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STATIONARY PROFILES OF DEGENERATE PROBLEMS WHEN A PARAMETER IS LARGE

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Abstract. The structure of positive solutions to nonlinear diffusion problems of the form $-\text{div }(|\nabla u|^{p-2}\nabla u) = \lambda f(u)$, in Ω, $u = 0$ on $\partial\Omega$, $p > 1$, $\Omega \subset \mathbb{R}^N$ a bounded, smooth domain, is precisely studied as $\lambda \to +\infty$, for a class of logistic-type nonlinearities $f(u)$. By logistic it is understood that $f(u)/u^{p-1}$ is decreasing in $u > 0$, $f(u) \sim mu^{p-1}$, $m > 0$, as $u \to 0+$, while f has a positive zero $u = u_0$ of order k. It is shown that the positive solution u_λ homogenizes towards u_0 as $\lambda \to +\infty$, and develops a boundary layer near $\partial\Omega$ whose width is exactly measured. On the other hand, the arising of "dead cores" $\{u_\lambda = u_0\}$ for λ large is shown in the parameters regime $k < p-1$, the distance $\text{dist}(\{u_\lambda = u_0\}, \partial\Omega)$ to $\partial\Omega$ being also exactly estimated as $\lambda \to +\infty$. Thus, earlier results in [12], [22] are substantially sharpened. In addition, suitable lower-order perturbations at infinity of the problem are studied.

1. Introduction. The present work is devoted to performing a detailed analysis of the *inner* and *boundary* behaviour of positive solutions to the following class of nonlinear diffusion problems:

$$\begin{cases} -\Delta_p u = \lambda f(u) & x \in \Omega \\ u = 0 & x \in \partial\Omega\,, \end{cases} \tag{P}$$

where the parameter $\lambda > 0$ grows to $+\infty$. In (P), $\Delta_p u = \text{div }(|\nabla u|^{p-2}\nabla u)$ stands for the p-Laplacian operator, defined in the distributional sense in $W_0^{1,p}(\Omega)$, $p > 1$, Ω being a $C^{2,\alpha}$ bounded domain of $\mathbb{R}^N$ for some $0 < \alpha < 1$ (cf. [18]).

As usual, the structure of the nonlinearity $f = f(u)$ exerts a crucial influence in the issues of existence, uniqueness and behaviour, with regard to

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λ, of positive solutions to (P). In this work we deal with smooth reaction terms $f = f(u)$ such that $f(u) \sim m\, u^{p-1}$ as $u \to 0+$, $m > 0$, and $f(u)/u^{p-1}$ is decreasing in $u > 0$, while f admits a (single) positive zero $u = u_0$ of order $k > 0$ (cf. hypothesis (Hf) in Section 2 for precise details). Distinguished examples and certain perturbations of this kind of problem will be also considered in Section 4.

Under such restrictions on f it is known in the literature (see [9] and further comments in Section 2) that (P) only admits positive solutions when λ exceeds the value $\lambda_{\min} := \sigma_{1,p}/m$, $\sigma_{1,p}$ being the first Dirichlet eigenvalue of $-\Delta_p$ (cf. Section 2). Moreover, a positive solution such as $u = u_\lambda(x)$ is unique in that range of λ (see also [20] for uniqueness when λ is large). In Theorem 1 of Section 2, existence and uniqueness for (P) are reviewed. In addition, a detailed account on the smoothness of u_λ together with the increasing and continuous dependence on λ of positive solutions to (P) is described. On the other hand, it is also pointed out that such solutions are uniformly estimated by the zero $u = u_0$ of f,

$$0 < u_\lambda(x) \leq u_0 \qquad x \in \Omega\,. \tag{E}$$

After these remarks, we are ready to draw the main features of the solutions of (P) to be considered in this paper. The first one consists in the homogeneity of u_λ when it converges to u_0 as $\lambda \to +\infty$, in other words, the validity of the limit

$$\lim_{\lambda \to +\infty} u_\lambda = u_0\,, \tag{H}$$

uniformly over compact sets $K \subset \Omega$. This type of "inner" behaviour of u_λ has been studied for the semilinear version (i.e., $\Delta_p = \Delta$) of (P) in a series of works, namely, [3], [5], [6], [12], [13], [29], and [30] (see also [28] for the case of Neumann boundary conditions and [25] for this sort of behaviour in the context of cooperative systems). In particular, continuing to the framework of (P) the results in [12] was the starting point for the present research.

On the other hand, in [22] (cf. also [19] for previous one-dimensional results) this phenomenology was studied for (P), in the parameters regime $0 < k < p - 1$ where (H) occurs in a singular way, namely, accompanied by the presence of dead cores. This means that for λ large enough the set $\mathcal{O}_\lambda := \{u_\lambda(x) = u_0\}$ becomes nonempty and (H) holds since $\mathcal{O}_\lambda$ progressively expands, almost covering Ω as $\lambda \to \infty$. Such a phenomenom belongs to the realm of free boundaries (cf. [8] for a general overview on the subject). The region $\mathcal{O}_\lambda$ is called a dead core, also a "quasi-solid" state in fluid mechanics ([8], [23]) while u_λ is termed as a "flat pattern" in [22].

