STATIONARY SIGN CHANGING SOLUTIONS FOR AN INHOMOGENEOUS NONLOCAL PROBLEM

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ABSTRACT. We consider the following nonlocal equation

$$\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{u(y)}{g(y)}dy - u(x) = 0 \qquad x \in \mathbb{R},$$

where J is an even, compactly supported, Hölder continuous probability kernel, g is a continuous function, bounded and bounded away from zero in $\mathbb{R}$. We prove the existence of a sign changing solution $q(x)$ which is strictly positive when $x > K$ and strictly negative for $x < -K$, provided that K is chosen large enough. The solution $q(x)$ so constructed verifies $a_1 \le \frac{q(x)}{x} \le a_2$ for positive constants a_1, a_2 and large $|x|$. In addition, we show that all solutions with polynomial growth are of the form $Aq(x) + Bp(x)$, where p is the unique normalized positive (bounded) solution of the equation. In the particular case where $g = 1$ we also construct solutions with exponential growth.

1. INTRODUCTION

The main purpose of this paper is to construct sign changing solutions of the following nonlocal linear inhomogeneous equation:

$$\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{u(y)}{g(y)}dy - u(x) = 0 \quad x \in \mathbb{R}, \tag{1.1}$$

where J is a Hölder continuous function which we assume nonnegative, even, supported in the unit interval $[-1,1]$ and with $\int_{\mathbb{R}} J(x)dx = 1$. The function g will be supposed to be continuous, bounded and bounded away from zero in $\mathbb{R}$.

Problem (1.1) is a generalization of the usual one

$$J * u - u = 0 \quad \text{in } \mathbb{R}, \tag{1.2}$$

where $*$ denotes the standard convolution. Equations involving the operator $J * u - u$ in $\mathbb{R}^N$ or in bounded domains of $\mathbb{R}^N$ have been extensively studied (see for instance [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [24], [25], [28], [29], [30], [32], [33] and [34]). However, the structure of the set of solutions of (1.2) is still not completely determined: it can be shown by a standard argument (see for instance the proof of Theorem 6 in page 28 of [23]) that harmonic functions are solutions of (1.2). Thus, we have solutions of the form $u(x) = ax + b$ for arbitrary constants a, b. The question of whether there are other solutions aside harmonic ones is not clear at the best of our knowledge. Let us mention in passing that solutions of (1.2) verify a sort of the so-called "restricted mean value property". That is, for every point $x \in \mathbb{R}$, the value of a solution at x coincides with its mean on a ball of radius 1 centered at x. It has been shown that solutions bounded by a harmonic function in absolute value are themselves harmonic (see [26] and [27]). We refer the reader to [31] for a survey on the subject.

But when the more general problem (1.1) is considered, the situation is even less clear. Of course, harmonic functions need not be solutions as in the case $g = 1$, since the constant functions are not even solutions. However, it was shown in [12] that, under certain conditions on the function g, a positive, bounded solution $p(x)$ of (1.1) exists, although it is not clear if this solution is unique (after some normalization) among the positive ones.

Moreover, and as far as we know, it is not known if sign changing solutions of (1.1) exist as for problem (1.2). Thus our objective in the present paper will be twofold: first establishing the existence of a sign changing solution $q(x)$ and second, proving that $p(x)$ and $q(x)$ expand the set of solutions with polynomial growth. We will also be involved with solutions with non polynomial growth when $g = 1$.

Let us first concrete our hypotheses on the kernel J and on the "step" function g. We will assume throughout the paper that J is an α-Hölder continuous function (with Hölder constant $L > 0$), which is nonnegative and even, supported in the unit interval $[-1, 1]$ and with unit integral, $\int_{\mathbb{R}} J(z)dz = 1$. An additional technical condition will be required in some places:

$$\text{(H1)} \qquad \begin{array}{l} \text{there exists } \eta > 0 \text{ such that for all} \\ 0 \le z_1 \le z_2 \text{ we have } J(z_2) \le \eta J(z_1). \end{array}$$

This hypothesis is met for instance when J is positive in $(-1, 1)$ and nonincreasing near 1. Concerning the function g we will suppose that it is continuous and satisfies

$$\underline{b} \le g(x) \le \overline{b} \quad \text{for all } x \in \mathbb{R}, \tag{1.3}$$

for some $\underline{b}, \overline{b} > 0$. Without further mention, we are always assuming that J and g verify these hypotheses.

In [12] the authors constructed a positive solution p for (1.1) which is bounded and bounded away from zero. A key element in the proof of the existence of such solution is the fact, proved in [12], that nonnegative solutions of (1.1) verify

$$\int_0^{\overline{b}} \int_{x-w}^{x+w} u(s) \int_{\frac{w}{g(s)}}^1 J(z)\, dz\, ds\, dw = B$$

for all $x \in \mathbb{R}$ and some positive constant B. As a slight generalization of this property, we will show in the present paper that u is a solution of (1.1) if and only if there exist constants A, B such that

$$\int_0^{\overline{b}} \int_{x-w}^{x+w} u(s) \int_{\frac{w}{g(s)}}^1 J(z)\, dz\, ds\, dw = Ax + B \tag{1.4}$$

in $\mathbb{R}$. Then it becomes clear that (1.4) implies that if p is a positive solution of (1.1), then $A = 0$ and $B > 0$. Formula (1.4) is a direct consequence of Proposition 4 that will be proved in Section 2.

In view of the above discussion, a natural question to ask is whether there exists a solution u of (1.1) for which, say, $A = 1$. In the particular case where g is constant, such a solution corresponds to $u(x) = Cx$, where C is a positive constant which can be explicitly evaluated. Our first result shows that this also happens for general functions g. Denote

$$F_u(x) = \int_0^{\overline{b}} \int_{x-w}^{x+w} u(s) \int_{\frac{w}{g(s)}}^1 J(z)\, dz\, ds\, dw.$$

Then we have:

Theorem 1. *Under the above stated hypotheses, there exists a solution q of* (1.1) *satisfying*

$$F_q(x) = x, \tag{1.5}$$

for every $x \in \mathbb{R}$ and

$$a_1 \le \frac{q(x)}{x} \le a_2, \text{ as } |x| \to \infty, \tag{1.6}$$

for some $a_1, a_2 > 0$.

We should point out that the uniqueness of such solutions is difficult to establish in general. However, we will show that the set of solutions of (1.1) with *polynomial* growth is expanded by $p(x)$ and $q(x)$. More precisely:

Theorem 2. *Assume that J, g are as before and let u be a solution of* (1.1) *verifying $|u(x)| \le C(|x|^k + 1)$ in $\mathbb{R}$ for some $C > 0$ and $k > 0$. If $F_u(x) = Ax + B$ then $u = Aq + Bp$.*

Finally, we show that the requirement that u has polynomial growth in the previous theorem is not a technical assumption. In the particular case where $g = 1$, we will construct some kernels J such that (1.1) has solutions with exponential growth, which in addition verify $F_u(x) = 0$. This shows in particular that the requirement that u is controlled by a harmonic function in the restricted mean value property cannot be dropped.

Theorem 3. *Let $g = 1$. Then there exists a C^1, nonnegative, compactly supported kernel J with unit integral such that* (1.1) *admits solutions of the form*

$$u(x) = e^{ax}\cos(bx), \quad u(x) = e^{ax}\sin(bx)$$

for $a, b \in \mathbb{R} \setminus \{0\}$. Moreover, $F_u(x) = 0$ when $x \in \mathbb{R}$.

The paper is organized as follows: in Section 2 we consider some preliminary properties of solutions of (1.1), in particular the proof of formula (1.4). Section 3 is dedicated to construct the so-called "approximate solutions" of (1.1). In Section 4 we obtain bounds for approximate solutions and in Section 5 the proofs of Theorems 1, 2 and 3 are performed.

