Eigenvalue Analysis for the p-Laplacian under Convective Perturbation *

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1 Introduction

In this paper we will undertake the analysis of the Dirichlet eigenvalue problem for the one-dimensional p-Laplacian when perturbed by convection terms on the unit interval $I = \{x : 0 < x < 1\}$. More precisely, we will be concerned with the problem,

$$\begin{cases} -(\psi_p(u'))' - c\ \psi_p(u') = \lambda\psi_p(u), \quad 0 < x < 1 \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

where, for $p > 1$, $\psi_p(z) := |z|^{p-2}z$ will stand for the odd extension of z^{p-1} while c will be assumed, without lost of generality, positive (the change $x \to -x$ reduces the case $c < 0$ to the present one).

A number $\lambda \in \mathbb{R}$ such that (1) possesses a nontrivial solution u is called an eigenvalue of (1), meanwhile any such a nontrivial solution u is said to be the associated eigenfunction. As usual in problems concerning the p-Laplacian, solutions to (1) will be understood in the weak Sobolev space $W_0^{1,p}(I)$ sense, so that u satisfies:

$$\int_0^1 \psi_p(u')\varphi'\ dx - c\int_0^1 \psi_p(u')\varphi\ dx = \lambda\int_0^1 \psi_p(u)\varphi\ dx \quad , \quad \forall\varphi\ \in W_0^{1,p}(I). \tag{2}$$

The main objetive of this work is to provide a complete and almost explicit description of the eigenvalues and eigenfunctions to (1).

The case where convection is absent ($c = 0$), namely,

$$\begin{cases} -(\psi_p(u'))' = \lambda\psi_p(u), \quad 0 < x < 1 \\ u(0) = u(1) = 0, \end{cases} \tag{3}$$

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has been completely studied recently (cf. [7],[14]). This is in strong contrast with its n-dimensional version,

$$\begin{cases} -\text{div}\,(|\nabla u|^{p-2}\nabla u) &= \lambda\,\psi_p(u) \qquad x\in\Omega \\ u &= 0 \qquad\qquad x\in\partial\Omega\,, \end{cases} \tag{4}$$

$\Omega\subset\mathbb{R}^n$ a bounded domain, where the knowledge of the complete structure of the spectrum is still an open problem (cf. [1], [3], [4], [5], [9], [10], [18]) while a number of partial results (most of them concerning to the first eigenvalue λ_1) has been obtained (cf. [2], [4], [6], [7], [8], [11], [16], [17], [18], [19]). Two specific works keeping a closer relation with the present research are [3], where partial results on the spectrum of a n-dimensional version of (1) are announced, and [20] where a complete Sturm-Liouville theory for the radial version of (4) is developed.

A direct approach to deal with the nonperturbed problem (3) is first showing that all possible eigenvalues are positive (cf. §2). Scaling as $\xi=\lambda^{1/p}x$ and using the homogeneity of the equation such problem can be reduced to study the nodal properties of the solution $\hat{u}=\hat{u}(\xi)$ to

$$\begin{cases} -(\psi_p(u'))'=\psi_p(u), \\ u(0)=0, \quad u'(0)=1\,, \end{cases} \tag{5}$$

with $'=d/d\xi$. The first integral $E(u,u')=(p-1)|u'(x)|^p+|u(x)|^p$ and a few of calculus can be then used to show that the solution $\hat{u}$ to (5) is periodic with period $\tau_p=4\pi(p-1)^{1/p}/(p\sin(\pi/p))$ and has zeros at $\sigma_n=n\tau_p/2$, $n\in\mathbb{Z}$. This shows that the eigenvalues of (3) are $\lambda_n=\sigma_n^p$, $n\in\mathbb{N}$, while all possible eigenfunctions associated to λ_n are multiple of $u_n(x)=\hat{u}(\sigma_n x)$, $0<x<1$. Hence, a complete picture of the features of (3) is obtained.

However, the convective case (1) can not be reduced to explicit integration, unlike what happens when $p=2$ and both (1) and (3) are linear. In that case, making $u=e^{-cx/2}v$ immediately reduces the former to the latter one (the harmonic oscillator) giving rise to the spectrum $\lambda_n=n^2\pi^2+c^2/4$, $n\in\mathbb{N}$, and corresponding eigenfunctions $u_n=e^{-cx/2}\sin n\pi x$. Such a reduction is not possible when $p\neq 2$.

In the present paper we are proceeding to give a finer as possible description of the problem (1) by showing that its main features can be red from the global structure of phase space to the equation,

$$-(\psi_p(u'))'-\gamma\psi_p(u')=\psi_p(u)\,, \tag{6}$$

when the parameter γ runs the range $0\le\gamma<p$.

Accordingly, the results and organization of the work rely upon a detailed analysis of (6). In Section 1 the smoothness of weak solutions to (6) is discussed (cf. Lemma 2.1) while the existence and uniqueness of global solutions in $\mathbb{R}^+$ to the Cauchy problem associated to (6) is obtained (cf. Theorem 2.3). It is shown in Section 3 that the phase space of the natural one-dimensional system associated to (6) exhibits, with regard to the parameter γ and the critical value $\gamma^*=p$, the same general features concerning structure of orbits (center, focus , node transitions) and convergence to the trivial solution. In particular, the characteristic equation to (6) is introduced (cf. Theorem 3.2). The eigenvalues and eigenfunctions to (1) are studied in Section 4 where a description of the simplicity, asymptotic distribution both regard $n\to+\infty$ and $c\to+\infty$ of the eigenvalues $\lambda_n(c)$ and qualitative properties of the eigenfunctions ϕ_n are given (cf. Theorem 4.1).

For later use it will be always designated by p', $p>1$, the Hölder conjugate of p, $1/p+1/p'=1$. It is very convenient to remark that $\psi_{p'}$ defines the inverse function of ψ_p.

2 The initial value problem.

In view of the proof of next Lemma 2.1 below, it will turn out that any possible eigenfunction $u \in W_0^{1,p}(I)$ will further assumed to satisfy $u', \psi_p(u') \in C^1(I)$, verifying the the equation

$$-(\psi_p(u'))' - c\ \psi_p(u') = \lambda\psi_p(u), \tag{7}$$

in a classical sense or equivalently,

$$-(e^{cx}\psi_p(u'))' = \lambda e^{cx}\psi_p(u). \tag{8}$$

Thus, a direct integration of (8) shows that if λ is an eigenvalue of (1) and u is an eigenfunction, then

$$\int_0^1 e^{cx}|u'|^p dx = \lambda \int_0^1 e^{cx}|u|^p dx\,.$$

This shows that all possible eigenvalues λ to (1) are positive.

As a consequence, the initial value problem for (7) reduces, after the scale change $\xi = \lambda^{\frac{1}{p}}x$, to the corresponding one for the equation (6), i. e.,

$$-(\psi_p(u'))' - \gamma\ \psi_p(u') = \psi_p(u),$$

where the parameter $\gamma = c\lambda^{-\frac{1}{p}}$ is positive. On the other hand, we will be interested in those solutions to (6) partially or wholly defined in the interval $[0,+\infty)$. More precisely, given an interval $J \subset [0,+\infty)$, the pair (u, J) is said to be a local weak solution of (6) if $u \in W^{1,p}_{loc}(J)$ and the following identity is fulfilled:

$$\int_J \psi_p(u')\varphi'\ dx - \gamma \int_J \psi_p(u')\varphi\ dx = \int_J \psi_p(u)\varphi\ dx \quad , \quad \forall\varphi \in C_0^1(J). \tag{9}$$

First of all, we have the following regularity result.

Lemma 2.1 *Let (u, J) be a nontrivial local weak solution of (6). Then u is of class C^2 for $1 < p \leq 2$, and of class C^2 in $J \setminus \{x \in J : u'(x) = 0\}$ in the case $p > 2$, where $\{x \in J : u'(x) = 0\}$ is, in addition, a discrete set in J. Furthermore, equation (6) is point-wise satisfied in J.*

Proof.