It is shown in Theorem 3 of Section 3 that positive solutions u_λ homogenize towards u_0 as $\lambda \to +\infty$. Observe that, as a consequence of (H) and the fact that solutions $u = u_\lambda(x)$ to (P) must satisfy the boundary condition $u_\lambda|_{\partial\Omega} = 0$, a second kind of behaviour arises. In fact, as λ becomes large, u_λ undergoes a progressively steeper transition from values close to (or exactly) u_0 to $u = 0$ on $\partial\Omega$, giving rise to a boundary layer. In particular,

$$\lim_{\lambda\to\infty} \frac{\partial u_\lambda}{\partial \nu} = -\infty\,, \tag{B}$$

uniformly on $\partial\Omega$, ν being the outward unit normal on $\partial\Omega$. To measure the "width" of the boundary layer is to provide as fine as possible an estimate of the order of magnitude of $\frac{\partial u_\lambda}{\partial \nu}$ as $\lambda \to +\infty$. Thus, to furnish an exact account of this "boundary" behaviour is the second objective of the present work. In Theorem 3 of Section 3, we also give an exact estimate of the boundary layer under the form $\frac{\partial u_\lambda}{\partial \nu} \sim -C_1\lambda^{\frac{1}{p}}$, where the expression of the constant C_1 is explicitly found in terms of f and p (see also [6] for a rough estimate in the case $p = 2$).

Theorem 3 provides a substantial extension of a series of results. Firstly, it extends to the case of the p-Laplacian operator the results on boundary layers of Section 2 in [12]. Moreover, even for the case $p = 2$ the technical convexity hypothesis on Ω there is dropped in our analysis. Therefore, the boundary layer estimates corresponding to the particular case $p = 2$ in our Theorem 3 can be used to remove everywhere in [12] the restriction on the convexity of Ω. Secondly, in [22] only the case where dead cores arise, i.e., in the parameter range $0 < k < p - 1$, is considered. However, our results on homogeneity and boundary layers are valid no matter what the relative values of k and p are. In fact, dead cores do not arise if $k \geq p-1$ (see Theorem 2, Section 2). Thus Theorem 3 fills the gap corresponding to $k \geq p - 1$ left open in [22].

Dead cores phenomenology for (P) is also studied in detail in the present work. Their existence was proved in [22] provided λ is larger than some critical value λ^* and when $0 < k < p-1$, such existence being excluded here in the complementary range $k \geq p-1$ (Theorem 2, Section 2). In Section 3, the results in [22] are sharpened in two directions. On one hand, the value λ^* is estimated in terms of the nonlinearity f and some geometrical parameters associated with Ω. On the other, and more importantly, the distance dist $(\mathcal{O}_\lambda, \partial\Omega)$ from $\mathcal{O}_\lambda$ to the boundary $\partial\Omega$ of Ω is precisely estimated in the form dist $(\mathcal{O}_\lambda, \partial\Omega) \sim C_2\lambda^{-\frac{1}{p}}$ as $\lambda \to +\infty$, providing an explicit expression for C_2 in terms of f and p. All these facts are collected in Theorem 4.

The results of Section 3 are applied in Section 4 to the family of logistic-type perturbed problems,

$$\begin{cases} -\Delta_p u = m\lambda|u|^{p-2}u - |u|^{q-1}u + g(u) & x \in \Omega \\ u = 0 & x \in \partial\Omega\,, \end{cases} \tag{LP}$$

where $q > p-1 > 0$, $m > 0$ and $g(u)$ is a C^α perturbation term which is both small when compared to u^{p-1} as $u \to 0+$ and to u^q as $u \to +\infty$. Observe that the case $g = 0$ in (LP), i.e.,

$$\begin{cases} -\Delta_p u = m\lambda|u|^{p-2}u - |u|^{q-1}u & x \in \Omega \\ u = 0 & x \in \partial\Omega\,, \end{cases} \tag{L}$$

is a sort of p-Laplacian version of the so-called logistic equation in population dynamics ($p = 2$; cf. [26], [27]), and also of Fisher's equation in population genetics ($p = q = 2$, cf. [11], [26]).

It is shown in Section 4 that (L) fits, after an appropriate scaling, with the model problem (P), and so Theorems 3 and 4 provide a precise account on boundary layers and dead cores to (L) (Corollaries 9, 10). By means of comparison with suitable logistic approximated problems (see Theorem 8) we will be able to describe how the positive solutions to the problem (LP) homogenize as $\lambda \to +\infty$, giving also a precise description of their boundary behaviour (see Corollary 9). However, it should be pointed out that regarding the issues of uniqueness of positive solutions and dead cores generation, (LP) is by no means a trivial perturbation of the logistic problem (L). In the case of uniqueness the main difficulty arises when one tries to use continuation methods based upon linearization, the major degeneration just arising in the presence of dead cores (Remarks 4.1, 4.3). The authors are coming back to this subject in a forthcoming paper [17].

Finally, it should also be remarked that a preliminary version of Theorem 3 for the case where Ω is convex was announced in [15].

2. Some basic results. In the present section our interest is centered on existence, uniqueness and dependence on parameters of positive solutions of the following kind of boundary value problems:

$$\begin{cases} -\Delta_p u = \lambda f(u) & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{P}$$

where $p > 1$, λ is a positive parameter and $\Omega \subset \mathbb{R}^N$ is a bounded domain of class $C^{2,\alpha}$ for a certain $0 < \alpha < 1$. As for the nonlinearity $f = f(u)$ it will be assumed that

(Hf-i) $f = f(u)$ is C^α in $u > 0$ for some $0 < \alpha \leq 1$, with $\lim_{u\to 0+} \frac{f(u)}{u^{p-1}} = m$ for some $0 < m < +\infty$.

(Hf-ii) $\frac{f(u)}{u^{p-1}}$ is decreasing in $u > 0$.

(Hf-iii) $u = u_0$ is a zero of order $k \in \mathbb{R}^+$. This means that for some $C_0 > 0$ the following relation holds:

$$\lim_{u\to u_0-} \frac{f(u)}{(u_0 - u)^k} = C_0. \tag{2.1}$$

All these conditions on f will be referred to as (Hf) in the sequel.

Remark 2.1. Observe that k is allowed to be any positive real number (the case $k = 1$, i.e., $u = u_0$ is a simple zero, generically occurs when f is C^1). On the other hand, no restrictions concerning the relative values of k and p are imposed.