2. Preliminary properties

In this section, we will collect some preliminary results which will be of constant use in the rest of the paper. They all deal with solutions of

$$\begin{aligned} &\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{u(y)}{g(y)}dy - u(x) = 0 \quad x \in [-M, M] \\ &u = u_0 \quad \text{in } \mathbb{R} \setminus [-M, M], \end{aligned} \tag{2.1}$$

where $M > 0$ and u_0 is a given function. We begin by considering the validity of formula (1.4) for solutions of (2.1). For a bounded, piecewise continuous function $u : \mathbb{R} \to \mathbb{R}$, we define

$$F_u(x) = \int_0^{\bar{b}} \int_{x-w}^{x+w} u(s) \int_{\frac{w}{g(s)}}^1 J(z)\, dz\, ds\, dw. \tag{2.2}$$

Observe that F_u is linear in the function u. The next result, which establishes the expressions for the derivatives of F_u, will be the essential tool in our proofs.

Proposition 4. *Suppose that $u:\mathbb{R}\to\mathbb{R}$ is continuous in $[-M,M]$, piecewise continuous in $\mathbb{R}$ and bounded. Then the function $F_u:[-M,M]\to\mathbb{R}$ belongs to $C^1[-M,M]\cap C^2(-M,M)$. Its derivatives are given by:*

$$F_u'(x)=\int_0^{\bar b} u(x+w)\int_{\frac{w}{g(x+w)}}^1 J(z)\,dzdw-\int_0^{\bar b} u(x-w)\int_{\frac{w}{g(x-w)}}^1 J(z)\,dzdw,$$

$x\in[-M,M]$, and

$$F_u''(x)=\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{u(y)}{g(y)}dy-u(x),$$

$x\in(-M,M)$. In particular, u is a solution of (2.1) if and only if there exist constants A, B such that $F_u(x)=Ax+B$ in $[-M,M]$.

Proof. Since u is bounded, we have that F_u is continuous $[-M,M]$. Also, it is clear from the fundamental theorem of calculus that F_u is C^1 in $[-M,M]$. Differentiating under the integral one gets for $x\in\mathbb{R}$

$$F_u'(x)=\int_0^{\bar b} u(x+w)\int_{\frac{w}{g(x+w)}}^1 J(z)dzdw-\int_0^{\bar b} u(x-w)\int_{\frac{w}{g(x-w)}}^1 J(z)dzdw. \tag{2.3}$$

Setting $r=x+w$ in the first integral of the right hand side of (2.3) and $r=x-w$ in the second one we get

$$F_u'(x)=\int_x^{x+\bar b} u(r)\int_{\frac{r-x}{g(r)}}^1 J(z)dzdr+\int_x^{x-\bar b} u(r)\int_{\frac{x-r}{g(r)}}^1 J(z)dzdr. \tag{2.4}$$

We now use the fact that J is symmetric, supported in the interval $[-1,1]$ and with unit integral to note that

$$\int_{\frac{x-r}{g(r)}}^1 J(z)dz=\int_{-1}^{\frac{r-x}{g(r)}} J(z)dz=1-\int_{\frac{r-x}{g(r)}}^1 J(z)dz.$$

Substituting this in the last integral of (2.4) we obtain

$$F_u'(x)=\int_{x-\bar b}^{x+\bar b} u(r)\int_{\frac{r-x}{g(r)}}^1 J(z)dzdr+\int_x^{x-\bar b} u(r)dr.$$

We differentiate again to get for $x\in(-M,M)$

$$F_u''(x)=u(x+\bar b)\int_{\frac{\bar b}{g(x+\bar b)}}^1 J(z)dz-u(x-\bar b)\int_{\frac{-\bar b}{g(x-\bar b)}}^1 J(z)dz$$
$$+\int_{x-\bar b}^{x+\bar b} u(r)J\left(\frac{r-x}{g(r)}\right)\frac{dr}{g(r)}+u(x-\bar b)-u(x).$$

Since $0<g\le\bar b$ we have $\frac{\bar b}{g}\ge 1$, $\frac{-\bar b}{g}\le -1$, so that

$$\int_{\frac{\bar b}{g(x+\bar b)}}^1 J(z)dz=0 \quad\text{and}\quad \int_{\frac{-\bar b}{g(x-\bar b)}}^1 J(z)dz=1.$$

Consequently

$$F_u''(x)=\int_{x-\bar b}^{x+\bar b} u(r)J\left(\frac{r-x}{g(r)}\right)\frac{dr}{g(r)}-u(x)$$

for $x\in(-M,M)$, as was to be shown. ∎

The next result is a kind of maximum principle for solutions of (2.1). From now on, p will stand for a positive solution of (1.1) with $F_p = 1$ in $\mathbb{R}$ (see [12] for details on its construction).

Lemma 5. *Suppose that* $u : [-M - \bar{b}, M + \bar{b}] \to \mathbb{R}$ *is bounded, piecewise continuous and continuous in* $[-M, M]$*, and satisfies* (2.1) *then*

$$\max_{x\in[-M,M]} \frac{u(x)}{p(x)} \leq \sup_{x\in[-M-\bar{b},M+\bar{b}]\setminus[-M,M]} \frac{u(x)}{p(x)},$$

and

$$\min_{x\in[-M,M]} \frac{u(x)}{p(x)} \geq \inf_{x\in[-M-\bar{b},M+\bar{b}]\setminus[-M,M]} \frac{u(x)}{p(x)}.$$

Proof. We prove the first inequality, since the second one follows in the same way. We observe that $z = u/p$ satisfies

$$\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right) \frac{z(y)p(y)}{g(y)p(x)}dy - z(x) = 0 \quad \text{for } x \in [-M, M]. \tag{2.5}$$

Let $m_0 = \max_{x\in[-M,M]} z(x)$, $m_1 = \sup_{x\in[-M-\bar{b},M+\bar{b}]\setminus[-M,M]} z(x)$, and take a point $x_0 \in [-M, M]$ where m_0 is attained. Setting

$$\lambda = \int_{-M}^{M} J\left(\frac{x_0-y}{g(y)}\right) \frac{p(y)}{g(y)p(x_0)}dy \in [0, 1],$$

and taking into account that

$$\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right) \frac{p(y)}{g(y)p(x)}dy = 1 \text{ for all } x \in \mathbb{R},$$

we obtain from (2.5):

$$m_0 \leq \lambda m_0 + (1 - \lambda)m_1. \tag{2.6}$$

Observe that if $\lambda = 1$ then (2.5) and the fact that $J\left(\frac{x-y}{g(y)}\right) > 0$ if $|x-y| < \underline{b}$ imply $u \equiv m_0$ in an interval centered at x_0 of length at least $2\underline{b}$. If we assume with no loss of generality that u is not constant, then x_0 can always be selected in such a way that $\lambda < 1$, and (2.6) gives the result. □

The next lemma establishes that, when g is constant for all $x \geq K$ and $x \leq -K$, the function p has limits in $\pm\infty$. This will be used in the construction of the approximated sign changing solutions of (1.1).