Let $u \in W^{1,p}(J)$ be a nontrivial local weak solution of (6). As $u \in W^{1,p}(J)$, it follows that u is absolutely continuous in J and therefore differentiable a.e. $x \in J$ (see [13]). In fact $u, u' \in L^p(J)$, so $\psi_p(u), \psi_p(u') \in L^{p'}(J)$. On the other hand as a consequence of (9), $-\gamma\psi_p(u') - \psi_p(u)$ represents the weak derivative of $\psi_p(u')$, and therefore $\psi_p(u') \in W^{1,p'}(J)$. As above, $\psi_p(u') \in C(J)$ and

$$\psi_p(u'(x)) = \psi_p(u'(x_0)) - \int_{x_0}^x \{\gamma\psi_p(u'(s)) + \psi_p(u(s))\}\, ds \qquad \forall x_0, x \in J. \tag{10}$$

¿From the relation (10), $\psi_p(u') \in C^1(J)$, and

$$u' = \psi_{p'}(\psi_p(u')) \in C(J). \tag{11}$$

Since ψ_p is differentiable in $\mathbb{R}$ for $p \geq 2$ and in $\mathbb{R} \setminus \{0\}$ for $1 < p < 2$, then both (11) and the fact that $\psi_p(u') \in C^1(J)$ imply that $u \in C^2(J)$ for $1 < p \leq 2$ and $u \in C^2(J \setminus \{x \in J : u'(x) = 0\})$ for $p > 2$.

Let us show now that $J \setminus \{x \in J : u'(x) = 0\}$ is discrete. By contradiction suppose that $J \setminus \{x \in J : u'(x) = 0\}$ is not discrete. Then there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset J$ such that $u'(x_n) = 0$ together with $x_n \to x_0$ and $x_0 \in J$. By continuity $u'(x_0) = 0$. If we define,

$$h(x) := e^{\gamma x} \ \psi_p(u'(x)),$$

then $h \in C^1(J)$ and $h(x_n) = h(x_{n+1}) = 0$. By Rolle's Theorem there exists $\xi_n \in (x_n, x_{n+1})$ such that $h'(\xi_n) = 0$.

On the other hand,

$$h'(x) = \gamma \ e^{\gamma x} \ \psi_p(u'(x)) + e^{\gamma x}(\psi_p(u'(x)))',$$

and therefore

$$h'(x) = - \ e^{\gamma x} \ \psi_p(u(x)).$$

In particular, $h'(\xi_n) = - \ e^{\gamma \xi_n} \ \psi_p(u(\xi_n)) = 0$, which implies that $u(\xi_n) = 0$ and consequently $u(x_0) = 0$. We are next showing that $u(x_0) = u'(x_0) = 0$ implies that $u \equiv 0$ in J. In fact, from the fact that x_0 is a double zero for u we obtain from (9),

$$u(x) = -\int_{x_0}^{x} \psi_{p'} \left(\int_{x_0}^{s} e^{\gamma(t-s)} \psi_p(u(t)) \ dt \right) \ ds \qquad x \in J \ . \tag{12}$$

If $\delta > 0$ are chosen so that $\{|x - x_0| < \delta\}$ fall into J for, say $\delta < \delta_0$, we find from (12) that

$$|u(x)| \le C(\delta) \sup_{|s-x_0|<\delta} |u(s)| \qquad \text{for} \qquad |x - x_0| < \delta \, ,$$

where $C(\delta) = O(\delta^2)$ as $\delta \to 0+$. Thus $u \equiv 0$ near $x = x_0$. This means that $\{\xi \in J : u(\xi) = u'(\xi) = 0\}$ is nonempty, open and closed in J and therefore $u \equiv 0$ in J. This contradicts the fact that u is nontrivial. ■

For later use, let us examine now in detail the uniqueness, local and global existence properties of local weak solutions of the initial value problem:

$$\begin{cases} -(\psi_p(u'))' - \gamma \ \psi_p(u') = \psi_p(u) \\ u(0) = u_0 \\ u'(0) = u'_0, \end{cases} \tag{13}$$

where $u_0, u'_0 \in \mathbb{R}$. Remark that the autonomous character of (13) allows us setting the initial position as $x_0 = 0$.

Remark 2.2 Notice that it has already seen that $u \equiv 0$ is the only possible response of (13) to the data $u_0 = u'_0 = 0$. On the other hand, observe that for $1 < p \le 2$ and $u_0 \ne 0$, the existence of a unique local solution is an immediate consequence of the general ode's theory ([15]). Indeed (13) is equivalent to the problem

$$\begin{cases} u' = \psi_{p'}(v) & u(0) = u_0 \\ v' = -\gamma \ v - \psi_p(u) & u'(0) = \psi_p(u'_0) \, , \end{cases} \tag{14}$$

so if $1 < p \le 2$ then $p' \ge 2$, and as $u(0) = u_0 \ne 0$, the general theory guarantees the existence of a unique local solution of class C^2 to (13). The same conclusion holds true for the complementary case $p \ge 2$ provided $u'(0) = u'_0 \ne 0$.

A full account on (13) is contained in our next result (see [20] for an alternative approach).

Theorem 2.3 *The IVP (13) admits for each* $u_0, u_0' \in \mathbb{R}$ *a unique solution in the interval* $[0, +\infty)$*, which is of class* C^2 *for* $1 < p \leq 2$*, meanwhile in the case* $p > 2$ *it is of class* C^2 *except in the discrete set of points where* u' *vanishes.*

PROOF.
Local existence and uniqueness. We will proceed separately in the cases $u_0' = 0$ and $u_0' \neq 0$.

Assume first that $u_0' = 0$. According to Remark 2.2 only the case $p > 2$ need to be studied (obviously it will be supposed that $u_0 \neq 0$). Due to the homogeneity of (13) no generality is lost if we assume $u_0 > 0$. So, let u be any possible local solution of (13). Using (8), u can be written in the integral form

$$u = u_0 - \int_0^x \psi_{p'} \left(\int_0^t e^{\gamma(s-t)} \psi_p(u(s))\, ds \right) dt. \tag{15}$$

Therefore if, for a conveniently chosen $\delta > 0$, we consider the operator $T : C([0,\delta]) \longrightarrow C([0,\delta])$ defined as:

$$T(h)(x) := u_0 - \int_0^x \psi_{p'} \left(\int_0^t e^{\gamma(s-t)} \psi_p(h(s))\, ds \right) dt,$$

it is clear that the fixed points of T characterize the solutions of (13).

Let us see that T admits a unique fixed point by showing that T is a contractive operator in some ball of $C([0,\delta])$. In fact, denoting by $\overline{B}_{\delta_1}(u_0) := \{h \in C([0,\delta]) : |h - u_0| \leq \delta_1\}$ ($|h| = \sup |h(\cdot)|$) and δ_1 small enough, we have

$$|T(h)(x) - u_0| \leq \frac{\delta}{\gamma^{1/(p-1)}} \left(|u_0| + \delta_1\right) \qquad \text{for} \qquad 0 \leq x \leq \delta\,,$$

whereby if we choose $\delta \leq \delta_1 \gamma^{1/(p-1)}/(|u_0| + \delta_1)$ then

$$|T(h)(x) - u_0| \leq \delta_1 \qquad \text{foreach} \qquad 0 \leq x \leq \delta\,.$$

Hence $T(\overline{B}_{\delta_1}(u_0)) \subset \overline{B}_{\delta_1}(u_0)$. On the other hand, for any pair $u_1, u_2 \in \overline{B}_{\delta_1}(u_0)$,

$$|T(u_1)(x) - T(u_2)(x)| = \left| \int_0^x \{\psi_{p'}(\int_0^t e^{\gamma(s-t)} \psi_p(u_1(s))\, ds) - \psi_{p'}(\int_0^t e^{\gamma(s-t)} \psi_p(u_2(s))\, ds)\}\, dt \right|.$$

Thus an application of the Mean Value Theorem shows that

$$|T(u_1)(x) - T(u_2)(x)| = \left| \int_0^x \frac{1}{p-1} |\xi(s)|^{\frac{2-p}{p-1}} \left[\int_0^t e^{\gamma(s-t)} \{\psi_p(u_1(s)) - \psi_p(u_2(s))\} ds \right] dt \right|,$$

where $\xi = \xi(t)$ takes intermediate values to the integrals $\int_0^t e^{\gamma(s-t)} \psi_p(u_i(s))\, ds$, $i = 1, 2$.

