Asymptotic Profiles of Nonlinear Diffusion Problems

J. García-Melián and J. Sabina de Lis

Departamento de Análisis Matemático, Universidad de La Laguna

C/ Astrofísico Francisco Sánchez s/n, 38271-La Laguna (Tenerife), SPAIN.

Abstract: Several aspects concerning the existence and uniqueness of positive solutions to a class of boundary value problems involving the p-Laplacian are discussed. Their limit behaviour as a parameter is large is also analyzed.

1. INTRODUCTION

The present work provides a description on the existence, uniqueness and limit profile structure as $\lambda \to +\infty$ of the positive radial solutions to the problem

$$\begin{cases} -\Delta_p u &= \lambda f(u) \qquad x \in B \\ \quad u &= 0 \qquad x \in \partial B, \end{cases} \tag{1.1}$$

where $B = \{x : |x| < R\} \subset \mathbb{R}^N$ and $\Delta_p := \text{div } (|\nabla u|^{p-2}\nabla u)$, $p > 1$, stands for the p-Laplacian defined in distributional sense in $W_0^{1,p}(B)$ with values in $W^{-1,p'}(B)$, $p' = p/(p-1)$.

It is well-known from the experience of the linear diffusion case $p = 2$ ($\Delta_p = \Delta$), the critical influence of the structure and growth properties as $u \to +\infty$ of the nonlinearity $f = f(u)$ on this kind of features. A scenario extensively studied when $p = 2$ is the case where f exhibits a positive zero $u = \overline{u}_0 > 0$ and positive solutions $u = u(x)$ to (1.1) are sought under the requirement that $0 < u(x) \le \overline{u}_0$ in B while $\overline{u} := \sup_B u \to \overline{u}_0-$ as $\lambda \to \infty$ (see, for instance, [1], [2], [3], [6], [7] and references therein). This is precisely the framework in which problem (1.1) will be studied here. As introductory facts, it should be first remarked that a local necessary condition on f in order that (1.1) exhibits positive solutions $u = u(x)$ with maximum $\overline{u} = \sup_B u \le \overline{u}_0$ satisfying $\overline{u} \to \overline{u}_0$ as $\lambda \to \infty$, is $f(u) > 0$ for $\overline{u}_0 - \eta \le u < \overline{u}_0$ and some $\eta > 0$. A second global necessary requirement on f is the integral condition,

$$F(u) < F(\overline{u}_0) \qquad 0 \le u < \overline{u}_0, \tag{1.2}$$

where $F(u) = \int_0^u f$. In fact, the one-dimensional version of (1.1), $-(|u'|^{p-2}u')' = \lambda f(u)$, $0 < x < R$, $u(0) = u(R) = 0$, can be used to check both assertions. For its use in the present note it is convenient to set the following terminology. It will be said that $u = \overline{u}_0$ is a zero of f with order $k > 0$ (k not necessarily an integer) if $f(\overline{u}_0) = 0$, f is continuous near $\overline{u}_0$ and C^1 in $0 < |u - \overline{u}_0| < \eta_1$ for some $\eta_1 > 0$ while, more importantly, either,

$$f'(u) = Ck|u - \overline{u}_0|^{k-2} + h(u) \quad \text{or} \quad f'(u) = -Ck\psi_k(u - \overline{u}_0) + h(u), \tag{1.3}$$

where $C \neq 0$, $\psi_k(z) = |z|^{k-2}z$ and $h = o(|u - \overline{u}_0|^{k-1})$ as $u \to \overline{u}_0$.

Sections 2 and 3 explains the importance of the balance between the order k of $\overline{u}_0$ and p with regard to the existence of positive radial solutions u to (1.1) close but smaller than $\overline{u}_0$. As a final remark it should be recalled that when $p \neq 2$, positive solutions to (1.1) need not to be radial.

2. NON DEGENERATE CASE

The parameter restriction $k \ge p - 1$ implies, through the strong maximum principle ([8]), the fact that solutions u to (1.1) with $u \le \overline{u}_0$, $\sup_B u \sim \overline{u}_0$, *never* attains the value $\overline{u}_0$ in B. A result of existence and uniqueness can be stated as follows.