Lemma 6. *Under the previous hypotheses on* J *and* g*, assume moreover that for some* $K > 0$ *we have* $g(x) = g(-K)$ *for* $x < -K$ *and* $g(x) = g(K)$ *when* $x > K$*. Then the limits*

$$p_\pm = \lim_{x\to\pm\infty} p(x)$$

exist and verify

$$\frac{1}{p_\pm} = \int_0^{g(\pm K)} \int_{-w}^{w} \int_{\frac{w}{g(\pm K)}}^{1} J(z)\, dz\, ds\, dw.$$

Proof. We will only prove that $\lim_{x\to\infty} p(x) = p_+$ as the proof of $\lim_{x\to-\infty} p(x) = p_-$ is similar. Let

$$\bar{p} = \limsup_{x\to\infty} p(x),$$

which is finite and positive, since p is bounded and bounded away from zero (see [12]). The lemma is a consequence of the following claim:

Claim: If $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} p(x_n) = \overline{p}$ then

$$p_n = \min\{p(y) : |x_n - y| \le g(K)\} \to \overline{p} \qquad \text{as } n \to \infty.$$

Assuming the claim for the moment, we will finish the proof of the lemma. For $\varepsilon > 0$, we have that $p(x) \le \overline{p} + \varepsilon$ if x is large enough. Recalling that $F_p(x) = 1$ we have, for $x_n \ge K + g(K)$,

$$\begin{aligned} 1 = F_p(x_n) &= \int_0^{g(K)} \int_{x_n-w}^{x_n+w} p(s) \int_{\frac{w}{g(K)}}^1 J(z)\,dz\,ds\,dw \\ &= \int_0^{g(K)} \int_{-w}^{w} p(x_n+s) \int_{\frac{w}{g(K)}}^1 J(z)\,dz\,ds\,dw \qquad (2.7) \\ &\le (\overline{p}+\varepsilon) \int_0^{g(K)} \int_{-w}^{w} \int_{\frac{w}{g(K)}}^1 J(z)\,dz\,ds\,dw = \frac{\overline{p}+\varepsilon}{p_+}, \end{aligned}$$

and letting $\varepsilon \to 0$ we have $p_+ \le \overline{p}$. Next, we also have from (2.7)

$$1 = F_p(x_n) \ge p_n \int_0^{g(K)} \int_{-w}^{w} \int_{\frac{w}{g(K)}}^1 J(z)\,dz\,ds\,dw = \frac{p_n}{p_+}$$

and we can let $n \to \infty$ and use the claim to obtain $\overline{p} = p_+$.

By a similar argument it can be proved that $\liminf_{x\to\infty} p = p_+$ and hence the lemma. Thus only the claim remains to be proved.

Proof of the Claim: Suppose, by contradiction, that there are sequences $x_n, y_n \to \infty$ such that $\lim_{n\to\infty} p(x_n) = \overline{p}$, $|x_n - y_n| \le g(K)$ and $\lim_{n\to\infty} p(y_n) = \overline{p} - d$ for some $d > 0$.

Notice that, since p is globally Hölder continuous with the same exponent as J, there exists $\delta > 0$ such that $|p(x_1) - p(x_2)| \le d/4$ whenever $|x_1 - x_2| \le \delta$.

On one hand, for large n we have $p(y_n) \le \overline{p} - d/2$, thus if y is such that $|y - y_n| \le \delta$, then $p(y) \le p(y_n) + d/4 \le \overline{p} - d/4$. On the other hand, given $\varepsilon > 0$ there exists N_0 such that $p(x) \le \overline{p} + \varepsilon$ if $x \ge N_0$. Then, for large enough n:

$$\begin{aligned} p(x_n) &= \int_{\mathbb{R}} J\left(\frac{x_n - y}{g(y)}\right) \frac{p(y)}{g(y)} dy = \int_{x_n-g(K)}^{x_n+g(K)} J\left(\frac{x_n-y}{g(K)}\right) \frac{p(y)}{g(K)} dy \\ &= \int_{x_n-g(K)}^{x_n+g(K)} 1_{\{|y_n-y|\le\delta\}} J\left(\frac{x_n-y}{g(K)}\right) \frac{p(y)}{g(K)} dy \\ &+ \int_{x_n-g(K)}^{x_n+g(K)} 1_{\{|y_n-y|>\delta\}} J\left(\frac{x_n-y}{g(K)}\right) \frac{p(y)}{g(K)} dy \\ &\le \int_{x_n-g(K)}^{x_n+g(K)} 1_{\{|y_n-y|\le\delta\}} J\left(\frac{x_n-y}{g(K)}\right) \frac{\overline{p}-d/4}{g(K)} dy \\ &+ \int_{x_n-g(K)}^{x_n+g(K)} 1_{\{|y_n-y|>\delta\}} J\left(\frac{x_n-y}{g(K)}\right) \frac{\overline{p}+\varepsilon}{g(K)} dy \\ &\le \overline{p} - d/4 \int_{x_n-g(K)}^{x_n+g(K)} 1_{\{|y_n-y|\le\delta\}} J\left(\frac{x_n-y}{g(K)}\right) \frac{1}{g(K)} dy + \varepsilon. \end{aligned}$$

Now notice that, by the hypotheses on J , $C_0 = \min\{J(z) : |z| \le 1 - \frac{\delta}{2g(K)}\}$ is strictly positive. Moreover, since

$$x_n + g(K) - \frac{\delta}{2} - (y_n - \delta) \ge g(K) + \frac{\delta}{2} - |x_n - y_n| \ge \frac{\delta}{2}$$

we have that the set $[x_n - g(K) + \frac{\delta}{2}, x_n + g(K) - \frac{\delta}{2}] \cap \{|y - y_n| \le \delta\}$ has measure $\ge \frac{\delta}{2}$. Hence

$$\int_{x_n - g(K)}^{x_n + g(K)} 1_{\{|y_n - y| \le \delta\}} J\left(\frac{x_n - y}{g(K)}\right) \ge \int_{x_n - g(K) + \frac{\delta}{2}}^{x_n + g(K) - \frac{\delta}{2}} 1_{\{|y_n - y| \le \delta\}} J\left(\frac{x_n - y}{g(K)}\right) \ge C_0 \frac{\delta}{2}.$$

It follows that, for every $\varepsilon > 0$ and sufficiently large n:

$$p(x_n) \le \overline{p} - d/4C_3 + \varepsilon,$$

and letting $n \to \infty$ we obtain a contradiction with $p(x_n) \to \overline{p}$. Hence the claim. □

We close this preliminary section by stating and proving a property of the function F_u given by (2.2) which will is very useful when dealing with solutions of (1.1).

Lemma 7. *Suppose that J satisfies* (H1)*. Then there exists C depending only on J and $\overline{b}$ such that for any bounded, piecewise continuous nonnegative function $h : \mathbb{R} \to \mathbb{R}$ we have*

$$F_h(x) \le C \int_{\mathbb{R}} J\left(\frac{x - y}{g(y)}\right) \frac{h(y)}{g(y)}\, dy \quad x \in \mathbb{R}.$$

Proof. We observe that by (H1), whenever x, y, w verify $0 \le |x - y|/g(y) \le w/g(y) \le 1$ we have that

$$\int_{\frac{w}{g(y)}}^{1} J(z)\, dz \le \eta J\left(\frac{w}{g(y)}\right) \le \eta^2 J\left(\frac{|x - y|}{g(y)}\right).$$

Hence

$$\begin{aligned} F_h(x) &\le \eta^2 \int_0^{\overline{b}} \int_{x-w}^{x+w} h(y) J\left(\frac{|x - y|}{g(y)}\right) dy\, dw \\ &\le \eta^2 \overline{b}^2 \int_{x - \overline{b}}^{x + \overline{b}} J\left(\frac{|x - y|}{g(y)}\right) \frac{h(y)}{g(y)}\, dy, \end{aligned}$$

as was to be proved. □

3. Construction of sign changing approximate solutions

In this section we will construct a sign changing solution for special types of functions g. Thus we will assume throughout the section that for some constant $K > 0$ we have $g(x) \equiv g(K)$ for all $x \ge K$ and $g(x) \equiv g(-K)$ for all $x \le -K$.