Since $p > 2$ and u_0 was chosen positive we arrive at,

$$|T(u_1)(x) - T(u_2)(x)| \leq \frac{\delta}{\gamma^{1/(p-1)} [1 - e^{-\gamma\delta}]^{p-2/p-1}} \left(\frac{u_0 + \delta_1}{u_0 - \delta_1} \right)^{p-2} |u_1 - u_2| \qquad x \in [0, \delta].$$

Hence, we achieve

$$|T(u_1) - T(u_2)| \leq O(\delta^{1/p}) |u_1 - u_2|\,,$$

as $\delta \to 0+$. Thus taking δ small enough we can conclude that T is contractive. Therefore, there exists a unique local solution in this case.

As for the situation $u_0' \neq 0$ the only case that need be considered is $u_0 = 0$ together with $1 < p < 2$ while u_0' can be taken positive. The alternative represenation

$$u = \int_0^x \psi_{p'}\left(e^{-\gamma t}\psi_p(u_0') - \int_0^t e^{\gamma(s-t)}\psi_p(u(s))\,ds\right)\,dt\,, \tag{16}$$

characterizes now any possible local solution $u = u(x)$, $0 \leq x \leq \delta$, $\delta > 0$ small. Since the derivative of u must be taken now into account, it is rather convenient to observe the second side in (16) as an operator $T(u)$ acting in the space $C^1[0,\delta]$. For $0 < \delta_1 < u_0'$ to be fixed we introduce the closed set $X_{\delta,\delta_1} = \{u \in C^1[0,\delta] : |u - h_0| + |u' - h_0'| \leq \delta_1\,, u(0) = 0\,, u'(0) = u_0'\}$, where $h_0(x) = u_0'\,x$. Then, it is easily found that

$$(u_0' - \delta_1)x \leq u(x) \leq (u_0' + \delta_1)x \qquad 0 \leq x \leq \delta\,, \tag{17}$$

for every $u \in X_{\delta,\delta_1}$. Thus,

$$|T(u)'(x) - u_0'| \leq \frac{1}{p-1}|\xi_1(x)|^{\frac{2-p}{p-1}}\left|\int_0^x e^{\gamma s}\psi_p(u(s))\,ds\right| + (1 - e^{-\frac{\gamma}{p-1}x})u_0'\,, \tag{18}$$

where $\xi_1(x) = \psi_p(u_0') - \theta_1(x)\int_0^x e^{\gamma s}\psi_p(u(s))\,ds$ and $0 < \theta_1(x) < 1$. By using (17) we obtain,

$$\xi_1(x) \geq \psi_p(u_0') - \frac{e^{\gamma\delta}}{p}(u_0' + \delta_1)^{p-1}x^p \geq \psi_p(u_0') - \frac{e^{\gamma\delta}}{p}(u_0' + \delta_1)^{p-1}\delta^p > 0\,,$$

provided $\delta > 0$ is small enough. Thus,

$$|T(u)'(x) - u_0'| \leq \frac{e^{\gamma\delta}\delta^p}{p(p-1)}(u_0')^{2-p}(u_0' + \delta_1)^{p-1} + (1 - e^{-\frac{\gamma}{p-1}x})u_0' = O(\delta)\,, \tag{19}$$

as $\delta \to 0+$. Thus, it follows easily from (19) that $|T(u) - h_0| + |T(u)' - u_0'| = O(\delta)$ as $\delta \to 0+$ and T keeps X_{δ,δ_1} invariant provided δ is chosen small.

Let us check now that T is contractive in X_{δ,δ_1}. In fact, for every pair $u_1, u_2 \in X_{\delta,\delta_1}$ we have,

$$T(u_1)'(x) - T(u_2)'(x) = \frac{1}{p-1}|\xi_2(x)|^{\frac{2-p}{p-1}}\int_0^x e^{\gamma(s-x)}(\psi_p(u_1) - \psi_p(u_2))\,ds\,,$$

where $\xi_2(x) = -\int_0^x e^{\gamma(s-x)}(\theta_2(x)\psi_p(u_1) - (1 - \theta_2(x))\psi_p(u_2))\,ds$ and $0 < \theta_2(x) < 1$.

¿From (17) we get the estimate,

$$|\xi_2(x)| \leq \frac{1}{p}(u_0' + \delta_1)^{p-1}\delta^p\,,$$

while,

$$|\int_0^x e^{\gamma(s-x)}(\psi_p(u_1) - \psi_p(u_2))\,ds| \leq \int_0^x e^{\gamma(s-x)}(p-1)|\xi_3(s)|^{p-2}\,ds\,|u_1 - u_2|\,,$$

being ξ_3 an intermediate function between u_1 and u_2 and $|u_1 - u_2| = \sup_{0\leq x\leq\delta}|u_1(x) - u_2(x)|$. By using once again (17) we arrive at,

$$|\int_0^x e^{\gamma(s-x)}(\psi_p(u_1) - \psi_p(u_2))\,ds| \leq (u_0' - \delta_1)^{p-2}\delta^{p-1}|u_1 - u_2|\,.$$

Thus,

$$|T(u_1)' - T(u_2)'| \leq \frac{1}{(p-1)p^{(2-p)/(p-1)}} \delta^{\frac{1}{p-1}} |u_1 - u_2| \,.$$

Therefore, $|T(u_1) - T(u_2)|_1 \leq O(\delta^{\frac{1}{p-1}})|u_1 - u_2|_1$ as $\delta \to 0+$, being $|u_1 - u_2|_1 = |u_1 - u_2| + |u_1' - u_2'|$. A convenient choice of δ proves that T is contractive in X_{δ,δ_1}. In particular, we obtain the uniqueness of local solutions to (13).

Global existence and uniqueness. Let us prove now that every local solution can be continuated over the whole interval $[0, +\infty)$. In fact, since the nonlinearity in (14) is continuous, standard ode's results ([15]) yield to a maximal continuation $\hat{u}$ of u on an interval $\hat{I} = [0, \omega)$ ($\hat{u} \equiv u$ in $[0, \delta)$). On the other hand, it is well-known that boundedness of $\hat{u}$ entails $\omega = +\infty$. Thus we are next showing that any local solution (u, J) to (6) keeps bounded in J. In fact, due to the conclusions of Lemma 2.1 it follows that the energy function

$$E\left(u(x), u'(x)\right) := \frac{p-1}{p}|u'(x)|^p + \frac{1}{p}|u(x)|^p, \tag{20}$$

is of class C^1 in J if $1 < p \leq 2$, meanwhile $E \in C(J) \cap C^1(J \setminus \{x \in J : u'(x) = 0\})$ provided $p > 2$. Since,

$$E'(x) = (p-1)\psi_p(u')u'' + \psi_p(u)u' = -\gamma|u'|^p < 0,$$

provided $u'(x) \neq 0$ and $\{u' = 0\}$ is discrete we conlude that E decreases in J. In particular,

$$E(u(x), u'(x)) = \frac{p-1}{p}|u'(x)|^p + \frac{1}{p}|u(x)|^p < \frac{p-1}{p}|u'(x_0)|^p + \frac{1}{p}|u(x_0)|^p, \qquad \forall x \in J, x > x_0\,,$$

and every local solution (u, J) to (6) keeps bounded. Therefore, $\omega = +\infty$.