The main examples of problems such as (P) will be treated in Section 4. Let us proceed now to introduce some basic facts. In the next results, the principal eigenvalue $\lambda = \sigma_{1,p}$ of the operator $-\Delta_p$ under Dirichlet conditions in Ω will play an important rôle. Specifically, consider the following eigenvalue problem (see for instance [14] for a general account):

$$\begin{cases} -\Delta_p u = \lambda\, |u|^{p-2} u & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{2.2}$$

$\Omega \subset \mathbb{R}^N$ being a bounded, $C^{2,\alpha}$ domain for some $0 < \alpha < 1$, and the solutions being considered in $W_0^{1,p}(\Omega)$. It is well known that (2.2) admits a first eigenvalue $\lambda = \sigma_{1,p}$, which is positive and can be variationally characterized as

$$\sigma_{1,p} := \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p \ dx}{\int_\Omega |u|^p \ dx}. \tag{2.3}$$

Moreover, $\sigma_{1,p}$ is simple and is the unique eigenvalue exhibiting a positive eigenfunction $\phi \in W_0^{1,p}(\Omega)$ (cf. [2]). It can be further shown that $\phi \in L^\infty(\Omega)$ and hence that $\phi \in C^{1,\beta}(\overline{\Omega})$ for some $0 < \beta < 1$ provided Ω is $C^{2,\alpha}$ (see [16] for a detailed account when (2.2) involves weights and $\partial\Omega$ is only of class $C^{1,\alpha}$). Here, ϕ_1 will stand for the normalization of ϕ so that $|\phi_1|_{\infty,\Omega} = 1$.

A word on notation: in some places $\sigma_{1,p}(\Omega)$ will be used to make explicit, if necessary, the dependence of $\sigma_{1,p}$ on Ω. For real intervals I with length l we will simply write $\sigma_{1,p}(l)$, using $\sigma_{1,p}(R)$ in the case of balls with radius R.

The following result provides an overview of some general features concerning (P). The information on existence and uniqueness is already known in the literature (cf. Remark 2.2 a) below). A proof of these facts is included here for completeness. On the other hand, a global estimate, together with a detailed description of the smoothness and dependence of positive solutions on λ, are obtained and will be extensively used in Section 3.

Theorem 1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{2,\alpha}$ for some $0 < \alpha < 1$. Assume that $f = f(u)$ satisfies hypotheses* (Hf). *Then problem* (P) *with $\lambda > 0$ exhibits the following features.*

(i) (P) *has a nonnegative solution $u \in W_0^{1,p}(\Omega)$, $u \neq 0$, only when the parameter λ satisfies*

$$\lambda > \frac{\sigma_{1,p}}{m}\,. \tag{2.4}$$

(ii) *If the relation* (2.4) *holds, then* (P) *has a unique positive solution $u = u_\lambda(x)$ which satisfies the estimate*

$$0 < u_\lambda(x) \leq u_0 \qquad \text{for each } x \in \Omega. \tag{2.5}$$

(iii) *There exists $0 < \beta < 1$ such that $u_\lambda \in C_0^{1,\beta}(\overline{\Omega}) := \{u \in C^{1,\beta}(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$. Moreover, for each $\lambda > \frac{\sigma_{1,p}}{m}$ there exists some $\epsilon > 0$ such that $u_\lambda \in C^{2,\gamma}(\overline{U_\epsilon})$ where $U_\epsilon = \{x \in \Omega : \text{dist}\,(x, \partial\Omega) < \epsilon\}$, and $\gamma = \min\{\alpha, k, p-1\}$.*

(iv) *The family of positive solutions $u = u_\lambda(x)$ bifurcates from $u = 0$ at $\lambda = \frac{\sigma_{1,p}}{m}$. Moreover, the mapping $\lambda \to u_\lambda$ is continuous and increasing from $\lambda \geq \frac{\sigma_{1,p}}{m}$ in $C_0^{1,\beta_0}(\overline{\Omega})$, for each $0 < \beta_0 < \beta$.*

Proof. Let us begin with assertion (i). If we assume that $u \in W_0^{1,p}(\Omega)$ is a weak nonnegative solution to (P), $u \neq 0$, corresponding to $0 < \lambda \leq \frac{\sigma_{1,p}}{m}$, then u itself can be used as a test function, thus arriving at

$$\int_\Omega |\nabla u|^p = \lambda \int_\Omega \frac{f(u)}{u^{p-1}} |u|^p \leq \frac{\sigma_{1,p}}{m} \int_\Omega \frac{f(u)}{u^{p-1}} |u|^p < \sigma_{1,p} \int_\Omega |u|^p\,.$$

However, such inequalities contradict the variational characterization (2.3) of $\sigma_{1,p}$. Thus positive solutions can only exist if $\lambda > \sigma_{1,p}/m$.

The existence of a positive solution to (P) for $\lambda > \frac{\sigma_{1,p}}{m}$ follows from, say, the method of sub- and supersolutions (cf. Chapter IV in [8] for the context of the p-Laplacian). In fact, and using the idea in [1], for $\lambda > \frac{\sigma_{1,p}}{m}$ fixed, a small $\delta < u_0$ can be found so that $\underline{u} = \delta\phi_1$ is a weak subsolution to (P). Since $\bar{u} = u_0$ is an obvious supersolution, there exists at least a positive solution to (P) satisfying (2.5).

To prove that the estimate in (2.5) holds for every nonnegative solution $u \in W_0^{1,p}(\Omega)$ to (P) corresponding to $\lambda > 0$, it suffices to take $\varphi = (u-u_0)^+$ as a test function. This leads to the expression

$$\int_\Omega |\nabla(u-u_0)^+|^p \, dx = \lambda \int_{\{u>u_0\}} f(u)(u-u_0) \, dx \leq 0.$$

Hence $(u-u_0)^+ = 0$ in Ω, and (2.5) holds.