The procedure consists in constructing a sequence of sign changing solutions $\{u_M\}$ satisfying

$$\int_{\mathbb{R}} J\left(\frac{x - y}{g(y)}\right) \frac{u(y)}{g(y)} dy - u(x) = 0 \quad x \in [-M, M]. \tag{3.1}$$

and

$$\int_0^{\overline{b}} \int_{x-w}^{x+w} u(s) \int_{\frac{w}{g(s)}}^{1} J(z)\, dz\, ds\, dw = x \quad \text{for all } x \in [-M, M]. \tag{3.2}$$

The result we will prove is the following:

Theorem 8. *Suppose that $g(x) \equiv g(K)$ for all $x \geq K$ and $g(x) \equiv g(-K)$ for all $x \leq -K$. Then there exists $M_0 > 0$ such that for all $M > M_0$ there exists a piecewise continuous function $u_M : \mathbb{R} \to \mathbb{R}$ satisfying* (3.1) *and* (3.2)*. Moreover u_M is continuous in $[-M, M]$ and*

$$u_M(x) = \alpha_M M p(x) \quad \text{for } x > M, \qquad u_M(x) = -\beta_M M p(x) \quad \text{for } x < -M, \tag{3.3}$$

with $\frac{1}{2} \leq \alpha_M, \beta_M \leq 3$.

For $M > K$ let $\tilde{v}_M : [-M, M] \to \mathbb{R}$ be the unique solution of

$$\int_{-M}^{M} J\left(\frac{x-y}{g(y)}\right) \frac{v(y)}{g(y)} dy - v(x) = \int_{-\infty}^{-M} J\left(\frac{x-y}{g(y)}\right) \frac{Mp(y)}{g(y)} dy - \int_{M}^{\infty} J\left(\frac{x-y}{g(y)}\right) \frac{Mp(y)}{g(y)} dy \tag{3.4}$$

in $C[-M, M]$, where p stands again for a positive solution with $F_p = 1$ in $\mathbb{R}$. We observe that $\tilde{v}_M$ exists since, by a simple computation, the spectral radius of the (compact) operator $Tv = \int_{-M}^{M} J\left(\frac{x-y}{g(y)}\right) \frac{v(y)}{g(y)} dy$ in $C[-M, M]$ is less than 1. The function $\tilde{v}_M$ is continuous in $[-M, M]$ by construction.

We now define the extension v_M of $\tilde{v}_M$ by $v_M(x) = \tilde{v}_M(x)$ if $x \in [M, M]$, $v_M(x) = -Mp(x)$ for $x < -M$ and $v_M(x) = Mp(x)$ for $x > M$. It is clear then that $v_M : \mathbb{R} \to \mathbb{R}$ is piecewise continuous and satisfies (3.1). Thus from Proposition 4 there exist constants A_M and B_M such that

$$F_{v_M}(x) = A_M x + B_M \quad \text{for all } x \in [-M, M]. \tag{3.5}$$

Our immediate aim is to establish lower and upper bounds for A_M and B_M. These will be used to construct the desired approximate solutions.

Proposition 9. *Let A_M and B_M be given by* (3.5)*. Then, for sufficiently large M,*

$$\frac{1}{2} \leq A_M \leq 1 \quad \textit{and} \quad |B_M| \leq \frac{M}{2}. \tag{3.6}$$

Proof. To keep notation simple we will denote v_M by v. We start observing that by the definition of v outside $[-M, M]$ and Lemma 5 we have

$$|v(x)| \leq Mp(x) \text{ in } [-M - \bar{b}, M + \bar{b}]. \tag{3.7}$$

Therefore, as $F_p(x) = 1$, we obtain

$$|F_v(x)| \leq F_{|v|}(x) \leq F_{Mp}(x) = M \quad \text{in } [-M, M].$$

If we assume $B_M \geq 0$ and evaluate $F_v(x) = A_M x + B_M$ at $x = \pm M$ we arrive at $|A_M| M \leq |A_M| M + B_M = ||A_M| M + B_M| = |\pm A_M M + B_M| = |F_v(\pm M)| \leq M$, from where $|A_M| \leq 1$ follows. When $B_M < 0$ we proceed similarly.

Thus $|A_M| \leq 1$. To refine this bound and prove (3.6) we will make use of the following claim.

Claim: For sufficiently large M, $F_v(M) \geq \frac{1}{2} M$ and $F_v(-M) \leq -\frac{1}{2} M$.

Assuming the claim for the moment, we will finish the proof of the proposition. From the claim we have that $A_M M + B_M \geq \frac{1}{2} M$ and $A_M M - B_M \geq \frac{1}{2} M$ so that $A_M \geq \frac{1}{2}$. To prove the inequalities for B_M we observe that $M \geq A_M M + B_M \geq \frac{1}{2} M + B_M$ which implies $B_M \le \frac{1}{2}M$, and $B_M \ge \frac{1}{2}M - A_M M \ge \frac{1}{2}M - M = -\frac{1}{2}M$. This concludes the proof.

Proof of the Claim: Let p_+ be as in Lemma 6. Given $0 < \varepsilon < p_+$, we can choose a large enough $M_1 > K$ so that $p_+ - \varepsilon \le p(z) \le p_+ + \varepsilon$ for all $z \ge M_1$. If $M \ge M_1 + g(K)$ then, by (3.7), $v(z) \le M(p_+ + \varepsilon)$ for all $z \ge M - g(K)$.

To simplify the notation, we set $H(w) = \int_{\frac{w}{g(K)}}^1 J(z)\,dz$. Observe that since $|A_M| \le 1$ we have that $F_v'(M) \le 1$. Hence

$$\begin{aligned}\int_0^{g(K)} v(M-w)H(w)\,dw &\ge \int_0^{g(K)} v(M+w)H(w)\,dw \ -1\\ &= M\int_0^{g(K)} p(M+w)H(w)\,dw \ -1.\end{aligned}$$

Thus, for large enough M, we obtain that

$$\int_0^{g(K)} v(M-w)H(w)\,dw \ge M(p_+ - \varepsilon)\int_0^{g(K)} H(w)\,dw - 1. \tag{3.8}$$

Denote $U_M = \{w \in [0, g(K)] : \ v(M-w) \le \frac{3}{4}p_+ M\}$. Then

$$\begin{aligned}\int_0^{g(K)} v(M-w)H(w)\,dw &\le \int_{U_M} \frac{3}{4}Mp_+ H(w)\,dw + \int_{[0,g(K)]\setminus U_M} v(M-w)H(w)\,dw\\ &\le \frac{3}{4}Mp_+\int_{U_M} H(w)\,dw + M(p_+ + \varepsilon)\int_{[0,g(K)]\setminus U_M} H(w)\,dw\\ &\le \frac{3}{4}Mp_+\int_0^{g(K)} H(w)\,dw + M\,(p_+ + \varepsilon)\int_{[0,g(K)]\setminus U_M} H(w)\,dw.\end{aligned}$$

From this last inequality and (3.8) we get

$$\frac{1}{4}p_+\int_0^{g(K)} H(w)\,dw \le \frac{1}{M} + \varepsilon\int_0^{g(K)} H(w)\,dw + \varepsilon\int_{[0,1]\setminus U_M} H(w)\,dw.$$

Then choosing M so large as to have $1/M \le \varepsilon\int_0^{g(K)} H(w)\,dw$ we obtain from this last inequality:

$$\int_{U_M} H(w)\,dw \le \frac{12\varepsilon}{p_+}\int_0^{g(K)} H(w)\,dw,$$

and the positivity of $H(w)$ implies then

$$|U_M| \le \varepsilon\, C, \tag{3.9}$$

where C is a positive constant that depends only on J and g.