Finally, the global uniqueness is an immediate consequence of the local uniqueness. In fact, if u_1 and u_2 are maximal solutions of (13) the set $\Gamma = \{x \in [0, +\infty) : u_1(x) = u_2(x) \text{ and } u_1'(x) = u_2'(x)\}$, is non empty and obviously closed. Suppose $x_0 > 0$ so that $x_0 \in \Gamma$. Then, either following standard ode's results if $u_0 u_0' \neq 0$ or otherwise using the preceeding local arguments we achieve that u_1 and u_2 further coincide in $|x - x_0|$ small. Therefore $\Gamma = [0, +\infty)$.

This concludes the proof of theorem 2.3. ■

Remark 2.4 The way in which a nontrivial solution u to (6) fails to be twice differentiable at a point x_0 where u' vanishes ($u(x_0) = u_0 \neq 0$) when $p > 2$ can be precisely described. In fact,

$$u'(x) = -\psi_{p'}\left(\int_0^x e^{\gamma(s-x)}\psi_p(u(s))\,ds\right)\,,$$

and so,

$$u'(x) = -u_0\ \psi_{p'}(x - x_0) + o(\psi_{p'}(x - x_0))\,, \tag{21}$$

as $x \to x_0$, being $p' = 1/(p-1)$. This clearly reveals that u can not exhibit a second derivative at $x = x_0$ if $p > 2$.

3 Phase space analysis

After the scale change $\xi = \lambda^{1/p}x$ the problem (1) can be written as,

$$\begin{cases} -(\psi_p(u'))' - \gamma\ \psi_p(u') = \lambda\psi_p(u), & 0 < \xi < \lambda^{1/p} \\ u(0) = u(\lambda^{1/p}) = 0\,, \end{cases} \tag{22}$$

where $' = d/d\xi$ and the parameter γ takes the precise value $\gamma = c\lambda^{-1/p}$. Thus, the strategy to analyze (22) will consists in finding out global solutions $u = u(\xi)$ to (13) corresponding to initial data $u_0 = 0$ and $u_0' \neq 0$ (see Theorem 2.3) so that vanish at the point $\xi = \lambda^{1/p}$ when the parameter γ matches the exact value $\gamma = c\lambda^{-1/p}$.

Since we are going to proceed by geometrical arguments, it should be remarked that, in view of Theorem 2.3, every global solution $u = u(x)$, $0 \leq x < +\infty$, to problem (13) gives raise to a solution $(u, v) = (u(x), u'(x))$ to the autonomous initial value problem,

$$\begin{cases} u' = v & u(0) = u_0 \\ v' = \frac{1}{1-p}|v|^{2-p}\{\psi_p(u) + \gamma\psi_p(v)\} & v(0) = u_0' \,, \end{cases} \tag{23}$$

keeping in mind that (u, v) undergoes singularities (and so does not solve the system) at those -discretely spread- points of $x \geq 0$ where u' vanishes, provided p falls in the range $p > 2$. In any case, (u, v) defines a piece-wise C^1 curve $\Gamma^+ = \Gamma^+(P_0) = \{(u(x), v(x)) : x \geq 0\}$, $P_0 = (u_0, u_0')$ which will be still referred here as to the semiorbit to (23) starting at P_0.

Remark 3.1 It should be stressed that the initial value problems (14) and (23) are not "completely" equivalent. In fact, every solution (u, v) to (14) always provides the solution $(u_1, v_1) = (u, \psi_{p'}(v))$ to (23) (this indeed may be taken as a definition for a global solution to (23) which incorporates the singularities arising at points x_0 where $u'(x_0) = 0$ when $p > 2$). However, when $1 < p < 2$ all points $(u_0, 0)$ are critical for the system in (23) and do not correspond to solutions of (14). Moreover, every fixed solution (u, v) to (14) so that $u'(x_0) = 0$ at some $x_0 > 0$ gives rise to infinitely many solutions $(\hat{u}_1, \hat{v}_1)$ to (23). In fact, given any $l > 0$ it is easily seen that $(\hat{u}_1, \hat{v}_1) = (u, \psi_{p'}(v))$ if $x \leq x_0$, $(\hat{u}_1, \hat{v}_1) = (u(x_0), 0)$ for $x_0 \leq x \leq x_0 + l$ while $(\hat{u}_1(\cdot), \hat{v}_1(\cdot)) = (u(\cdot - l), \psi_{p'}(v(\cdot - l)))$ beyond $x_0 + l$ defines a solution to (23).

Since we are dealing henceforth with the equation (23), all possible solutions (u, v) to that equation will be understood as coming from a unique solution $(\tilde{u}, \tilde{v})$ to (14) by the relation $(u, v) = (\tilde{u}, \psi_{p'}(\tilde{v}))$. Thus, such solutions (u, v) to (23) cross through points $(u_0, 0)$ without stopping if $1 < p < 2$ or plainly go beyond the singularities $(u_0, 0)$ provided $p > 2$.

Our next result gives a detailed description of the phase space of equation (6) through the associated first order system (23). As a main conclusion it will be seen that the equation (6) exhibits, as γ ranges $\mathbb{R}^+$, the same kind of behaviour as the linear situation $p = 2$. Namely, a transition "stable-focus" to "stable-node" when γ acrosses the precise value $\gamma^* = p$.

Theorem 3.2 *Let $\gamma \geq 0$ and $P_0 = (u_0, v_0) \in \mathbb{R}^2$ be arbitrary. Then every semiorbit $\Gamma^+(P_0)$ of (23) is bounded. Moreover, the qualitative global behaviour of all those orbits can be depicted with regard $\gamma \geq 0$ in the following terms,*

i) If $\gamma = 0$ all semiorbits $\Gamma^+(P_0)$ are closed, $(u, v) = (0, 0)$ is a "center", and such semiorbits are parametrized by periodic solutions with commom period $T_p = 4\pi(p-1)^{1/p}/(p\sin(\pi/p))$ (cf. [7], [14]).

ii) For $0 < \gamma < \gamma^* = p$, $(u, v) = (0, 0)$ is a stable focus type point to (23), every semiorbit $\Gamma^+(P_0)$ exponentially decays to $(0, 0)$, turning clockwise around $(0, 0)$ as $x \to \infty$, i. e.,

$$\lim_{x \to +\infty} (u(x), v(x)) = (0, 0) \,.$$

iii) At $\gamma = \gamma^* = p$*,* $(u, v) = (0, 0)$ *is a stable double-node type singular point to (23), the half lines* $\{v = -u : u > 0\}$ *and* $\{v = -u : u < 0\}$ *are orbits to (23) and every semiorbit* $\Gamma^+(P_0)$ *decays exponentially towards* $(0, 0)$ *in such a way that*

$$\lim_{x\to+\infty} \frac{v(x)}{u(x)} = -1\,.$$

iv) If $\gamma > \gamma^* = p$ *then* $(u, v) = (0, 0)$ *is a stable node type singular point in the sense that the four half lines* $\Gamma_{i,+(-)} = \{v = m_i\ u : u > 0\ \ (\text{respectively } u < 0)\}$*,* $i = 1, 2$*, where* $m_1 < m_2 < 0$ *are the unique roots of,*

$$(p - 1)|m|^p + \gamma\psi_p(m) + 1 = 0\,, \tag{24}$$

are orbits to (23). Moreover, every semiorbit $\Gamma^+(P_0)$ *different from* $\Gamma_{1,+}$ *and* $\Gamma_{1,-}$ *exponentially decays to* $(0, 0)$ *so that keeps either tangent to* $\Gamma_{2,+}$ *or to* $\Gamma_{2,-}$ *as* $x \to +\infty$*, i. e.,*

$$\lim_{x\to+\infty} \frac{v(x)}{u(x)} = m_2\,. \tag{25}$$

Remark 3.3 Equation (24) is the generalization to $p \neq 2$ of the characteristic equation in the linear case $p = 2$. A possible way to arrive to (24) is to look for explicit solutions of the form $u = e^{mx}$ to (6),

$$(\psi_p(u'))' + \gamma\psi_p(u') + \psi_p(u) = 0\,.$$

A second way to produce (24) is looking for stright-line solutions $v = m\ u$ to the orbital equation,

$$\frac{dv}{du} = -\frac{1}{p-1}\{\psi_p(u/v) + \gamma\}v\,.$$

associated to (23).