Before undertaking the uniqueness assertion in (ii) let us make some remarks. If $u \in W_0^{1,p}(\Omega)$ and $u \neq 0$ is any nonnegative solution to (P), it has been found that in particular $u \in L^\infty(\Omega)$. Thus, the $C^{1,\alpha}$ estimates in [10] and [32] allow us to state that, for any $\Omega' \subset \overline{\Omega'} \subset \Omega$ some $0 < \beta' < 1$ exists for which $u \in C^{1,\beta'}(\overline{\Omega'})$. By using a sort of Schwarz reflection with respect to $\partial\Omega$ (see [31], [4]) it can be shown that actually $u \in C^{1,\beta}(\overline{\Omega})$ for some $0 < \beta < 1$. Finally, a careful analysis of the dependence of the Hölder exponent β on the L^∞ estimates of solutions in Theorem 1 of [24], together with a scaling of λ in (P), reveal that β can, indeed, be chosen not depending on λ.

The uniqueness follows now from the following argument in [9]. If $u_1, u_2 \in W_0^{1,p}(\Omega)$ are positive solutions to (P) then the strong maximum principle in [33] implies $u_i > 0$ in Ω, $\frac{\partial u_i}{\partial \nu} < 0$ on $\partial\Omega$, $i = 1, 2$, and so $\frac{u_1}{u_2}, \frac{u_2}{u_1} \in L^\infty(\Omega)$. Thus, the following outstanding inequality in [9] holds:

$$I := \left\langle \frac{(-\Delta_p u_1)}{u_1^{p-1}} - \frac{(-\Delta_p u_2)}{u_2^{p-1}}, u_1^p - u_2^p \right\rangle \geq 0 \tag{2.6}$$

(cf. [2] for an alternative proof of (2.6)). However, a direct computation leads to

$$I = \int_\Omega \Big(\frac{f(u_1)}{u_1^{p-1}} - \frac{f(u_2)}{u_2^{p-1}}\Big)(u_1^p - u_2^p) \, dx \leq 0\,. \tag{2.7}$$

Due to (Hf-ii), the expression in (2.7) is negative provided that $u_1 \neq u_2$. Thus the only consistent option with (2.6) is $u_1 = u_2$ in Ω, and (ii) is shown.

To complete the proof of assertion (iii) observe that, due to Hopf's maximum principle ([33], [31]), $|\nabla u| \geq c > 0$ if $\mathrm{dist}(x, \partial\Omega) < \epsilon$ and $\epsilon > 0$ is small enough (ϵ will depend on λ), if u is the positive solution to (P). Since $u \in C^{1,\beta}(\overline{\Omega})$, then $-\Delta_p u = \lambda f(u)$ can be read in U_ϵ as a uniformly elliptic equation with Hölder coefficients. Schauder's theory then gives $u \in C^2(\overline{U_\epsilon})$ at least. But now the coefficients in the equation are C^1. Since $f \circ u \in C^{\gamma_0}(\overline{\Omega})$, $\gamma_0 = \min\{\alpha, k, p-1\}$, then $u \in C^{2,\gamma}(\overline{U_\epsilon})$.

Finally, let us conclude with the proof of (iv). As a first remark, the increasing character of u_λ is a consequence of uniqueness and the fact that u_{λ_1} is a subsolution to problem (P) with $\lambda = \lambda_2$ if $\lambda_1 < \lambda_2$. As for the continuity, let $\lambda_n \to \bar{\lambda}$, $\bar{\lambda} \geq \frac{\sigma_{1,p}}{m}$, with associated positive solutions u_{λ_n}. In view of the uniform estimate (2.5), a constant C_1 can be found (cf. [24], Theorem 1) such that $|u_{\lambda_n}|_{1,\beta} \leq C_1$. Hence, for any $1 < \beta_0 < \beta$ a subsequence $u_{\lambda_{n'}}$ of u_{λ_n} exists so that $u_{\lambda_{n'}} \to \bar{u}$ in $C_0^{1,\beta_0}(\overline{\Omega})$. Thus $\bar{u}$ must be a nonnegative weak solution to (P) with $\lambda = \bar{\lambda}$. Now, two cases are possible. If $\bar{\lambda} > \frac{\sigma_{1,p}}{m}$ then $u_{\lambda_{n'}} \geq u_{\bar{\lambda}-\epsilon} > 0$ in Ω, for some small ϵ and n' large enough. Thus $\bar{u}$ is positive and necessarily $\bar{u} = u_{\bar{\lambda}}$. The uniqueness also implies that the whole sequence $u_{\lambda_n} \to u_{\bar{\lambda}}$ in $C_0^{1,\beta_0}(\overline{\Omega})$. If finally $\bar{\lambda} = \frac{\sigma_{1,p}}{m}$ then $\bar{u}$ must necessarily be zero, in view of assertion (ii). Again, the whole sequence $u_{\lambda_n} \to 0$. Thus the result on bifurcation from zero is also shown. □

Remarks 2.2. a) An alternative proof for the existence result in (ii), by means of variational techniques, can be found in [9].

b) The conclusions of Theorem 1 hold true if $m = +\infty$ in (Hf-i). In this case the value $\frac{\sigma_{1,p}}{m}$ must be replaced by zero in the assertions of the theorem.

c) By further exploiting inequality (2.6) (see Lemma 1 in [2]), it can be shown that the requirement that $\frac{f(u)}{u^{p-1}}$ is decreasing in (Hf-ii) can be replaced by the slightly weaker one that $\frac{f(u)}{u^{p-1}}$ is nonincreasing in $\mathbb{R}^+$. In fact, all the assertions remain unchanged if $u_m := \sup\{u \geq 0 : \frac{f(u)}{u^{p-1}} = m\}$ equals zero. If, on the contrary, $0 < u_m$ (and so $u_m < u_0$) then nonnegative solutions $u \in W_0^{1,p}(\Omega)$, $u \neq 0$, do not exist if either $\lambda < \frac{\sigma_{1,p}}{m}$ or $\lambda = \frac{\sigma_{1,p}}{m}$ but $\sup_\Omega u > u_m$. However, for $\lambda = \frac{\sigma_{1,p}}{m}$ a family of positive solutions appears. Namely, $u_{\frac{\sigma_{1,p}}{m},t} = t\ \phi_1$ with $0 < t \leq u_m$.