With the aid of this estimate we will be able to give a lower bound for $F_v(M)$. Observe that, for big enough M,

(3.10)

$$\begin{aligned}F_v(M) &= \int_0^{g(K)}\int_M^{M+w} Mp(s)\,ds\,H(w)\,dw + \int_0^{g(K)}\int_{M-w}^M v(s)\,ds\,H(w)\,dw\\ &= \int_0^{g(K)}\int_M^{M+w} Mp(s)\,ds\,H(w)\,dw + \int_0^{g(K)}\int_{M-w}^M v(s)1_{\{v(s)>\frac{3Mp_+}{4}\}}\,ds\,H(w)\,dw\\ &\quad + \int_0^{g(K)}\int_{M-w}^M v(s)1_{\{v(s)\le\frac{3Mp_+}{4}\}}\,ds\,H(w)\,dw.\end{aligned}$$

Then, using that $\frac{3Mp_+}{4} \geq \frac{3M(p(s)-\varepsilon)}{4}$ and $v(s) \geq -M(p_+ + \varepsilon)$ we get, from (3.9) and (3.10)

$$\begin{aligned} F_v(M) \geq& \frac{3M}{4} \int_0^{g(K)} \int_M^{M+w} p(s)\, ds\, H(w)\, dw + \frac{3M}{4} \int_0^{g(K)} \int_{M-w}^{M} p(s)\, ds\, H(w)\, dw \\ &- \frac{3M\varepsilon}{4} \int_0^{g(K)} \int_{M-w}^{M} ds\, H(w)\, dw - M(p_+ + \varepsilon)|U_M| \int_0^{g(K)} \int_{M-w}^{M} ds\, H(w)\, dw \\ \geq& \frac{3M}{4} - \varepsilon M C(J, g). \end{aligned}$$

This implies $F_v(M) \geq \frac{1}{2}M$ for big enough M. It is shown with a similar procedure that $F_v(-M) \leq -\frac{1}{2}M$, so the proof of the claim is finished. □

To conclude the section, we will prove Theorem 8.

Proof of Theorem 8. Set $u_M = (v_M - B_M p)/A_M$. Clearly u_M satisfies (3.1), (3.2) and (3.3) with $\alpha_M = (M - B_M)/(MA_M)$, $\beta_M = (M + B_M)/(MA_M)$. The proof is finished using the bounds (3.6) in Proposition 9. □

4. Bounds for approximate solutions

This section is dedicated to obtain bounds for the approximate solutions of (2.1) in the interval $[-M, M]$. They will be essential for the obtention of a sign-changing solution of (1.1).

For a bounded, piecewise continuous function u, denote $u^+ = \max\{u, 0\}$ and $u^- = \max\{0, -u\}$, the positive and negative parts, respectively.

Theorem 10. *Assume that J verifies hypothesis* (H1) *and let $\{u_M\}_{M \geq M_0}$ be a family of piecewise continuous functions defined in $\mathbb{R}$, such that each u_M is a solution of* (2.1) *in the interval $[-M, M]$ with $F_{u_M}(x) = x$ in $[-M, M]$. Suppose in addition that there exist positive constants C_1, C_2 such that for $M \geq M_0$:*

$$F_{u_M^+}(x) \leq C_1 M, \quad F_{u_M^-}(x) \leq C_2 M \quad \text{in } [-M, M]$$

and $u_M(x) \leq 0$ for $x < -M$, $u_M(x) \geq 0$ when $x > M$. Then there exists a constant C depending only on J, $\underline{b}$, $\overline{b}$, C_1 and C_2 such that for every $\kappa > 0$ and $M \geq 2\kappa + 8\overline{b}$:

$$|u_M(x)| \leq C(|x| + 1) \quad \text{when } |x| \leq \kappa. \tag{4.1}$$

Moreover, there exists $N > 0$ depending only on J, $\underline{b}$, $\overline{b}$, C_1 and C_2 such that for every $\kappa > N$ and $M \geq 2\kappa + 8\overline{b}$ it holds

$$\begin{aligned} u_M^-(x) &\leq C \quad \text{if} \quad N \leq x \leq \kappa \\ u_M^+(x) &\leq C \quad \text{if} \quad -\kappa \leq x \leq -N. \end{aligned} \tag{4.2}$$

The proof of Theorem 10 is rather involved, so it will be based on several lemmas. An important observation is that, whenever u is a solution of (2.1), the positive and negative parts of u verify

$$u^+(x) \leq \int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right) \frac{u^+(y)}{g(y)}\, dy \tag{4.3}$$

and

$$u^-(x) \leq \int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right) \frac{u^-(y)}{g(y)}\, dy$$

when $x \in [-M, M]$.

From now on, to keep the notation simple, the letter C will denote a positive constant depending on J, $\underline{b}$ and $\overline{b}$, not necessarily the same everywhere. Also, denote for simplicity $F_+ = F_{u^+}$, $F_- = F_{u^-}$.

Lemma 11. *Let u be a solution of* (2.1)*. Then there exists a constant C depending only on J, $\underline{b}$ and $\overline{b}$ such that for all $x \in [-M+\overline{b}, M-\overline{b}]$ the following hold:*

(i) $u^+(x) \le C \sup_{y\in[x-\overline{b},x+\overline{b}]} F_+(y)$;

(ii) $u^-(x) \le C \sup_{y\in[x-\overline{b},x+\overline{b}]} F_-(y)$.

Proof. We only prove case (i) since (ii) is analogous. Observe that since u^+ is positive, we have

$$\begin{aligned} F_+(x) &\ge \int_{\frac{\underline{b}}{4}}^{\frac{\underline{b}}{2}} \int_{x-w}^{x+w} u^+(y) \int_{\frac{w}{g(y)}}^{1} J(z)\,dz\,dy\,dw, \\ &\ge \frac{\underline{b}}{4} \int_{x-\frac{\underline{b}}{4}}^{x+\frac{\underline{b}}{4}} u^+(y)\,dy \int_{\frac{1}{2}}^{1} J(z)\,dz, \end{aligned}$$

so that there exists a constant C depending only on J and $\underline{b}$ such that

$$\int_{x-\frac{\underline{b}}{4}}^{x+\frac{\underline{b}}{4}} u^+(y)\,dy \le C F_+(x). \tag{4.4}$$

Since the interval $[x-\overline{b}, x+\overline{b}]$ can be covered by a finite number m of intervals of the form $[z_i - \frac{\underline{b}}{4}, z_i + \frac{\underline{b}}{4}]$ (where m depends only on $\underline{b}$ and $\overline{b}$), we obtain from (4.4) that

$$\int_{x-\overline{b}}^{x+\overline{b}} u^+(y)\,dy \le \sum_{i=1}^{m} \int_{z_i-\frac{\underline{b}}{4}}^{z_i+\frac{\underline{b}}{4}} u^+(y)dy \le C \sum_{i=1}^{m} F_+(z_i) \le C \sup_{y\in[x-\overline{b},x+\overline{b}]} F_+(y). \tag{4.5}$$

Taking into account that u solves (3.1), we finally obtain from (4.5):

$$u^+(x) \le \frac{\|J\|_\infty}{\underline{b}} \int_{x-\overline{b}}^{x+\overline{b}} u^+(y)\,dy \le C \sup_{y\in[x-\overline{b},x+\overline{b}]} F_+(x),$$

where C only depends on J, $\underline{b}$ and $\overline{b}$. This concludes the proof. □

Lemma 12. *Let u be a solution of* (2.1) *verifying $F_+(x) \le C_1 M$, $F_-(x) \le C_2 M$ in $[-M, M]$ for some constants C_1, C_2, and $u(x) \le 0$ if $x < -M$, $u(x) \ge 0$ if $x > M$. Then there exists a constant C depending only on J, $\underline{b}$, $\overline{b}$, C_1 and C_2 such that for every interval $I \subset [-M/2, M/2]$ with $|I| \le \underline{b}/4$:*

(i) *If* $|\{x \in I \ / \ u(x) \ge 0\}| \ge \underline{b}/8$ *then* $\displaystyle\int_I u^-(x)\,dx \le C$;

(ii) *If* $|\{x \in I \ / \ u(x) \le 0\}| \ge \underline{b}/8$ *then* $\displaystyle\int_I u^+(x)\,dx \le C$.