As for the discussion of solutions of (24), setting $h(m) = (p - 1)|m|^p + \gamma\psi_p(m) + 1$ one finds that $h'(m) = p(p - 1)|m|^{p-2}(m + \gamma/p)$, so $m = -\gamma/p$ is an absolute minimum with $h(-\gamma/p) = 1 - (\gamma/p)^p$, and since $h \to +\infty$ as $|m| \to +\infty$, then (24) has exactly two simple zeros $m_1 < m_2 < 0$ (respectively, the single solution $m = -1$, no solutions) provided $\gamma > p$ ($\gamma = p$, $0 \leq \gamma < p$). In particular, when $\gamma > p$, $h > 0$ either if $m < m_1$ or $m > m_2$ while $h < 0$ between m_1 and m_2. Since $m_1 < -\gamma/p < m_2 < 0$ then $h'(m_1) < 0$, $h'(m_2) > 0$.

Proof of Theorem 3.2.

As has been pointed out, every semiorbit $\Gamma^+(P_0)$ is parametrized by a unique bounded and piece-wise C^1 (fully C^1 if $1 < p \leq 2$) solution $(u(x), v(x))$ to (23) such that $(u(0), v(0)) = (u_0, u_0')$ and $0 \leq x < +\infty$ (cf. Theorem 2.3 and Remark 3.1). Thus, if $\rho_0 > 0$ and $0 \leq \varphi_0 < 2\pi$ are chosen so that $u_0 = \rho_0 \cos\varphi_0$, $v_0 = \rho_0 \sin\varphi_0$ then unique piece-wise C^1 functions (repectively, C^1 if $1 < p \leq 2$) $\rho = \rho(x)$ and $\varphi = \varphi(x)$ can be found such that,

$$\begin{cases} u(x) = \rho(x)\cos\varphi(x) \\ v(x) = \rho(x)\sin\varphi(x)\,, \end{cases} \tag{26}$$

for $0 \leq x < +\infty$ and verifying $\rho(0) = \rho_0$, $\varphi(0) = \varphi_0$. The polar coordinates expression (26) for the solution (u, v) (often termed in ode's circles as Prüfer's transformation, cf. [15]) leads to the corresponding equations for ρ and φ which have the form,

$$\begin{cases} \rho' = \rho\ \Phi_1(\varphi) \\ \varphi' = -\Phi_2(\varphi)\,, \end{cases} \tag{27}$$

where the functions Φ_i, $i=1,2$ are given by,

$$\begin{aligned}\Phi_1(\varphi) &= \sin^2\varphi\ \{\cot\varphi-(p'-1)\{\psi_p(\cot\varphi)+\gamma\}\}\\ \Phi_2(\varphi) &= \sin^2\varphi\ \{1+(p'-1)\cot\varphi\{\psi_p(\cot\varphi)+\gamma\}\}\,,\end{aligned}$$

with $p'=p/(p-1)$. It should be observed that points $(\rho,n\pi)$, $n\in\mathbb{Z}$ are critical points to (27) when $1<p<2$ while give raise to the singularities in the solutions to (14) and (23), when $p>2$, as already pointed out in §2 and Remark 3.1.

Let us consider first the points *i)* and *ii)* and so $0\le\gamma<p$. Since Φ_2 can be written as,

$$\begin{aligned}\Phi_2(\varphi) &= (p'-1)\sin^2\varphi\ |\cot\varphi|^p\ \{(p-1)|\tan\varphi|^p+\psi_p(\tan\varphi)+1\}\\ &= (p'-1)\sin^2\varphi\ |\cot\varphi|^p\ h(\tan\varphi)\end{aligned} \tag{28}$$

we see from the analysis of h in Remark 3.3 that $\Phi_2(\varphi)>0$ for $\varphi\neq n\pi$, $n\in\mathbb{Z}$. Let us study now the behaviour of $\Gamma^+=\Gamma^+(P_0)$, $P_0=(u_0,v_0)$. No generality is lost if we assume $v_0\neq0$. Otherwise, the expression (21) for $v=v(x)$,

$$v(x)=-u_0\ \psi_{p'}(x)+o(\psi_{p'}(x)),\qquad x\sim0+\,, \tag{29}$$

ensures that Γ^+ exhibits a point $\tilde P_0=(\tilde u_0,\tilde v_0)$ close to P_0 with $\tilde v_0\neq0$ and then $\tilde P_0$ can be used instead P_0.

Assume that $v_0>0$ (the reasoning is the same if $v_0<0$) so that $0<\varphi_0<\pi$. Then, the equation $\varphi'=-\Phi_2(\varphi)$ shows that Γ^+ meets a first point $P_1=(u_1,0)$, $u_1>0$, then meets a second point of the form $P_2=(0,v_2)$, $v_2<0$, to arrive at a point $P_3=(0,v_3)$, $v_3>0$. The three arcs P_0P_1, P_1P_2 and P_2P_3 are runned by Γ^+ in finite "times" x , respectively, T_1, T_2 and T_3. To compute the T_i's observe that Γ^+ reachs, after a time x, the position $\varphi(x)$ given by,

$$x=(p-1)\int_{\varphi(x)}^{\varphi_0}\frac{ds}{\sin^2 s|\cot s|^p h(\tan s)}\,,$$

where $h(m)=(p-1)|m|^p+\gamma\psi_p(m)+1$, so

$$x=(p-1)\int_{\cot\varphi_0}^{\cot\varphi(x)}\frac{dt}{|t|^p h(1/t)}=(p-1)\int_{\cot\varphi_0}^{\cot\varphi(x)}\frac{dt}{|t|^p+\gamma t+(p-1)}\,,$$

where it must be remarked that since $0\le\gamma<p$, the function $|t|^p h(1/t)=|t|^p+\gamma t+(p-1)$ never vanishes. So, the values for T_i, $i=1,2,3$ are provided by the convergent integrals,

$$T_1=(p-1)\int_{\cot\varphi_0}^{\infty}\frac{dt}{|t|^p+\gamma t+(p-1)}\,,$$

$$T_2=(p-1)\int_{0}^{\infty}\frac{dt}{|t|^p-\gamma t+(p-1)}\,,$$

and

$$\begin{aligned}T_3 &= (p-1)\int_0^\infty\frac{dt}{|t|^p+\gamma t+(p-1)}+(p-1)\int_0^\infty\frac{dt}{|t|^p-\gamma t+(p-1)}\\ &= 2(p-1)\int_0^\infty\frac{t^p+p-1}{(t^p+p-1)^2-\gamma^2t^2}\,.\end{aligned} \tag{30}$$

For later use let us define now the function

$$T_p(\gamma) = 2(p-1)\int_0^\infty \frac{t^p+p-1}{(t^p+p-1)^2-\gamma^2t^2}\,. \qquad (31)$$

The previous argument also shows that every semiorbit Γ^+ starting at $P_0 = (0, v_0) \in ov^+ := \{(0, v) : v > 0\}$ return to a point $\tilde{P}_0 = (0, \tilde{v}_0) \in ov^+$ after describing an arc $P_0\tilde{P}_0$ around $(0, 0)$, and always delaying the *same* time $2T_p(\gamma)$ (the function $v_0 \to \tilde{v}_0$ defines a Poincaré's mapping). In fact, the π periodicity of Φ_2 in φ implies that every orbit to (23) need the same time x in travelling from ov^+ to $ov^- := \{(0, v) : v < 0\}$ as in travelling from ov^- to ov^+. Therefore, the semiorbit $\Gamma^+(P_0)$ will produce after P_3 a sequence interesections P_n to ov^+, $P_n = (0, v_n)$ with the v_n's recursively defined by

$$v_{n+1} = v_n e^{F(\gamma)}\,, \qquad (32)$$

where,

$$F(\gamma) = \int_{-\pi/2}^{\pi/2} \frac{\cot\varphi - (p'-1)\{\psi_p(\cot\varphi)+\gamma\}}{1+(p'-1)\cot\varphi\{\psi_p(\cot\varphi)+\gamma\}}\,d\varphi\,. \qquad (33)$$

The existence of the sequence $\{v_n\}$ explains the vortex like behaviour of all orbits around $(0, 0)$ when $0 \le \gamma < p$.