THEOREM 1. *Assume that* $f = f(u) \in C^1[0, \overline{u}_0]$ *exhibits a zero* $u = \overline{u}_0$ *of order* k *with* $C > 0$ *in (1.3) satisfying the condition (1.2). If* k *and* p *keep the relation* $k \geq p - 1$ *then there exist positive constants* λ_0 *and* η_2 *such that (1.1) exhibits, for each* $\lambda \geq \lambda_0$, *a* unique *positive radial solution* $u = u_\lambda(x)$ *with* $\overline{u}_0 - \eta_2 < \sup_B u < \overline{u}_0$. *In addition,*

$$\lim_{\lambda \to +\infty} u_\lambda(x) = \overline{u}_0, \tag{2.1}$$

uniformly over compact sets of B. *Moreover, the following exact asymptotic profile estimate holds,*

$$\lim_{\lambda \to +\infty} \lambda^{-1/p} \frac{du_\lambda}{dr}_{|r=|x|=R} = -(p' F(\overline{u}_0))^{1/p}, \tag{2.2}$$

with $F(\overline{u}_0) = \int_0^{\overline{u}_0} f$ *and* $p' = p/(p-1)$.

REMARKS. The relevant fact of uniqueness in Theorem 1 is achieved by showing that positive radial solutions $u = u(x) \sim \overline{u}_0$ to (1.1) must be isolated with index 1 as $\lambda \to +\infty$ ([5]). Radial symmetry is instrumental in this point to deal with the linearization of (1.1) which degenerates at the critical points of such solutions u, $x_0 \in B$, $\nabla u(x_0) = 0$. In fact, such x_0's must be all $x_0 = 0$. However, to extend the uniqueness assertion in Theorem 1 to the framework of general domains $\Omega \subset \mathbb{R}^N$ seems to be a harder open problem.

3. DEAD CORES

The complementary parameters regime $k < p - 1$ is characterized by the appearance of the so-called dead cores, i.e. nonempty regions $\{u(x) = \overline{u}_0\} \subset B$ where a positive solution u to (1.1) achieves the value $\overline{u}_0$. The main features of this case are next described.

THEOREM 2. *Let* $f = f(u) \in C[0, \overline{u}_0] \cap C^1\{u : f(u) \neq 0\}$ *be such that all its possible zeros satisfy (1.3). Let* $u = \overline{u}_0$ *be one of such zeros which satisfies in addition (1.2),* $C > 0$ *in (1.3) while* $k < p - 1$. *Then, problem (1.1) admits a positive radial solution* $u = u_\lambda(x)$ *for* $\lambda \geq \bar{\lambda}$ *with* $\overline{u}_0 - \eta_3 < \sup_B u_\lambda \leq \overline{u}_0$, *where* $\bar{\lambda}$ *and* η_3 *are certain positive constants. Moreover, such a solution is* unique *under the restriction*

$$u_\lambda \to \overline{u}_0 \qquad \text{as} \quad \lambda \to \infty \ , \tag{3.1}$$

uniformly on compact sets of B. *On the other hand, there exists* $\lambda^* \geq \bar{\lambda}$ *such that the region* $\mathcal{O}_\lambda = \{x : u_\lambda(x) = \overline{u}_0\}$ *is nonempty for* $\lambda \geq \lambda^*$ *while* $\mathcal{O}_\lambda = \{x : |x| \leq r(\lambda)\}$ *where* $r(\lambda)$ *can be exactly estimated as*

$$r(\lambda) \sim R - \lambda^{-1/p} T_0 \qquad \text{as} \quad \lambda \to +\infty \ ,$$

with $T_0 = (p')^{-1/p} \int_0^{\overline{u}_0} (F(\overline{u}_0) - F(s))^{-1/p} ds$. *Finally, the asymptotic estimate (2.2) for* $\frac{du_\lambda}{dr}|_{r=R}$ *holds also true.*

REMARKS. Due to the presence of dead cores the linearization of (1.1) is now out of use. Therefore one must work directly with the radial o.d.e. version of (1.1) (cf. [4]). Moreover, an extension of Theorem 2 permits also to show the uniqueness of *all* positive solutions, radial or not, to the logistic case of (1.1) where $f(\lambda, u) = \lambda u^{p-1} - u^q + o(u^q)$ as $u \to +\infty$, $q > p - 1$, replaces the term $\lambda f(u)$ in (1.1) (cf. [4]).