Proof. Again, we only prove (i), the other case being similar. Observe first that $F_+'' \ge 0$ in $(-M, M)$ thanks to (4.3) and Proposition 4. Also, since $u^+ = 0$ if $x < M$ we have $F_+'(-M) \ge 0$, and it follows that $F_+'(x) \ge 0$ in $[-M, M]$.

If we assume that $F_+' > 2C_1$ somewhere in $[-M/2, M/2]$, then we would have $F_+'(x) > 2C_1$ for all $x \in [M/2, M]$. Hence

$$2C_1 \frac{M}{2} < \int_{M/2}^{M} F_+'(x)\,dx = F_+(M) - F_+(M/2) \le F_+(M),$$

which contradicts our assumption. Thus $F_+'(x) \le 2C_1$ for $x \in [-M/2, M/2]$, and we arrive at

$$\int_{-M}^{M/2} \left(\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right) \frac{u^+(y)}{g(y)}\, dy - u^+(x) \right) dx = \int_{-M}^{\frac{M}{2}} F_+''(x)\, dx \le F_+'(\frac{M}{2}) \le 2C_1.$$

Next we observe that if $u(x) \ge 0$ then

$$u^+(x) = u(x) = \int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right) \frac{u^+(y) - u^-(y)}{g(y)}\, dy$$

so that

$$\int_{-M}^{\frac{M}{2}} 1_{\{u\ge 0\}} \int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right) \frac{u^-(y)}{g(y)}\, dy\, dx \le 2C_1. \tag{4.6}$$

Finally, if $I \subset [-M/2, M/2]$ is an interval with $|I| \le \underline{b}/4$, then from (4.6)

$$\int_I 1_{\{u\ge 0\}} \int_I J\left(\frac{x-y}{g(y)}\right) \frac{u^-(y)}{g(y)}\, dy\, dx \le 2C_1,$$

and since for $x, y \in I$ we have $|x-y| \le \underline{b}/2 \le \frac{g(y)}{2}$ then if $|I \cap \{u(x) \ge 0\}| \ge \underline{b}/8$ we conclude that

$$\frac{\underline{b}}{8} \min_{|z|\le \frac{1}{2}} J(z) \int_I u^-(y)\, dy \le 2C_1,$$

and (i) follows. □

With these preliminaries at hand, we can finally prove Theorem 10.

Proof of of Theorem 10. Fix M such that $M \ge 2\kappa + 8\overline{b}$. In the remainder of the proof, we will drop the subindex M for simplicity of notation, and set $u = u_M$.

Choose $x_0 \in [-\kappa, \kappa]$, and consider an interval I with the following properties: $[x_0 - 3\overline{b}, x_0 + 3\overline{b}] \subset I \subset [x_0 - 4\overline{b}, x_0 + 4\overline{b}]$ and the length of I equals $N\underline{b}/4$ for some positive integer N which depends only on $\overline{b}$ and $\underline{b}$ (this is always possible: we could take N as the integer part of $32\overline{b}/\underline{b}$, for instance). Next, divide I in closed subintervals $I_j = [a_j, b_j]$ with $j = 1, \dots, N$ of length $l = \underline{b}/4$ such that $a_j < b_j = a_{j+1}$ for $j = 1, \dots, N-1$.

We will say that $I_j \in \mathcal{I}_+$ if $|\{x \in I_j \ / \ u(x) \ge 0\}| \ge \underline{b}/8$, and $I_j \in \mathcal{I}_-$ if $|\{x \in I_j \ / \ u(x) \le 0\}| \ge \underline{b}/8$. There are three cases to consider

(i) $I_j \in \mathcal{I}_+$ for all $j = 1, \dots, N$;

(ii) $I_j \in \mathcal{I}_-$ for all $j = 1, \dots, N$;

(iii) There exist two intervals I_j, I_{j+1} such that $I_j \in \mathcal{I}_+$ and $I_{j+1} \in \mathcal{I}_-$, or $I_j \in \mathcal{I}_-$ and $I_{j+1} \in \mathcal{I}_+$.

If (i) holds, then by Lemma 12 (i) we have $\int_{I_j} u^-(x)\, dx \le C$ for all j, where C depends only on J, $\underline{b}$, $\overline{b}$, C_1 and C_2. Then, for all $x \in [x_0 - 2\overline{b}, x_0 + 2\overline{b}]$ we have

$$u^-(x) \le \int_{x-\overline{b}}^{x+\overline{b}} J\left(\frac{x-y}{g(y)}\right) \frac{u^-(y)}{g(y)}\, dy \le \frac{CN}{\underline{b}} \|J\|_\infty = C. \tag{4.7}$$

Using (4.7) it easily follows that $F_-(x) \le C$ for all $x \in [x_0 - \overline{b}, x_0 + \overline{b}]$. Hence $F_+(x) \le x + C$ for all $x \in [x_0 - \overline{b}, x_0 + \overline{b}]$. Thanks to Lemma 11:

$$u^+(x_0) \le C(x_0 + \overline{b}). \tag{4.8}$$

Then due to (4.7) and (4.8), (5.3) at x_0 follows in this case. When (ii) holds we can proceed similarly, to obtain that

$$u^+(x_0) \le C, \quad u^-(x_0) \le C(-x_0 + \bar{b}). \tag{4.9}$$

Now we consider case (iii) and suppose that $I_j \in \mathcal{I}_+$ and $I_{j+1} \in \mathcal{I}_-$. (the case $I_j \in \mathcal{I}_-$ and $I_{j+1} \in \mathcal{I}_+$ follows in the same way). Set $\delta = \min_{\{x \in I_j \ / \ u(x) \ge 0\}} F_+''(x)$. Then we have

$$F_+'(b_j) = F_+'(a_j) + \int_{a_j}^{b_j} F_+''(x)\,dx \ge \delta |\{x \in I_j \ / \ u(x) \ge 0\}| \ge \frac{\delta \underline{b}}{8},$$

since $F_+' \ge 0$ and $F_+'' \ge 0$. Therefore

$$F_+'(x) \ge \frac{\delta \underline{b}}{8} \quad \text{for all } x \ge b_j, \tag{4.10}$$

and since $F_+(M) \le C_1 M$ we conclude that

$$C_1 M \ge F_+(M) \ge \int_{x_0+3\bar{b}}^{M} F_+'(x)dx \ge \frac{\delta \underline{b}}{8}(M - x_0 - 3\bar{b}).$$

Then, since $x_0 \le \kappa \le M/2 - 4\bar{b}$ we have

$$\delta \le \frac{16 C_1}{\underline{b}}. \tag{4.11}$$

Next take $y_0 \in I_j$ such that $F_+''(y_0) = \delta$ and $u(y_0) \ge 0$. Observe that since $u(y_0) = u^+(y_0)$ we have

$$\int_{\mathbb{R}} J\left(\frac{y_0 - y}{g(y)}\right) \frac{u^-(y)}{g(y)}\,dy = \int_{\mathbb{R}} J\left(\frac{y_0 - y}{g(y)}\right) \frac{u^+(y)}{g(y)}\,dy - u^+(y_0) = \delta.$$

Thus, by Lemma 7 and (4.11) there exists C depending only on J, $\bar{b}$ and $\underline{b}$ such that $F_-(y_0) \le C$.