For later use, it is now very convenient to state some few properties of the functions $T_p(\gamma)$ and $F(\gamma)$. Namely,

a) $T = T_p(\gamma)$ is a C^∞ mapping on $0 \le \gamma < p$ such that $T_p(0) = 2\pi(p-1)^{1/p}/(p\sin(\pi/p))$.

b) $T = T_p(\gamma)$ is increasing in $0 \le \gamma < p$.

c) $\lim_{\gamma\to p-} T_p(\gamma) = +\infty$. Moreover,

$$T(p-\epsilon^2) \sim \frac{\pi}{\sqrt{p'}}\frac{1}{\epsilon}\,, \qquad (34)$$

as $\epsilon \to 0+$, with $p' = p/(p-1)$.

d) $F = F(\gamma)$ is C^∞ and decreasing in $0 \le \gamma < p$ while $F(0) = 0$.

Therefore, by setting $F(0) = 0$ in (32) it follows that all orbits to (23) are closed and $2T_p(0) = 4\pi(p-1)^{1/p}/(p\sin(\pi/p))$ periodic if $\gamma = 0$ (cf. [7], [14]), and hence *i)*. Similarly, the fact that $F(\gamma) < 0$ for $0 < \gamma < p$ together with (32) entails that all orbits exponentially decay towards $(0, 0)$ turning around the origin when $0 < \gamma < p$ and *ii)* is proven.

Let us sketch now the proof of the announced properties of T and F. For *a)* notice that a few variable changes lead to $T_p(0) = (2/p)(p-1)^{1/p}\,B(1-(1/p),(1/p))$ with B the eulerian beta function what implies the result. On the other hand,

$$T'(\gamma) = 2(p-1)\int_0^\infty \frac{2t(t^p+p-1)\gamma}{[(t^p+p-1)^2-\gamma^2t^2]^2}$$

being the integral convergent since the denominator never vanishes if $0 \le \gamma < p$. Thus T is increasing. As for (34) notice that

$$(p'-1)T_p(\gamma) = \int_0^\infty \frac{dt}{t^p+\gamma t+(p-1)} + \int_0^\infty \frac{dt}{t^p-\gamma t+(p-1)} = I_1(\gamma)+I_2(\gamma)\,.$$

It is easily seen that $I_1(\gamma)$ is convergent and decreasing for $\gamma > 0$, in particular is $O(1)$ as $\gamma \to \infty$. As for $I_2(\gamma)$, given any fixed $\delta > 0$ small enough and setting $\gamma = p - \epsilon^2$ we find that

$$\epsilon I_2(p-\epsilon^2) \sim \epsilon \left\{ \int_{1-\delta}^{1} + \int_{1}^{1+\delta} \right\} \frac{dt}{t^p - (p-\epsilon^2)t + (p-1)}$$

as $\epsilon \to 0+$. To show *c)* it suffices with proving that $\epsilon I_2(p-\epsilon^2) \sim \pi/\sqrt{p(p-1)}$.

Now observe that $t^p - pt + p - 1 = (p(p-1) + o(1))(t-1)^2$ as $t \to 1$. This implies,

$$(p(p-1) - \eta)(t-1)^2 \leq t^p - pt + p - 1 \leq (p(p-1) + \eta)(t-1)^2\,,$$

if $|t-1| < \delta$ where $\eta = o(1)$ as $\delta \to 0+$. Hence,

$$\left\{ \int_{1-\delta}^{1} + \int_{1}^{1+\delta} \right\} \frac{dt}{t^p - (p-\epsilon^2)t + (p-1)} \geq \left\{ \int_{1-\delta}^{1} + \int_{1}^{1+\delta} \right\} \frac{dt}{[(p(p-1)+\eta](t-1)^2 + \epsilon^2(1+\delta)}.$$

Thus, for $\delta > 0$ fixed we get,

$$\epsilon \left\{ \int_{1-\delta}^{1} + \int_{1}^{1+\delta} \right\} \frac{dt}{t^p - (p-\epsilon^2)t + (p-1)} \geq \frac{2}{\sqrt{1+\delta}\sqrt{p(p-1)+\eta}} \arctan \frac{\delta\sqrt{p(p-1)+\eta}}{\epsilon\sqrt{1+\delta}}\,.$$

Therefore,

$$\liminf_{\epsilon \to 0+} \epsilon I_2(p-\epsilon^2) \geq \frac{\pi}{\sqrt{1+\delta}\sqrt{p(p-1)+\eta}}\,,$$

which, as $\delta \to 0+$, implies

$$\liminf_{\epsilon \to 0+} \epsilon I_2(p-\epsilon^2) \geq \frac{\pi}{\sqrt{p(p-1)}}\,.$$

In the same way, from

$$\left\{ \int_{1-\delta}^{1} + \int_{1}^{1+\delta} \right\} \frac{dt}{t^p - (p-\epsilon^2)t + (p-1)} \leq \left\{ \int_{1-\delta}^{1} + \int_{1}^{1+\delta} \right\} \frac{dt}{[(p(p-1)-\eta](t-1)^2 + \epsilon^2(1-\delta)}\,.$$

we get the complementary estimate,

$$\limsup_{\epsilon \to 0+} \epsilon I_2(p-\epsilon^2) \leq \frac{\pi}{\sqrt{1-\delta}\sqrt{p(p-1)-\eta}}\,.$$

By making $\delta \to 0+$ it follows that $\lim_{\epsilon \to 0+} \epsilon I_2(p-\epsilon^2) = \pi/\sqrt{p(p-1)}$, which implies (34).

To show *d)* notice that $F(0) = 0$ follows easily from the fact that the integral in (33) is odd in $\cot\varphi$ as $\gamma = 0$. Observe also that F can be written as

$$F(\gamma) = -\int_{-\infty}^{\infty} \frac{g(z)}{h(z)} \frac{dz}{1+z^2}\,,$$

with h as in Remark 3.3 and $g(z) = \gamma|z|^p - (p-1)\psi_p(z) + z$. Since h never vanishes if $0 \leq \gamma < p$ then I converges while it is immediately seen that

$$F'(\gamma) = -(p-1) \int_{-\infty}^{\infty} \frac{|z|^{2p-2}}{h(z)^2} dz\,, \tag{35}$$

and so F decreases in $0 \le \gamma < p$.

Finally let us undertake the proof of *iii)* and *iv)*. It suffices with proceeding with *iv)* (an entirely similar argument leads to *iii)*). Thus, let $(\rho(x), \varphi(x))$ the polar parametrization (26) of an arbitrary semiorbit $\Gamma^+ = \Gamma^+(P_0)$. First observe that the angular part of equation (27) can be written as,

$$(\psi_p(\tan\varphi))' = -h(\tan\varphi)\,. \tag{36}$$

Since (36) is π periodic in φ it is enough with studying the behaviour of Γ^+ in, say $v > 0$. So, let $m_1 < m_2 < 0$ the zeros of h with $m_i = \tan\tilde{\varphi}_i$, $i = 1, 2$, $-\pi/2 < \tilde{\varphi}_1 < \tilde{\varphi}_2 < 0$. As already seen (cf. Remark 3.3) the stationary solutions $\varphi = \tilde{\varphi}_i$, $i = 1, 2$ to (36) give raise the orbits $\Gamma_{i,+}$, $i = 1, 2$. Thus, assume that the initial position $-\pi/2 < \varphi_0 < \pi/2$ verifies $\varphi_0 \notin \{\tilde{\varphi}_1, \tilde{\varphi}_2\}$. According to the profile of h (cf. Remark 3.3) it follows that $\varphi(x) \to \tilde{\varphi}_2$ as $x \to \infty$ in a decreasing (respectively, increasing) way according to whether $\varphi_0 > \tilde{\varphi}_2$ or $\tilde{\varphi}_1 < \varphi_0 < \tilde{\varphi}_2$. If, otherwise, $-\pi/2 < \varphi_0 < \tilde{\varphi}_1$ then Γ^+ reaches the value $\varphi = -\pi/2$ after x_0 units of "time" with

$$x_0 = (p-1)\int_{-\infty}^{\varphi_0} \frac{|z|^{p-2}}{h(z)}\;dz\,.$$

Then, $\varphi < -\pi/2$ for $x > x_0$ close to x_0 and by the previous discussion φ decreases to $\tilde{\varphi}_2 - \pi$ as $x \to \infty$. In both cases the asymptotic rate (25) is satisfied.