If $\lambda > \frac{\sigma_{1,p}}{m}$, uniqueness holds, and the continuity assertion in (iv) and the smoothness results (iii) all remain true. Nevertheless one now obtains $\lim_{\lambda \to \frac{\sigma_{1,p}}{m}+} u_\lambda = u_m \phi_1$ in $C^{1,\beta_0}(\overline{\Omega})$.

Finally, it should be remarked that the set of nontrivial solutions can be parametrized in this case as $(\lambda(s), u_{\lambda(s)})$ with $\lambda(s) = \frac{\sigma_{1,p}}{m}$, $u_{\lambda(s)} = s\phi_1$ if $0 \leq s \leq u_m$ while $\lambda(s) = \frac{\sigma_{1,p}}{m} + (s - u_m)$, $u_{\lambda(s)} = u_\lambda$ for $s > u_m$. In this case a *vertical* bifurcation at $\lambda = \frac{\sigma_{1,p}}{m}$ occurs. Notice that the few bifurcation results available for the p-Laplacian operator (see for instance Theorem 1.1 in [7]) do not really provide, even locally, explicit information on the nature and behaviour of bifurcated solutions.

The estimate (2.5) ensures that $u = u_0$ is the maximum possible value expected for the positive solutions of (P). An evidence of the degeneracy of Δ_p is the fact that positive solutions $u = u_\lambda$ can meet the value $u = u_0$ in a nonempty subdomain of Ω, as will be studied in full detail later (see Section 3). However, such a behaviour does not appear for a certain regime of the parameters p and k (the order of $u = u_0$). In fact, the following result holds.

Theorem 2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{2,\alpha}$ for some $0 < \alpha < 1$, and assume that $f = f(u)$ satisfies* (Hf). *If the following relation between p and the order k of $u = u_0$ as zero of f holds,*

$$k \geq p - 1, \tag{2.8}$$

then the positive solution $u = u_\lambda(x)$ to (P) satisfies

$$0 < u_\lambda(x) < u_0 \qquad x \in \Omega \tag{2.9}$$

for each $\lambda > \frac{\sigma_{1,p}}{m}$.

Remark 2.3. As will be seen later, (2.8) will turn out also to be a necessary condition for the strict inequality (2.9) to hold for each $\lambda > \frac{\sigma_{1,p}}{m}$.

Proof of Theorem 2. Let $u = u_\lambda(x)$ be the positive solution to (P) corresponding to some $\lambda > \frac{\sigma_{1,p}}{m}$. If we set $v = u_0 - u$ in (P), then via (2.5) v_λ turns out to be a nonnegative solution of

$$-\Delta_p v + \lambda f(u_0 - v) = 0 \qquad x \in \Omega\,. \tag{2.10}$$

Now, $f(u_0 - v) = C_0 v^k h_1(v)$, with h_1 a continuous, positive function in $0 < u < u_0$ and $\lim_{v \to 0+} h_1(v) = 1$. Since (2.10) leads to

$$-\Delta_p v + \lambda C_0 \sup_{0 \leq v \leq u_0} h_1 \; v^k \geq 0 \qquad x \in \Omega,$$

and in view of (2.8), the integral $\int_{0+}^{a} \frac{dv}{(v^{k+1})^{1/p}}$ with $a > 0$ diverges, then the strong maximum principle in [33] implies $v > 0$, which is the desired conclusion. $\square$

3. Boundary layers and dead cores exact estimates. In the present section we will be involved in proving that the unique positive solution u_λ to (P) satisfies $u_\lambda \to u_0$ uniformly on compact sets $K \subset \Omega$, as $\lambda \to +\infty$, where u_0 is the unique positive zero of f. This means that u_λ becomes "nearly flat" over any compact $K \subset \Omega$ as $\lambda \to +\infty$. However, while due to Theorem 2, that flat character is partial in the case $k \geq p-1$, k the order of u_0, since $u_\lambda(x) < u_0$ in Ω, it will also be shown that u_λ is "completely" flat over compact sets of Ω when λ is large enough, provided that $k < p-1$.

On the other hand, the convergence $u_\lambda \to u_0$ on compact sets of Ω as $\lambda \to +\infty$ entails, as was pointed out in Section 1, that $\frac{\partial u_\lambda}{\partial \nu} \to -\infty$ at $\partial\Omega$ as $\lambda \to \infty$. Thus, the more relevant information about the corresponding boundary layer near $\partial\Omega$ will be now ascertained.

Let us next introduce the first result of this section.

Theorem 3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{2,\alpha}$ for $0 < \alpha < 1$, and assume that $f \in C^\alpha(\mathbb{R}^+)$ satisfies (Hf). If $u = u_\lambda(x)$ stands for the unique positive solution to*

$$\begin{cases} -\Delta_p u = \lambda f(u) & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \qquad (P)$$

when $\lambda > \frac{\sigma_{1,p}}{m}$, then the following properties are satisfied.