Also, we are assuming that $u^- = 0$ if $x > M$, so that $F_-'(M) \le 0$ and thus $F_-' \le 0$ in $[-M, M]$, that is, F_- is nonincreasing. Hence $F_-(x) \le C$ for every $x \in [y_0, M]$. This entails $F_+(x) \le x + C$ in $[y_0, M]$. We may apply Lemma 11 again to obtain a constant C such that

$$u^-(x) \le C, \quad u^+(x) \le C(x + \bar{b}) \qquad \text{for all } x \in [y_0 + \bar{b}, M]. \tag{4.12}$$

A symmetric argument shows that

$$u^+(x) \le C, \quad u^-(x) \le C(-x + \bar{b}) \qquad \text{for all } x \in [-M, y_0 - \bar{b}]. \tag{4.13}$$

By Lemma 5 we see that (4.1) holds in $[-M, M]$.

Finally notice that the election of the intervals depends on x_0, and since x_0 was arbitrary, the proof of (4.1) is concluded.

It only remains to prove (4.2). We observe that if $x_0 > \bar{b}$, then case (ii) above cannot hold thanks to (4.9). When cases (i) or (iii) hold, the first inequality in (4.2) follows from (4.7) and (4.12), respectively. The second inequality is proved in the same way. This concludes the proof. □

5. Existence of sign changing solutions. Uniqueness

This final section will be devoted to prove Theorems 1, 2 and 3. We begin by showing the existence of a solution $u : \mathbb{R} \to \mathbb{R}$ to (1.1) satisfying

$$F_u(x) = x \text{ for all } x \in \mathbb{R}. \tag{5.1}$$

We first prove the existence of such solution in the case that g is constant for large values of x, i. e. for a fixed $K > 0$, $g(x) \equiv g(K)$ for all $x \geq K$ and $g(x) \equiv g(-K)$ for all $x \leq -K$. In this case the solution u can be obtained as the limit as $M \to \infty$ of a sequence of solutions $\{u_M\}$ to the approximated problems (3.1) provided by Theorem 8. For a general g, we first truncate at a value K and then pass to the limit as $K \to \infty$.

Proof of Theorem 1. First we assume that for some $K > 0$, $g(x) \equiv g(K)$ when $x \geq K$, $g(x) \equiv g(-K)$ for $x \leq -K$. We may apply Theorem 8 to obtain a family of solutions $\{u_M\}_{M \geq M_0}$ of (3.1) verifying (3.2).

Observe that by Lemma 5 we have

$$0 \leq u^+(x) \leq \alpha_M M p(x), \quad 0 \leq u^-(x) \leq \beta_M M p(x) \quad \text{for all } x \in \mathbb{R}, \tag{5.2}$$

so that $0 \leq F_{u_M^+}(x) \leq \alpha_M M F_p(x) = \alpha_M M \leq 3M$, $0 \leq F_{u_M^-}(x) \leq \beta_M M F_p(x) = \beta_M M \leq 3M$. Also, since $u^+(x) = 0$ for all $x < -M$ and $u^-(x) = 0$ for all $x > M$ we can apply Theorem 10 to obtain that, for every $\kappa > 0$ and large enough M:

$$|u_M(x)| \leq C(|x| + 1) \qquad \text{for } |x| \leq \kappa, \tag{5.3}$$

where C does not depend on M (nor on K). In particular, (5.3) implies that for every sequence $M_n \to \infty$ the sequence u_{M_n} is locally bounded. Using (3.1), we obtain that

$$|u_{M_n}(x) - u_{M_n}(x')| \leq \frac{1}{\underline{b}^{1+\alpha}} L|x - x'|^\alpha \int_{-M-\overline{b}}^{M+\overline{b}} |u_{M_n}(y)| dy,$$

where α and L are the Hölder exponent and Hölder constant of J, respectively, so that u_{M_n} is equicontinuous in compact sets, and by Arzelá-Ascoli theorem and a standard diagonal procedure we arrive at the convergence (passing to a subsequence) $u_{M_n} \to u$ uniformly in compacts of $\mathbb{R}$. It follows easily that u is a solution of (1.1) verifying (5.1) and

$$|u(x)| \leq C(|x| + 1) \qquad \text{in } \mathbb{R}. \tag{5.4}$$

Moreover, passing to the limit in the first inequality of (4.2) in Theorem 10 we obtain that $u^- \leq C$ if $x \geq N$, where N only depends on J, $\underline{b}$ and $\overline{b}$. Then, thanks to Lemma 7:

$$\begin{aligned} x = F_u(x) \leq F_{u^+}(x) &\leq C \int J\left(\frac{x-y}{g(y)}\right) \frac{u^+(y)}{g(y)} dy \\ &= C\left(u(x) + \int J\left(\frac{x-y}{g(y)}\right) \frac{u^-(y)}{g(y)} dy\right) \leq C(u(x) + D) \end{aligned} \tag{5.5}$$

if $x \geq N + \overline{b}$, where D is a positive constant, depending only on J, $\underline{b}$ and $\overline{b}$. Then (5.4) and (5.5) imply (1.6) for large positive x. The case of large negative x is dealt with in the same way.

The existence result for a general function g is similarly shown. We truncate g to obtain a function g_K which verifies $g_K(x) = g(x)$ if $x \in [-K, K]$, $g_K(x) = g(-K)$ for $x < -K$ and $g_K(x) = g(K)$ for $x > K$. Of course, the function g_K still verifies $\underline{b} \leq g_K \leq \overline{b}$. By the first part of the proof, there exists a solution u_K of (1.1) verifying $F_{u_K}(x) = x$ in $\mathbb{R}$ and (5.4), where C does not depend on K. Thus for a sequence $K_n \to \infty$, the sequence $\{u_{K_n}\}$

is locally uniformly bounded, and we conclude as before that there exists a continuous function q such that $u_{K_n} \to q$ uniformly on compacts. Then q is a solution of (1.1), and it verifies $F_q(x) = x$ and $|q(x)| \leq C(|x|+1)$. It also follows passing to the limit in (5.4) and (5.5) that q verifies (1.6). This concludes the proof. □

We now turn to the question of characterizing the solutions of (1.1) with polynomial growth. But before proving Theorem 2 in its full generality, let us consider a restricted version of uniqueness, valid only for solutions which are bounded on one side.

Proposition 13. *Let u be a solution of* (1.1) *which is bounded from above or from below. Then $u = \theta p$ for some constant θ.*

Proof. It suffices for the proof with assuming that u is bounded from above, since otherwise we change u to $-u$.

We will make use of a sweeping argument. For $t > 0$ let $z_t(x) = p(x) - tu(x)$. Since p is bounded away from zero and u is bounded from above, we have that $z_t > 0$ in $\mathbb{R}$ for small enough t. Define

$$\theta = \sup\{t > 0 : \ z_t(x) > 0 \text{ in } \mathbb{R}\}.$$

Then clearly $z_\theta \geq 0$ in $\mathbb{R}$. Also notice that z_θ cannot be bounded away from zero in $\mathbb{R}$. If it were, we would have $z_{\theta+\varepsilon} > 0$ in $\mathbb{R}$ for small ε, contradicting the maximality of θ.

Thus two cases are possible: either $z_\theta(x_0) = 0$ for some $x_0 \in \mathbb{R}$ or $z_\theta > 0$ in $\mathbb{R}$ and there exists an unbounded sequence $\{x_n\}$ such that $z_\theta(x_n) \to 0$.

