μ, v_μ) to

$$\begin{cases} \Delta u = u^p v^q & \text{in} \quad \Omega \\ \Delta v = u^r v^{s+\varepsilon} & \text{in} \quad \Omega \\ u = +\infty, \ v = \mu & \text{on} \quad \partial\Omega. \end{cases} \tag{4.1}$$

Problem (4.1) has a unique solution thanks to Theorem 1 (iii), since $(p-1)(s+\varepsilon-1) > qr$. According to Remark 3 b), $\inf v_\mu \to +\infty$ as $\mu \to +\infty$. Choose μ large enough so that $\inf v_\mu \geq 1$. It is then clear that (u_μ, v_μ) is a supersolution to (P). It is not hard to show that $(\mu^{-\frac{q}{p-1}} U_{p,0}, \mu)$ is a subsolution. Let us see that they are ordered. Indeed, notice that $\Delta v_\mu \geq 0$ with $v = \mu$ on $\partial\Omega$ implies $v \leq \mu$. Thus $\Delta u_\mu \leq \mu^q u_\mu^q$, and Lemma 8 implies $u_\mu \geq \mu^{-\frac{q}{p-1}} U_{p,0}$. Hence, Theorem A.3 in the Appendix gives the existence of at least a positive solution (u, v) to (P) with the boundary conditions (SF), provided μ is large enough. But then, as stated in Theorem 3 (iii), problem (P) has a solution (u, v) with $u = +\infty$, $v = \mu$ on $\partial\Omega$ for every $\mu > 0$, by means of a simple scaling. □

Proof of Theorem 5. Denote $\Omega_\rho = \rho\Omega$, and let V_μ be the unique solution to

$$\begin{cases} \Delta V = U_{p,0}^r V^s & \text{in} \quad \Omega_\rho \\ V = \mu & \text{on} \quad \partial\Omega_\rho \end{cases}$$

(of course $U_{p,0}$, V_μ depend on ρ, but we are not making explicit this dependence for the moment). It is not hard to check that $(U_{p,0}, V_\mu)$ is a supersolution provided that $\inf V_\mu \geq 1$, and that (as in the proof of Theorem 4 (ii)) $(\mu^{-\frac{q}{p-1}} U_{p,0}, \mu)$ is a subsolution such that $U_{p,0} \geq \mu^{-\frac{q}{p-1}} U_{p,0}$, $V_\mu \leq \mu$ in Ω for $\mu \geq 1$. Thus, for $\inf V_\mu \geq 1$ Theorem A.3 in the Appendix implies the existence of a solution with $u = +\infty$, $v = \mu$ on $\partial\Omega$. As before, there is a solution for every $\mu > 0$.

Hence we only need to show that $\inf V_\mu \geq 1$ if ρ is sufficiently small. It is known (see [7]) that, as $\mu \to +\infty$, V_μ converges to the unique solution to

$$\begin{cases} \Delta V = U_\rho^r V^s & \text{in} \quad \Omega_\rho \\ V = +\infty & \text{on} \quad \partial\Omega_\rho, \end{cases}$$

which will be denoted by V_ρ (with a slight change of notation, we are denoting by U_ρ the unique solution to $\Delta u = u^p$ in Ω_ρ, $u = +\infty$ on $\partial\Omega_\rho$). It is straightforward to check that $U_\rho(x) = \rho^{-\frac{2}{p-1}} U_1(x/\rho)$. It then follows that $V_\rho(x) = \rho^{-\frac{2(p-1-r)}{(p-1)(s-1)}} V_1(x/\rho)$, and it becomes clear that we can achieve $\inf V_\rho > 1$ if ρ is small enough. Thus $\inf V_\mu > 1$ if μ is large and ρ is small enough, and the Theorem is proved. □

5. Some related problems

We consider in this section some variants of the blow-up problem (P) which can be studied with the same techniques. For the sake of simplicity, we only treat the subcritical case and the boundary conditions (I), but the rest of the cases can be treated similarly.

5.1. *Problem (P) with weights*

A problem which is closely related to (P) is the following systems involving weights:

$$\begin{cases} \Delta u = a(x) u^p v^q \\ \Delta v = b(x) u^r v^s \end{cases} \quad \text{in } \Omega \tag{P1}$$

where a, b are Hölder–continuous positive functions, and $p, s > 1$, $(p-1)(s-1) > qr$. The weights a, b can be unbounded, but a growth near the boundary must be prescribed. Concretely, we are assuming $a(x) \sim C_1 d(x)^{\gamma_1}$, $b(x) \sim C_2 d(x)^{\gamma_2}$ when $d(x) \to 0$, for some positive constants C_1, C_2 and real numbers γ_1, γ_2. With a similar approach as in Section 3, we can prove:

Theorem 11. *Problem (P1) admits a positive solution (u, v) with $u = v = +\infty$ on $\partial\Omega$ if and only if $\gamma_1, \gamma_2 > -2$ and*

$$\frac{q}{s-1} < \frac{2+\gamma_1}{2+\gamma_2} < \frac{p-1}{r}.$$

This solution is unique and it verifies:

$$\lim_{x\to x_0} d(x)^{\alpha} u(x) = \left(\frac{(\alpha(\alpha+1))^{s-1} C_2^q}{(\beta(\beta+1))^q C_1^{s-1}} \right)^{\frac{1}{(p-1)(s-1)-qr}}$$

$$\lim_{x\to x_0} d(x)^{\beta} v(x) = \left(\frac{(\beta(\beta+1))^{p-1} C_1^r}{(\alpha(\alpha+1))^r C_2^{p-1}} \right)^{\frac{1}{(p-1)(s-1)-qr}}$$

$$\lim_{x\to x_0} d(x)^{\alpha+1} \nabla u(x) \nu(x_0) = \alpha \left(\frac{(\alpha(\alpha+1))^{s-1} C_2^q}{(\beta(\beta+1))^q C_1^{s-1}} \right)^{\frac{1}{(p-1)(s-1)-qr}}$$

$$\lim_{x\to x_0} d(x)^{\beta+1} \nabla v(x) \nu(x_0) = \beta \left(\frac{(\beta(\beta+1))^{p-1} C_1^r}{(\alpha(\alpha+1))^r C_2^{p-1}} \right)^{\frac{1}{(p-1)(s-1)-qr}}$$

for every $x_0 \in \partial\Omega$, where $\nu(x_0)$ stands for the exterior unit normal to $\partial\Omega$ at x_0 and

$$\alpha = \frac{(\gamma_1+2)(s-1) - (\gamma_2+2)q}{(p-1)(s-1)-qr}, \quad \beta = \frac{(\gamma_2+2)(p-1) - (\gamma_1+2)r}{(p-1)(s-1)-qr}.$$

5.2. More general nonlinearities in (P)

We are showing next that a slightly larger class of problems can be studied with the same type of arguments. Thus we are considering the system

$$\begin{cases} \Delta u = f(u, v) \\ \Delta v = g(u, v) \end{cases} \quad \text{in } \Omega \tag{P2}$$

where f and g are locally Lipschitz functions which roughly speaking behave like powers. More precisely, we are imposing on f and g the following conditions: there exist positive numbers $p, s > 1$, $q, s > 0$ such that

(i) f is positive and increasing as a function of v. Moreover, f/u^p is increasing and f/v^q decreasing as functions of u and v, respectively.

(ii) g is positive and increasing as a function of u. Moreover, g/v^s is increasing and g/u^r decreasing as functions of v and u, respectively.

(iii) $(p-1)(s-1) > qr$.

We remark that these conditions are enough for an analogue of Lemma 10 to hold. However, they do not seem enough for existence nor for obtaining the boundary behaviour of solutions. We need to prescribe instead an asymptotic growth involving the same powers as in (i)-(iii).

Theorem 12. *Assume f and g verify (i), (ii) and (iii), and the following growth conditions:*

(a) $f(1,t) \geq Bt^q$, $f(t,1) \leq At^p$ for large t;

(b) $g(1,t) \leq At^s$, $g(t,1) \geq Bt^r$ for large t;

for some positive constants A and B. Then problem (P2) has at least a positive solution (u,v) such that $u = v = +\infty$ on $\partial\Omega$. If moreover

$$\lim_{s,t\to+\infty} \frac{f(s,t)}{s^p t^q} = \lim_{s,t\to+\infty} \frac{g(s,t)}{s^r t^s} = 1,$$

then (u,v) is unique, and the limits (1.1) in Theorem 2 hold.

We finally consider a particular case where conditions (i)–(iii) and (a),(b) can be better understood. More precisely, we take special functions f, g of the form $f(u,v) = u^p v^q f_1(u) g_1(v)$, $g(u,v) = u^r v^s f_2(u) g_2(v)$. Then:

Corollary 13. *Assume f_1, f_2, g_1, g_2 are locally Lipschitz functions in $(0,+\infty)$ such that*

(1) f_1 is increasing, positive and $\lim_{t\to+\infty} f_1(t) = 1$, while g_1 is decreasing, positive, $\lim_{t\to+\infty} g_1(t) = 1$ and $g_1(t)t^q$ is increasing;

(2) f_2 is decreasing, positive, $\lim_{t\to+\infty} f_2(t) = 1$ and $f_2(t)t^r$ is increasing while g_2 is increasing, positive and $\lim_{t\to+\infty} g_2(t) = 1$;

(3) $(p-1)(s-1) > qr$.