(i) *If u_0 designates the positive zero of f then, for each compact set $K \subset \Omega$, the following limit,*

$$\lim_{\lambda \to +\infty} u_\lambda(x) = u_0 \qquad x \in K, \tag{3.1}$$

holds uniformly in K.

(ii) *If, for any $x \in \partial\Omega$, $\nu(x)$ stands for the outward unit normal at x, then the limit*

$$\lim_{\lambda \to +\infty} \lambda^{-\frac{1}{p}} \frac{\partial u_\lambda}{\partial \nu}(x) = -\Big(\frac{p}{p-1}\Big)^{\frac{1}{p}} F(u_0)^{\frac{1}{p}}, \tag{3.2}$$

holds uniformly on $\partial\Omega$, where $F(u_0) = \int_0^{u_0} f(s)\, ds$.

Remark 3.1. Theorem 3 extends substantially the results of Section 2 in [12] for the case of the Laplacian Δ ($p = 2$), since nonlinearities f in (Hf) are more general than the polynomial ones considered there. In addition, the restriction on the convexity of Ω imposed in [12] is removed here allowing us to deal with general smooth domains.

We will also prove that, provided $k < p-1$, the solution u_λ to (P) is completely flat. More precisely, the positive solution u_λ, $\lambda > \frac{\sigma_{1,p}}{m}$, is called a "flat solution" to (P) if the set $\mathcal{O}_\lambda = \{x \in \Omega : u_\lambda(x) = u_0\}$ is nonempty. As a consequence of Theorem 2, flat solutions can only appear when the order k of u_0 and p keep the relation above.

Our next result states that such a flatness phenomenom appears when λ exceeds some critical value $\lambda^* > \frac{\sigma_{1,p}}{m}$, providing in addition a geometrical estimate of such a value λ^*. On the other hand, an exact asymptotic estimate, as $\lambda \to +\infty$, of the distance dist $(\mathcal{O}_\lambda, \partial\Omega)$ from the dead core to the boundary $\partial\Omega$ of Ω is also given (see Remark 3.2). Before introducing it, let us first set some notation.

We define the inner radius R_Ω of the domain Ω as the radius of the largest ball contained in Ω; i.e., $R_\Omega = \sup\{R > 0 : B_R(x_0) \subset \Omega\,, \text{for some } x_0 \in \Omega\}$. Also, the "thickness" L_Ω of Ω is defined as the minimum distance between parallel hyperplanes enclosing Ω. More precisely, for each $\eta \in S^{N-1} := \{x : |x| = 1\}$ an $x_\eta \in \partial\Omega$ exists so that $\Omega \subset \{x : (x - x_\eta)\eta \geq 0\}$. Set l_η as the minimum $l > 0$ with $\Omega \subset \{x : 0 \leq (x - x_\eta)\eta \leq l\}$. Then $L_\Omega = \inf_{\eta \in S^{N-1}} l_\eta$.

Theorem 4. *Let $\Omega \subset \mathbb{R}^N$ be a smooth, bounded domain of class $C^{2,\alpha}$, $0 < \alpha < 1$, and $f = f(u)$ a C^α function satisfying hypothesis* (Hf), *while $p > 1$ and the order k of the zero u_0 of f keep the relation*

$$0 < k < p - 1\,. \tag{3.3}$$

Then there exists a first *positive value λ^* of λ, $\lambda^* > \frac{\sigma_{1,p}}{m}$, such that for each $\lambda \geq \lambda^*$ the positive solution u_λ to (P) is a flat solution. In other words, $\mathcal{O}_\lambda = \{x \in \Omega : u_\lambda(x) = u_0\} \neq \emptyset$ for all $\lambda \geq \lambda^*$. Moreover,*

(i) *The following estimate of λ^* holds:*

$$C(f,p)\frac{1}{L_\Omega^p} \leq \lambda^* \leq \ C(f,p)\Big(\frac{C_\theta}{2R_\Omega}\Big)^p\,, \tag{3.4}$$

where to simplify $C(f,p) := \frac{1}{p'}(2I(u_0-))^p = \frac{1}{p'}\Big(\int_0^{u_0} \frac{2\ ds}{(F(u_0)-F(s))^{\frac{1}{p}}}\Big)^p$, $\theta = \frac{N-1}{p-1}$, and $C_\theta = \theta^{\frac{\theta}{\theta-1}}$ if $p \neq N$ while $C_\theta = e$ for $p = N$.

(ii) *If for $\lambda > \lambda^*$, $d(\lambda) =$ dist $(\mathcal{O}_\lambda, \partial\Omega)$ designates the distance from the dead core $\mathcal{O}_\lambda$ to the boundary $\partial\Omega$ of Ω, then the following exact* *asymptotic estimate holds:*

$$\lim_{\lambda\to+\infty} \lambda^{\frac{1}{p}}\ d(\lambda) = \frac{C(f,p)^{\frac{1}{p}}}{2}\,. \tag{3.5}$$

Remark 3.2. a) Observe that the exact value for λ^* in the one-dimensional case $0 < x < l$ (see below) is obtained from (3.4) by setting $L_\Omega = 2R_\Omega = l$ and doing formally $N \to 1$ in the expression for C_θ.

b) It should be remarked that the existence of flat solutions to problems (L) and (P) for λ larger than some λ^* was stated in Theorems 1 and 2, respectively, in [22], which are the main results in that work. However, no estimate for λ^* such as that in (3.4) is given. More importantly, the information about the distance dist $(\mathcal{O}_\lambda, \partial\Omega)$ from the dead core to the boundary obtained in [22] says that $0 <$ dist $(\mathcal{O}_\lambda, \partial\Omega) \le C\ \lambda^{-1/p}$ as $\lambda \to +\infty$, while no explicit information is given about C. In contrast to that, our estimate (3.5) elucidates the exact asymptotic behaviour of such a distance. Thus, Theorem 4 furnishes a considerable improvement of Theorems 1 and 2 in [22] (see Section 4 below).