4. A COUNTEREXAMPLE

It can be further shown ([4]) that the *uniqueness* of positive solutions holds in annuli $A = \{x : 0 < a < |x| < R\}$ under the less restrictive assumption (cf. Theorem 1) that $\sup_A u \to \overline{u}_0$ as

$\lambda \to +\infty$ when $k < p-1$. However, our next result says that the stronger convergence condition (3.1) can not be relaxed in this way in Theorem 2 for the case of balls B.

THEOREM 3. *Fix arbitrary positive numbers $p > 1$, $0 < \overline{u}_1 < \overline{u}_0$, $0 < k_1, k_0 < p-1$ together with $d_0 \geq 0$. Then, there exists $\overline{u}_1 < \overline{u}_2 < \overline{u}_0$ such that problem (1.1) corresponding to the bistable-like nonlinearity,*

$$f(u) = -(\psi_{k_1+1}(u-\overline{u}_1))(u-\overline{u}_2)\psi_{k_0+1}(u-\overline{u}_0), \tag{4.1}$$

admits, aside *the family of radial positive solutions $u = u_\lambda(r)$ obtained in Theorem 2, a* second *family of radial positive solutions $u = \tilde{u}_\lambda(r)$ in the ball B such that* $\sup_B \tilde{u}_\lambda = \overline{u}_0$ *for λ large. Moreover,*

$$\tilde{u}_\lambda(r) = \begin{cases} \overline{u}_0 & \text{for} \quad 0 \leq r \leq r_0(\lambda) := \lambda^{-1/p} d_0 \\ \in (\overline{u}_1, \overline{u}_0) & \text{for} \quad r_0(\lambda) < r < \tilde{r}_0(\lambda) = \lambda^{-1/p}(d_0 + \tilde{T}_0) \\ \overline{u}_1 & \text{for} \quad \tilde{r}_0(\lambda) \leq r \leq r_1(\lambda) < R, \end{cases}$$

for some positive $\tilde{T}_0$, while

$$R - r_1(\lambda) \sim \lambda^{-1/p} T_1 \qquad \text{as} \quad \lambda \to +\infty,$$

with $T_1 = (p')^{-1/p} \int_0^{\overline{u}_1} (F(\overline{u}_1) - F(s))^{-1/p} ds$, $F(u) = \int_0^u f$, $f = f(u)$ given by (4.1).

REMARKS. The key point in the proof of Theorem 3 is the fact that radial solutions $u = u_\lambda(r)$ to (1.1) arise, after scaling λ and a convenient phase translation, from the *minimal* solution to the initial value problem $-((r+d)^{N-1}\psi_p(u'))' = \lambda(r+d)^{N-1} f(u)$, $u(0) = \overline{u}_0, u'(0) = 0$, where $d \to +\infty$ and $r_0(\lambda) = \lambda^{-1/p} d$ measures the width of a dead core. The crucial fact is that such problem can be considered as a smooth perturbation as $d \to +\infty$ of its one-dimensional version $N = 1$. The same holds when $\overline{u}_1$ replaces $\overline{u}_0$ in such problem.

ACKNOWLEDGMENTS. This work has been supported by a DGES project under contract PB96-0621.

REFERENCES

1. ANGENENT S. B., Uniqueness of the solution of a semilinear boundary value problem, *Math. Ann.* **272** (1985) 129-138 .
2. DANCER E. N., On the number of positive solutions of weakly non-linear elliptic equations when a parameter is large, *Proc. London Math. Soc.* (3) **53** (1986) 429-452.
3. FRAILE J., LÓPEZ-GÓMEZ J., SABINA DE LIS J., On the global structure of the set of positive solutions of some semilinear elliptic equations, *J. Differential Equations.* (1) **123** (1995) 180-212.
4. GARCIA-MELIÁN J., SABINA DE LIS J., Uniqueness to quasilinear problems for the p-Laplacian in radially symmetric domains, *Nonlinear Anal.*, to appear (1999).
5. GARCIA-MELIÁN J., SABINA DE LIS J., Radially symmetric solutions to quasilinear problems, in preparation.
6. SWEERS G., On the maximum of solutions for a semilinear elliptic equations, *Proc. Roy. Soc. Edinburgh Sect. A* **108** (1988) 357-370.
7. SMOLLER J., WASSERMAN A., On the Monotonicity of the Time-Map, *J. Differential Equations* **77** (1989) 287-303.
8. VAZQUEZ J. L., A strong maximum principle for some quasilinear elliptic equations, *Appl. Math. and Optimization* **12** (1984) 191-202.