Then there exists a unique positive solution (u,v) to the system

$$\begin{cases} \Delta u = u^p v^q f_1(u) g_1(v) \\ \Delta v = u^r v^s f_2(u) g_2(v) \end{cases} \quad \text{in } \Omega$$

such that $u = v = +\infty$ on $\partial\Omega$. Moreover, (u,v) verifies the estimates (1.1) of Theorem 2.

Remark 4. As concrete choices for the functions f_1, g_2 we can take $t^\delta/(1+t^\delta)$, for $\delta > 0$ or $\tanh t$. Likewise, for f_2 and g_1 valid functions are $1 + e^{-t}$ or $1 + t^{-\delta}$, $\delta > 0$.

APPENDIX

We collect in this Appendix some results about the method of sub and supersolutions for the system (P)

$$\begin{cases} \Delta u = u^p v^q \\ \Delta v = u^r v^s \end{cases} \quad \text{in } \Omega$$

which need to be used along the paper. We recall that since (P) is of competitive type, $(\underline{u},\underline{v})$ is a subsolution provided $\Delta\underline{u} \geq \underline{u}^p\underline{v}^q$ and $\Delta\underline{v} \leq \underline{u}^r\underline{v}^s$ in Ω. As always, a supersolution $(\overline{u},\overline{v})$ is defined by reversing the inequalities. Because the exponents p,q,r,s need not be integers, all sub and supersolutions are assumed to be nonnegative.

We begin with the case of finite boundary conditions (F), that is $u = f(x)$, $v = g(x)$, where f,g are continuous and positive functions defined on $\partial\Omega$. Since this case is rather standard, we are omitting the proof (see [29]).

Theorem A. 1. *Assume $(\underline{u},\underline{v})$ is a subsolution and $(\overline{u},\overline{v})$ a supersolution to (P) with $\underline{u} \leq f(x) \leq \overline{u}$, $\underline{v} \geq g(x) \geq \overline{v}$ on $\partial\Omega$ and $\underline{u} \leq \overline{u}$, $\underline{v} \geq \overline{v}$ in Ω. Then problem (P) has at least a solution (u,v) with $\underline{u} \leq u \leq \overline{u}$, $\underline{v} \geq v \geq \overline{v}$ in Ω and $u = f(x)$, $v = g(x)$ on $\partial\Omega$.*

We now come to the case of infinite boundary conditions (I). The next result is a consequence of Theorem A.1, with the procedure used in Lemma 4 of [15] (see also Lemma 1 in [14]).

Theorem A. 2. *Assume $(\underline{u},\underline{v})$ is a subsolution and $(\overline{u},\overline{v})$ a supersolution to (P) with $\underline{u} = \overline{u} = \underline{v} = \overline{v} = +\infty$ on $\partial\Omega$ and $\underline{u} \leq \overline{u}$, $\underline{v} \geq \overline{v}$ in Ω. Then problem (P) has at least a solution (u,v) with $\underline{u} \leq u \leq \overline{u}$, $\underline{v} \geq v \geq \overline{v}$ in Ω. In particular, $u = v = +\infty$ on $\partial\Omega$.*

Proof. Let $\delta > 0$ and in $\Omega_\delta := \{x \in \Omega : \ d(x) > \delta\}$ consider the problem

$$\begin{cases} \Delta u = u^p v^q \\ \Delta v = u^r v^s \end{cases} \quad \text{in } \Omega_\delta$$

with boundary data $u = \underline{u}$, $v = \underline{v}$. Then by Theorem A.1, there exists a solution (u_δ, v_δ) such that $\underline{u} \leq u_\delta \leq \overline{u}$, $\underline{v} \geq v_\delta \geq \overline{v}$ in Ω_δ. This in turn gives bounds for u_δ and v_δ, and it is standard to conclude the existence of a sequence $\delta_n \to 0$ such that $u_{\delta_n} \to u$, $v_{\delta_n} \to v$ in $C^2_{\text{loc}}(\Omega)$. It follows that (u,v) is a solution to (P) and $\underline{u} \leq u \leq \overline{u}$, $\underline{v} \geq v \geq \overline{v}$ in Ω. □

We finally consider the case in which $u = +\infty$ but $v = \mu$ on $\partial\Omega$, for a positive real number μ. We remark that in this situation, unlike the previous case, an estimate on the behaviour of the supersolution near $\partial\Omega$ is essential. The existence proof will be based on a "nonlinear version" of the method of sub and supersolution.