As for the behaviour of the radial part $\rho(x)$ assume, say that $\tilde{\varphi}_2 < \varphi_0 < \pi/2$ (so $\varphi(x) \downarrow \tilde{\varphi}_2$ as $x \uparrow \infty$). Then, by integrating (27) we find,

$$\rho(x) = \rho_0 \exp\left\{-\int_{\varphi_0}^{\varphi(x)} \frac{\cot\varphi - (p'-1)\{\psi_p(\cot\varphi) + \gamma\}}{(p'-1)|\cot\varphi|^p h(\tan\varphi)}\;d\varphi\right\},$$

or, equivalently,

$$\rho(x) = \rho_0 \exp\left\{-\int_{\tan\varphi_0}^{\tan\varphi(x)} \frac{g(z)}{(1+z^2)h(z)}\;dz\right\}, \tag{37}$$

where, as previously introduced, $g(z) = z\{\gamma\psi_p(z) + 1\} - (p-1)\psi_p(z)$. Observe now that $g(m_2) = -(p-1)(\psi_{p+2}(m_2) + \psi_p(m_2)) > 0$ since $h(m_2) = 0$ with $m_2 < 0$. On the other hand the integral in (37) can be estimated as,

$$\int_{\tan\varphi_0}^{\tan\varphi(x)} \frac{g(z)}{(1+z^2)h(z)}\;dz \sim -\frac{g(m_2)}{(1+m_2^2)h'(m_2)}\log(\tan\varphi(x) - m_2)\,, \tag{38}$$

as $x \to \infty$. Finally, $\tan\varphi \downarrow m_2$ as $x \uparrow \infty$. From (36) we achieve,

$$(\tan\varphi)' = -\frac{1}{p-1}\frac{h(\tan\varphi)}{|\tan\varphi|^{p-2}}\,,$$

what implies that

$$\tan\varphi - m_2 \sim C_0\; e^{-\frac{1}{p-1}\frac{h'(m_2))}{|m_2|^{p-2}}}\,,$$

as $x \to \infty$, being C_0 a certain constant. Therefore,

$$\rho \sim \rho_0 C_1\; e^{m_2 x} \qquad \text{as } x \to \infty\,, \tag{39}$$

where $C_1 = C_0^{(p-1)|m_2|^{p-1}/h'(m_2)}$, and ρ exponentially decays toward 0 as $x \to \infty$, as desired. This concludes the proof of Theorem 3.2.

■

Remark 3.4 Estimate (39) shows that the leading root $m = m_2$ of the "characteristic" equation (6), $h(m) = 0$, plays the same role as in the linear case, regarding the asymptotic decay of solutions in the stable node case $\gamma \geq p$.

4 The eigenvalue problem

The main features concering the eigenvalues of (1) are next depicted in the following result.

Theorem 4.1 *For every positive $c \in \mathbb{R}^+$ the eigenvalue problem (1),*

$$\begin{cases} -(\psi_p(u'))' - c\psi_p(u') = \lambda\psi_p(u) & 0 < x < 1 \\ u(0) = u(1) = 0\,, \end{cases}$$

admits as a whole set of eigenvalues an unbounded increasing sequence $\{\lambda_n = \lambda_n(c)\}$ of positive numbers such that,

$$\left(\frac{c}{p}\right)^p < \lambda_1 < \cdots < \lambda_n < \cdots \tag{40}$$

In addition, the following properties are satisfied,

i) Every λ_n is simple in the sense that every eigenfunction u_n corresponding to (1) for $\lambda = \lambda_n$ is a multiple of a fixed (normalized) eigenfunction ϕ_n.

ii) Each eigenfunction u_n corresponding to $\lambda = \lambda_n$ exhibits $n-1$ equally spaced zeros in the interval $0 < x < 1$.

iii) $\lambda_n = \lambda_n(c,p)$ is a smooth function of c and p. Moreover, λ_n is increasing in $c > 0$ while $\lambda_n(c) \sim (c/p)^p$ as $c \to +\infty$. In a more accurate way the following asymptotic estimate holds,

$$\lambda_n(c) \sim \frac{1}{(1-(p-1)\pi^2 n^2 \frac{1}{c^2})^p}\left(\frac{c}{p}\right)^p = \left(1 + p(p-1)\pi^2 n^2 \frac{1}{c^2} + o(\frac{1}{c^2})\right)\left(\frac{c}{p}\right)^p, \tag{41}$$

as $c \to +\infty$. On the other hand, the asymptotic behaviour of λ_n for n large is dictated by,

$$\lambda_n(c) \sim (p-1)\left\{\frac{2\pi}{p\sin\frac{\pi}{p}}\right\}^p n^p\,, \tag{42}$$

as $n \to +\infty$.

iv) Let $\tilde{\phi}_n$ the n-th eigenfunction normalized so as

$$\lambda_n^{-\frac{1}{p}}\tilde{\phi}_n'(0) = 1 \qquad (' = \frac{d}{dx})\,. \tag{43}$$

According to ii), $x_n = \frac{n-1}{n}$ defines the maximum zero of $\tilde{\phi}_n$ in $0 < x < 1$. Then, the following estimate holds true,

$$(-1)^{n+1}\tilde{\phi}_n(x_n) \sim e^{-\frac{c}{p}}\,, \tag{44}$$

as $n \to +\infty$.

Remark 4.2 *a)* Estimate (41) should be compared with the explicit expression for λ_n provided in the linear case $p = 2$ by the formula $\lambda_n = \frac{c^2}{4} + n^2\pi^2$. Notice also that the right side of (42) is the n-th eigenvalue of (3), i. e. $c = 0$ in (1). Thus (42) asserts that the eigenvalues of (1) become arbitrarily close of those of (3) as $n \to +\infty$.

b) In the linear situation $p = 2$ the normalized eigenfunction $\tilde{\phi}_n$ is explicitely given by

$$\tilde{\phi}_n(x) = \frac{1}{n\pi}\sqrt{\frac{c^2}{4} + n^2\pi^2} e^{-\frac{cx}{2}} \sin n\pi x\,.$$

Thus observe that the estimate (44) is immediately obtained. In fact,

$$(-1)^{n+1}\tilde{\phi}_n(\frac{n-1}{n}) = \sqrt{1 + \frac{c^2}{4\pi n}} e^{-\frac{(n-1)c}{2n}} \sim e^{-\frac{c}{2}}\,,$$

as $n \to +\infty$.

Proof of Theorem 4.1.

Let $\tilde{u} = \tilde{u}(x)$ be any possible eigenfunction to (1) corresponding to some eigenvalue λ which must be positive (cf. §2). Then, by making the scaling $\xi = \lambda^{1/p}x$, $u = u(\xi)$ obtained as $u(\xi) = \tilde{u}(\lambda^{-1/p}\xi)$ defines a solution of (22) corresponding to the precise value $\gamma = c\lambda^{-1/p}$.

Consider now such a possible eigenfunction $u = u(\xi)$ as the solution to the initial value problem,

$$\begin{cases} (\psi_p(u'))' + \gamma\psi_p(u') + \psi_p(u) = 0 \\ u(0) = 0, u'(0) = u'_0\,, \end{cases} \tag{45}$$

with $' = d/d\xi$ and a certain $u'_0 \neq 0$. As has been shown, $u = u(\xi)$ is defined and bounded in the whole of $\mathbb{R}^+$ (cf. Theorem 2.3).