Theorems 3 and 4 will be proved with a common strategy after several steps. We will study first in detail the one-dimensional version of (P). Some auxiliary results, corresponding to the radial cases $\Omega = B_R(O) := \{x \in \mathbb{R}^N : |x| < R\}$ (the ball) and $\Omega = A(a,R) := \{x \in \mathbb{R}^N :\ 0 < a < |x| < R\}$ (the annulus) will be shown then later in order to proceed to the proof of the theorems.

Lemma 5. *Let $f = f(u)$ be a C^α function in $u > 0$ which satisfies* (Hf)*. Then, the boundary value problem,*

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda f(u) & \quad 0 < x < l \\ \quad u(0) = u(l) = 0, \end{cases} \tag{3.6}$$

admits for each $\lambda > \lambda_{\min}(l) := \frac{p-1}{m}\left(\frac{2}{l}\right)^p \left(\frac{(\pi/p)}{\sin(\pi/p)}\right)^p$ a unique positive classical solution $u = u_\lambda(x)$ such that $\lim_{\lambda\to+\infty} u_\lambda(x) = u_0$ uniformly on compact sets of $0 < x < l$, with u_0 the positive zero of f. Moreover,

$$u_\lambda'(0) = -u_\lambda'(l) \sim \Big(\frac{p}{p-1}\Big)^{\frac{1}{p}} \lambda^{\frac{1}{p}} F(u_0)^{\frac{1}{p}} \qquad \text{as } \lambda \to +\infty\,, \tag{3.7}$$

with $F(u_0) = \int_0^{u_0} f(s)\,ds$*. In addition if* $k < p-1$*, dead cores just appear when* $\lambda \geq \lambda^*(l) := C(f,p)/l^p$*, and then* $\mathcal{O}_\lambda = \{x : u_\lambda(x) = u_0\} = [d(\lambda), l - d(\lambda)]$*, where* $d(\lambda) = \lambda^{-\frac{1}{p}} C(f,p)^{1/p}/2$*.*

Proof. Firstly, the term classical solution $u = u(x)$ of the equation

$$-(|u'|^{p-2}u')' = \lambda f(u) \tag{3.8}$$

in an interval $I \subset \mathbb{R}$ must be understood as a C^1 function u so that $|u'|^{p-2}u' \in C^1(I)$ and satisfies (3.8). Elementary arguments show that any weak solution $u \in W^{1,p}_{\mathrm{loc}}(I)$ of (3.8) is also, and modulo a zero-measure set, a classical solution.

Thus, if u is a classical solution to (3.6), then $u_1 = u(x)$, $v_1 = |u'|^{p-2}u'$ is a solution in I of

$$\begin{cases} u_1' = |v_1|^{p'-2} v_1 \\ v_1' = -\lambda f(u_1), \end{cases} \tag{3.9}$$

p' being the Hölder conjugate of p, $\frac{1}{p} + \frac{1}{p'} = 1$. Since the first integral $E(u_1, v_1) = \frac{|v_1|^{p'}}{p'} + \lambda F(u_1)$, $F(u) = \int_0^u f(s)\,ds$, remains constant along solutions to (3.9), for every solution to (3.8) in an interval I, a constant C exists so that

$$|u'(x)|^p + p'\lambda\, F(u(x)) = C \qquad x \in I\,. \tag{3.10}$$

We are ready now to perform a detailed "ad hoc" analysis of the existence, uniqueness and asymptotic behaviour in λ of the positive solutions to (3.6). Firstly, any positive solution $u = u(x)$ is related to its maximum $\bar{u} = \sup_{0<x<l} u(x)$ by means of

$$|u'|^p = p'\lambda(F(\bar{u}) - F(u))\,. \tag{3.11}$$

Since F decreases beyond $u = u_0$, (3.11) entails the estimate $\bar{u} \leq u_0$. On the other hand, $\bar{u}$ is the only critical value of $u = u(x)$. So, if $x_1 = \min\{x : u = \bar{u}\}$, $x_2 = \max\{x : u = \bar{u}\}$, then u increases before x_1, decreases after x_2, while $u = \bar{u}$ in $x_1 \leq x \leq x_2$. Thus, it follows from (3.11) that

$$\int_0^{u(x)} \frac{ds}{(F(\bar{u}) - F(s))^{\frac{1}{p}}} = (p'\lambda)^{\frac{1}{p}} x \qquad 0 < x < x_1\,, \tag{3.12}$$

while

$$\int_0^{u(x)} \frac{ds}{(F(\bar{u}) - F(s))^{\frac{1}{p}}} = (p'\lambda)^{\frac{1}{p}} (l - x) \qquad x_2 < x < l\,. \tag{3.13}$$

In addition (3.12–13) imply that u is symmetric with regard to $\frac{l}{2}$ and

$$\int_0^{\bar{u}} \frac{ds}{(F(\bar{u}) - F(s))^{\frac{1}{p}}} = (p'\lambda)^{\frac{1}{p}} x_1 \,. \tag{3.14}$$

Observe now that the degenerate case $x_1 < x_2$ can only occur if $\bar{u} = u_0$ since $u(x) = \bar{u}$ in $x_1 < x < x_2$ and equation (3.9) leads to $f(\bar{u}) = 0$. Moreover, due to the fact that the integral in (3.14) diverges at $\bar{u} = u_0$ when $k \geq p-1$ it turns out that such a degenerate case can only occur when $k < p-1$.