In the former case, since z_θ is also a solution of (1.1), then

$$\int_{\mathbb{R}} J\left(\frac{x_0 - y}{g(y)}\right) \frac{z_\theta(y)}{g(y)} dy = 0,$$

and we conclude that $z_\theta = 0$ in a neighborhood of x_0 of length at least $2\underline{b}$. A standard connectedness argument would then provide with $z_\theta = 0$ in $\mathbb{R}$, that is, $u = \theta p$, as was to be shown.

In the latter case, since z_θ is positive, we may apply the Harnack inequality in [18] finitely many times to obtain a positive constant C such that $z_\theta(s) \leq C z_\theta(x_n)$ in $|s - x_n| \leq \bar{b}$. Hence

$$F_{z_\theta}(x_n) = \int_0^{\bar{b}} \int_{x_n - w}^{x_n + w} z_\theta(s) \int_{\frac{w}{g(s)}}^1 J(z) dz ds dw \leq C z_\theta(x_n). \tag{5.6}$$

On the other hand, since $z_\theta \geq 0$, it also follows that $F_{z_\theta} \geq 0$, and by Proposition 4, F_{z_θ} has to be a nonnegative constant. Letting $n \to \infty$ in (5.6) implies $F_{z_\theta} = 0$ in $\mathbb{R}$, which leads to $z_\theta = 0$, so that $u = \theta p$. This concludes the proof. □

We can now proceed to prove Theorem 2.

Proof of Theorem 2. Let u be a solution of (1.1) verifying $|u(x)| \leq C(|x|^k + 1)$ and $F_u(x) = Ax + B$ in $\mathbb{R}$ for some constants A, B. Let $v = u - Aq - Bp$, so that v is a solution of (1.1) verifying $|v(x)| \leq C(|x|^k + 1)$ and $F_v = 0$. The proof reduces to show that $v = 0$.

If v is bounded from above, then thanks to Proposition 13 we have $v = \theta p$ for some $\theta \in \mathbb{R}$. Thus $0 = F_v = \theta$ and $v = 0$ follows. Hence we may assume in what follows that v is unbounded from above. For $n \in \mathbb{N}$, define

$$c_n = \sup_{[-n\bar{b}, n\bar{b}]} \frac{v(x)}{p(x)},$$

and observe that $c_n \to \infty$. Now choose a point $x_n \in [-n\bar{b}, n\bar{b}]$ where the supremum is attained. Thanks to Lemma 5, we may take $x_n \in [-n\bar{b}, -(n-1)\bar{b}] \cup [(n-1)\bar{b}, n\bar{b}]$.

Since $c_n = v(x_n)/p(x_n)$:

$$c_n \int J\left(\frac{x_n - y}{g(y)}\right) \frac{p(y)}{g(y)} dy \le \int J\left(\frac{x_n - y}{g(y)}\right) \frac{v(y)}{g(y)} dy$$

so that

$$\int_{\{v < c_n p\}} J\left(\frac{x_n - y}{g(y)}\right) \frac{c_n p(y) - v(y)}{g(y)} dy \le \int_{\{v \ge c_n p\}} J\left(\frac{x_n - y}{g(y)}\right) \frac{v(y) - c_n p(y)}{g(y)} dy.$$

Next, we notice that the last integral is performed in a set contained in $[x_n - \bar{b}, x_n + \bar{b}] \subset [-(n+1)\bar{b}, (n+1)\bar{b}]$, so that $v \le c_{n+1} p$ there. Thus

$$\int_{\{v < c_n p\}} J\left(\frac{x_n - y}{g(y)}\right) \frac{c_n p(y) - v(y)}{g(y)} dy \le (c_{n+1} - c_n) p(x_n).$$

We can now use Lemma 7 to obtain that $F_{(c_n p - v)^+}(x_n) \le D(c_{n+1} - c_n) p(x_n)$ for some positive constant D which does not depend on n. Then,

$$\begin{aligned} 0 = F_{-v}(x_n) &= F_{c_n p - v}(x_n) - c_n \\ &\le F_{(c_n p - v)^+}(x_n) - c_n \le D(c_{n+1} - c_n) - c_n. \end{aligned}$$

We deduce that $c_{n+1} \ge (1+a) c_n$ for some $a > 0$, and iterating this inequality we arrive at $c_n \ge C(1+a)^n$ for some constant $C > 0$. Finally, since $|x_n| \le n\bar{b}$, this yields

$$v(x_n) \ge C(1+a)^{\frac{|x_n|}{\bar{b}}}.$$

Taking into account that $|x_n| \to \infty$, we reach a contradiction with the polynomial growth of v. This contradiction concludes the proof. □

Proof of Theorem 3. Let H be a C^1, even, compactly supported, nonnegative function with unit integral. For small $\varepsilon > 0$ define

$$J_\varepsilon(x) = \frac{1}{4\varepsilon}\left(H\left(\frac{x+1}{\varepsilon}\right) + H\left(\frac{x+\frac{1}{2}}{\varepsilon}\right) + H\left(\frac{x-\frac{1}{2}}{\varepsilon}\right) + H\left(\frac{x-1}{\varepsilon}\right)\right).$$

Then J_ε is also a C^1, compactly supported, nonnegative function with unit integral. We look for solutions of $J_\varepsilon * u - u = 0$ of the form $e^{\alpha x}$ for some $\alpha \in \mathbb{C}$. This leads to the equation $F(\alpha, \varepsilon) = 0$, where

$$\begin{aligned} F(\alpha, \varepsilon) := \frac{1}{4\varepsilon}\Bigg(&\int H\left(\frac{z+1}{\varepsilon}\right) e^{\alpha z} dz + \int H\left(\frac{z+\frac{1}{2}}{\varepsilon}\right) e^{\alpha z} dz \\ &+ \int H\left(\frac{z-\frac{1}{2}}{\varepsilon}\right) e^{\alpha z} dz + \int H\left(\frac{z-1}{\varepsilon}\right) e^{\alpha z} dz \Bigg) - 1. \end{aligned}$$

It is not hard to check that F is continuous with respect to both variables α, ε when ε is small enough, and continuously differentiable with respect to α. Moreover:

$$F(\alpha, 0) = \frac{1}{4}\left(e^{-\alpha} + e^{-\frac{1}{2}\alpha} + \varepsilon^{\frac{1}{2}\alpha} + e^{\alpha}\right) - 1.$$

Setting $t = e^{\frac{1}{2}\alpha}$, the equation $F(\alpha, 0) = 0$ is easily solved, and we obtain the roots $\alpha = 0$ (double) and $\alpha_{\pm} = 2\log\left(\frac{3\pm\sqrt{5}}{2}\right) + 2\pi i$ (simple). Thus

$$\frac{\partial F}{\partial \alpha}(\alpha_{\pm}, 0) \neq 0,$$

and we may apply the implicit function theorem to obtain that the equation $F(\alpha, \varepsilon) = 0$ is solvable in a neighborhood of $\alpha_{\pm}$ for small ε. This entails, by taking real and imaginary parts, that the equation $J_\varepsilon * u - u = 0$ in $\mathbb{R}$ has solutions of the form $e^{ax}\cos(bx)$, $e^{ax}\sin(bx)$ for small enough ε, where $a \sim 2\log\left(\frac{3\pm\sqrt{5}}{2}\right)$, $b \sim 2\pi$. The proof is concluded. □

Remark 1. The kernel J can be constructed strictly positive in $(-1-\varepsilon, 1+\varepsilon)$, by simply adding to J_ε a suitable function G_ε which is of order ε^γ for some $\gamma > 1$ and small ε.

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