Now observe that from *iii) - iv)* of Theorem 3.2 we already know that every nontrivial solution $u = u(\xi)$ to (45) keeps its sign so that $u'_0u(\xi) > 0$ in $\xi > 0$ while decays to 0 as $e^{m_2\xi}$ as $\xi \to +\infty$ provided that $\gamma \geq p$. This means that any possible eigenvalue λ must satisfy $0 < \gamma = c\lambda^{-1/p} < p$. In other words, the following inequality holds,

$$\left(\frac{c}{p}\right)^p < \lambda\,. \tag{46}$$

On the other hand, if $0 \leq \gamma < p$ we infer from the proof of Theorem 3.2 that every nontrivial solution u to (45) vanishes in $\mathbb{R}^+$ exactly at the points $\xi_n = nT_p(\gamma)$, $n \in \mathbb{N}$. Therefore $\lambda > 0$ is an eigenvalue to (1) if and only if λ satisfies the equation

$$nT_p\left(\frac{c}{\lambda^{\frac{1}{p}}}\right) = \lambda^{\frac{1}{p}}\,, \tag{47}$$

for some positive integer $n \in \mathbb{N}$. We are next showing that (47) admits, for each $n \in \mathbb{N}$ and each $c \geq 0$, a unique solution $\lambda = \lambda_n(c)$.

Before that, observe that it follows from the uniqueness statement in Theorem 2.3 and the homogeneity of (45) that the solution u to (45) can be written as $u = u'_0\phi$ being ϕ the solution corresponding to $u'_0 = 1$. From this remark and equation (47) we conclude that inserting $\gamma = c\lambda_n(c)^{-1/p}$ in (45) all of its solutions are multiple of ϕ while vanish in the interval $0 \leq \xi \leq \lambda_n^{1/p}(c)$

exactly at the points $\xi_k = kT_p(c\lambda_n(c)^{-1/p})$, $0 \le k \le n$. In other words, we achieve the points *i)*, *ii)* of Theorem 4.1.

Let us show now the existence and uniqueness of solutions to (47) for each $n \in \mathbb{N}$ and $c \geq 0$. The case $c = 0$ is immediate since (47) directly implies that $\lambda_n|_{c=0} = n^{1/p}T_p(0)^{1/p}$ (cf. §1). As for $c > 0$ by setting $\zeta = c\lambda^{-1/p}$, equation (47) can be read as

$$n\zeta T_p(\zeta) = c\,. \tag{48}$$

¿From (46) solutions to (48) need only be searched in the interval $0 < \zeta < p$. However, the existence and uniqueness to (48) easily follows from the fact that the function $\zeta T_p(\zeta)$ increases from 0 to $+\infty$ as ζ increases from 0 to p.

Thus, let $\zeta = \zeta_n(c)$ be the unique solution to (48). It is then clear that $\zeta_n \to 0$ as $n \to +\infty$. It also follows from (48) that ζ_n is smooth and increases in c from 0 to p as c increases from 0 to $+\infty$. Since $\lambda_n(c)^{1/p} = nT_p(\zeta_n(c))$ and T_p is incresing, the same holds true for λ_n with regard c. That representation for λ_n together with the fact $\zeta_n \to 0$ lead to $\lambda_n(c) \sim n^pT_p(0)^p$ as $n \to +\infty$, what proves (42). On the other hand, the fact that $\zeta_n(c) \uparrow p$ as $c \uparrow +\infty$ entails

$$\lambda_n(c) = \left(\frac{c}{\zeta_n(c)}\right)^p \sim \left(\frac{c}{p}\right)^p, \qquad \text{as} \quad c \to +\infty\,.$$

This partially proves *iii)*. To show (41) observe that (48) can be written as,

$$T_p(p - (\sqrt{p - \zeta_n(c)})^2) = \frac{c}{n\zeta_n(c)}\,.$$

Taking into account the fact that $\zeta_n(c) \to p$ as $c \to +\infty$ and the estimate (34) for T_p we arrive at,

$$\frac{\pi}{\sqrt{p'}\sqrt{p - \zeta_n}} \sim \frac{c}{np}\,,$$

as $c \to +\infty$. Thus,

$$\zeta_n \sim p - p(p-1)n^2\pi^2c^{-2}\,,$$

as $c \to \infty$. In other words,

$$\lambda_n^{\frac{1}{p}} \sim \frac{1}{1 - (p-1)n^2\pi^2c^{-2}}\left(\frac{c}{p}\right) \qquad c \uparrow +\infty\,,$$

what proves (34).

Finally, let us show (44). First observe that if $\tilde{\phi}_n$ stands for the n-th eigenfunction, normalized according (43), then $\phi_n(\xi) = \tilde{\phi}_n(\lambda_n^{-1/p}\xi)$ is the solution to (45) corresponding to $u_0' = 1$ and $\gamma = c\lambda_n^{-1/p}$.

If ϕ_n is observed as the solution $(u, v) = (\phi_n, \phi_n')$ to the system (23), after representing it under the polar coordinates form (ρ, φ) (26) we obtain,

$$(-1)^{n+1}\tilde{\phi}_n(\frac{n-1}{n}) = (-1)^{n+1}\phi_n(\frac{n-1}{n}\lambda_n^{\frac{1}{p}}) = \rho(-(n-1)\pi)\,,$$

where $\rho = \rho(\varphi)$ is explicitely given by $\rho(\varphi) = \exp\left\{\int_\varphi^{\pi/2} \Phi_1/\Phi_2 \; d\varphi\right\}$. So define $\rho_n = \rho(-(n-1)\pi)$. By using the definition (33) of $F(\gamma)$ together with the π periodicity of Φ_1/Φ_2 we find the expression,

$$\log\rho_n = \left\{\int_{\frac{\pi}{2}-(n-1)\pi}^{\frac{\pi}{2}} + \int_{-(n-1)\pi}^{\frac{\pi}{2}-(n-1)\pi}\right\}\frac{\Phi_1(\varphi)}{\Phi_2(\varphi)}\;d\varphi = (n-1)F(\zeta_n(c)) + J\,,$$

where $J = \int_0^{\pi/2} \Phi_1/\Phi_2 \, d\varphi$. Since $\zeta_n \to 0$ and $n\zeta_n \sim (n-1)\zeta_n$ as $n \to +\infty$ we conclude that $(n-1)\zeta_n \sim c/T_p(0)$. Taking into account that $F(0) = 0$ (cf. the proof of Theorem 3.2) we infer that

$$\log \rho_n = (n-1)\left(F(\zeta_n) + \frac{1}{n-1} J\right) \sim c\frac{F'(0)}{T_p(0)}\,.$$

¿From the expression (35) for the derivative of F we obtain,

$$F'(0) = -2(p'-1)\int_0^{+\infty} \frac{dt}{[1+(p'-1)t^p]^2}\,.$$

Some calculus leads to

$$F'(0) = -\left(\frac{2}{p}\right)(p'-1)^{1-\frac{1}{p}} B\left(2-\frac{1}{p}, \frac{1}{p}\right)\,.$$

Since, as already shown $T_p(0) = (2/p)(p'-1)^{-1/p}B(1-1/p, 1/p)$ (see the proof of Theorem 3.2) we achieve,

$$\begin{aligned} \frac{F'(0)}{T_p(0)} &= -(p'-1)\frac{B(2-\frac{1}{p},\frac{1}{p})}{B(1-\frac{1}{p},\frac{1}{p})} \\ &= -(p'-1)\frac{\Gamma(2-\frac{1}{p})\Gamma(\frac{1}{p})}{\Gamma(2-\frac{1}{p})\Gamma(\frac{1}{p})} = -\frac{p'-1}{p'} = -\frac{1}{p}\,. \end{aligned}$$

This shows that

$$\log \rho_n \sim -\frac{c}{p}\,,$$

as $n \to \infty$ what shows (44). Thus, the proof of Theorem 4.1 is concluded.

■.

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