Theorem A. 3. *Assume $(\underline{u},\underline{v})$ is a subsolution and $(\overline{u},\overline{v})$ a supersolution to (P) with $\underline{u} = \overline{u} = +\infty$, $\underline{v} \geq \mu \geq \overline{v}$ on $\partial\Omega$ and $\underline{u} \leq \overline{u}$, $\underline{v} \geq \overline{v}$ in Ω. Assume moreover that $\overline{u} \leq Cd(x)^{-\gamma}$ for some positive constant C and $\gamma < 2/r$. Then problem (P) has at least a solution (u,v) with $\underline{u} \leq u \leq \overline{u}$, $\underline{v} \geq v \geq \overline{v}$ in Ω and $u = +\infty$, $v = \mu$ on $\partial\Omega$.*

Proof. Since $\underline{v}$ is a bounded and positive function in $\overline{\Omega}$, the problem

$$\begin{cases} \Delta u = \underline{v}^q u^p & \text{in} \quad \Omega \\ u = +\infty & \text{on} \quad \partial\Omega \end{cases}$$

has a unique positive solution, which we denote by u_1. Moreover, $\Delta\underline{u} \geq \underline{v}^q\underline{u}^p$ in Ω, so an easy variant of Lemma 8 gives $\underline{u} \leq u_1$. Likewise, $\Delta\overline{u} \leq \overline{v}^q\overline{u}^p \leq \underline{v}^q\overline{u}^p$ in Ω, and so $\overline{u} \geq u_1$. We now define v_1 as the unique solution to

$$\begin{cases} \Delta v = u_1^r v^s & \text{in} \quad \Omega \\ v = \mu & \text{on} \quad \partial\Omega \ , \end{cases} \tag{A.1}$$

which exists thanks to Lemma 3 in [7] (see also Remark (1a) there), since $0 < u_1 \le C d(x)^{-\gamma r}$ with $\gamma r < 2$. It is not hard to see that $\underline{v} \ge v_1 \ge \overline{v}$ in Ω. We continue this procedure and define u_2 as the unique solution to

$$\begin{cases} \Delta u = v_1^q u^p & \text{in} \quad \Omega \\ u = +\infty & \text{on} \quad \partial\Omega \ . \end{cases} \tag{A.2}$$

Then it follows as before that $\underline{u} \le u_2 \le \overline{u}$ in Ω. In addition, $\Delta u_1 = \underline{v}^q u_1^p \ge v_1^q u_1^p$, and hence $u_1 \le u_2$.

We can recursively define v_n as the unique solution to (A.1) replacing u_1 by u_n, and u_n as the unique solution to (A.2) replacing v_1 by v_{n-1}. In this way, we obtain two sequences $\{u_n\}$ and $\{v_n\}$ such that u_n is increasing, v_n is decreasing, $\underline{u} \le u_n \le \overline{u}$ and $\underline{v} \ge v_n \ge \overline{v}$ in Ω. It is standard to conclude that there exists a subsequence (labelled again by u_n and v_n) such that $u_n \to u$, $v_n \to v$ in $C^2_{\text{loc}}(\Omega)$, where (u, v) is a solution to (P) and $\underline{u} \le u \le \overline{u}$, $\underline{v} \ge v \ge \overline{v}$ in Ω. As a consequence, $u = +\infty$ on $\partial\Omega$, but it is not immediate that $v = \mu$ on $\partial\Omega$. To ensure this, we need to use the following important result, which can be found in [18] (cf. Theorem 4.9 and Exercise 4.6 there).

Lemma A. 4. *Let Ω be a C^2 bounded domain of $\mathbb{R}^N$ and $u \in C^2(\Omega)$ a solution to the problem $\Delta u = f$ in Ω with $u = 0$ on $\partial\Omega$, where $f \in C(\Omega)$ is such that $\sup_\Omega d(x)^\gamma |f(x)| < +\infty$ for some $1 < \gamma < 2$. Then there exists a positive constant C depending only on Ω and γ such that*

$$\sup_\Omega d(x)^{\gamma-2} |u(x)| \le C \sup_\Omega d(x)^\gamma |f(x)|.$$

Now let $w_n = \mu - v_n$. Then $\Delta w_n = -u_n^r (\mu - w_n)^s$ in Ω, $w_n = 0$ on $\partial\Omega$. Since $u_n \le C d^{-\gamma r}$ in Ω and w_n is uniformly bounded, Lemma A.4 implies that $|w_n| \le C' d^{2-\gamma r}$. This inequality also holds for $w = \mu - v$, and thus $w = 0$ on $\partial\Omega$, that is $v = \mu$ on $\partial\Omega$. □

Remarks 5. a) Theorem A.3 continues to be valid if a more general boundary condition $v = \mu(x)$, with $\mu(x)$ continuous and positive on $\partial\Omega$, is imposed. Indeed, we only have to notice that if z is the unique harmonic function such that $z = \mu(x)$ on $\partial\Omega$, and we set $w = z - v$, then it follows as above that $w = 0$ on $\partial\Omega$, and thus $v = \mu(x)$ on $\partial\Omega$.

b) Theorem A.3 (respectively Lemma A.4) does not hold if the condition $\gamma < 2/r$ (resp. $\gamma < 2$) is violated.

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