It is now convenient to set the notation $I(u) = \int_0^u \frac{ds}{(F(u)-F(s))^{\frac{1}{p}}}$. Then the following properties of the integral I hold:

i) $I = I(u)$ increases in $0 < u < u_0$.

ii) $I(0+) = \lim_{u\to 0+} I(u) = \left(\frac{p}{m}\right)^{\frac{1}{p}} \frac{(\pi/p)}{\sin(\pi/p)}$.

iii) $I(u_0-) = \lim_{u\to u_0-} I(u) = \begin{cases} +\infty & k \geq p-1 \\ < +\infty & k < p-1\,. \end{cases}$

In fact, i) follows from the expression $I(u) = \int_0^1 \left(\int_s^1 \sigma^{p-1} h(u\sigma)\, d\sigma\right)^{-1/p} ds$, $0 < u < u_0$, and the decreasing character of $h(u) := f(u)/u^{p-1}$.

From (3.14) and the previous properties it follows that $I(0+) < I(\bar{u}) < (p'\lambda)^{\frac{1}{p}} \frac{l}{2}$. Thus, positive solutions to (3.6) are only possible when $\lambda > \frac{1}{p'}\left(\frac{2}{l} I(0+)\right)^p := \lambda_{\min}(l)$. In passing, observe that the number $m\ \lambda_{\min}(l)$ coincides with the value for $\sigma_{1,p}(l)$ computed in [19].

The positive solutions $u = u(x)$ to (3.6) can now be described in terms of their maxima $\bar{u}$ as follows.

A) If $k \geq p-1$, (3.6) admits for each $\lambda_{\min} < \lambda < +\infty$ a unique positive solution $u = u_\lambda(x)$ whose maximum $\bar{u} = \bar{u}_\lambda$ is given by the unique solution to the equation

$$I(\bar{u}) = (p'\lambda)^{\frac{1}{p}} \frac{l}{2} \tag{3.15}$$

(the uniqueness follows from property i)). Observe also that the solution $\bar{u} = \bar{u}_\lambda$ to (3.15) is increasing in λ with $\lambda_{\min} < \lambda < +\infty$, $\lim_{\lambda\to\lambda_{\min}} \bar{u}_\lambda = 0$ and $\lim_{\lambda\to+\infty} \bar{u}_\lambda = u_0$. In addition, the solution to (3.6) can be constructed from $\bar{u}_\lambda$ by means of

$$\int_0^{u(x)} \frac{ds}{(F(\bar{u}_\lambda) - F(s))^{\frac{1}{p}}} = (p'\lambda)^{\frac{1}{p}} x\,, \qquad 0 < x < \frac{l}{2}\,. \tag{3.16}$$

Notice that $x_1 = x_2 = \frac{l}{2}$ in this case.

B) If $k < p-1$ two options are possible. The first one corresponds to the regime $\lambda_{\min} < \lambda \leq \lambda^*(l) := \frac{1}{p'}\left(\frac{2}{l}I(u_0-)\right)^p = C(f,p)/l^p$. In this case (3.15) exhibits an increasing family of solutions $\bar{u} = \bar{u}_\lambda$, with $\bar{u}_\lambda \to 0+$ if $\lambda \to \lambda_{\min}+$ while $\bar{u}_\lambda \to u_0$ as $\lambda \to \lambda^*-$. As in the previous case, the positive solution to (3.6) can be recovered from its maximum value $\bar{u}_\lambda$ by means of (3.16).

If $\lambda \geq \lambda^*$, then $d(\lambda) := (p'\lambda)^{-\frac{1}{p}} I(u_0-) = \lambda^{-\frac{1}{p}} C(f,p)^{1/p}/2 \leq \frac{l}{2}$, and from (3.12–13) we can conclude that the unique solution $u = u(x)$ to (3.6) is given by

$$\int_0^{u(x)} \frac{ds}{(F(u_0)-F(s))^{\frac{1}{p}}} = (p'\lambda)^{\frac{1}{p}} x\,, \qquad 0 < x < d(\lambda)\,, \tag{3.17}$$

while $u \equiv u_0$ if $d(\lambda) \leq x \leq \frac{l}{2}$ (recall that u must be symmetric).

To achieve (3.7) observe that (3.11) yields $u'_\lambda(0) = (p'\lambda)^{\frac{1}{p}} F(\bar{u}_\lambda)^{\frac{1}{p}} \sim (p'\lambda)^{\frac{1}{p}} F(u_0)^{\frac{1}{p}}$ as $\lambda \to +\infty$ if $k \geq p-1$; meanwhile, $u'_\lambda(0) = (p'\lambda)^{\frac{1}{p}} F(\bar{u}_\lambda)^{\frac{1}{p}} = (p'\lambda)^{\frac{1}{p}} F(u_0)^{\frac{1}{p}}$ for $\lambda \geq \lambda^*$ if $k < p-1$. Thus, (3.7) is shown.

To conclude the convergence assertion observe that, given $\epsilon > 0$ small enough we have both from (3.16) and (3.17) the following:

$$\begin{aligned} \operatorname{dist}\bigl(\{x : u(x) \geq u_0 - \epsilon\}, \{0,l\}\bigr) &= \frac{1}{(p'\lambda)^{1/p}} \int_0^{u_0-\epsilon} \frac{ds}{(F(\bar{u}_\lambda)-F(s))^{\frac{1}{p}}} \\ &\sim \frac{1}{(p'\lambda)^{1/p}} \int_0^{u_0-\epsilon} \frac{ds}{(F(u_0)-F(s))^{\frac{1}{p}}}\,, \qquad \text{as } \lambda \to +\infty. \end{aligned}$$

Hence, this distance vanishes as $\lambda \to +\infty$.

Finally notice that for $\lambda \geq \lambda^*(l)$ defined above $\mathcal{O}_\lambda = \{x : u_\lambda(x) = u_0\} = [d(\lambda), l-d(\lambda)]$, and therefore Lemma 5 is proved. $\square$

Remark 3.3. It should be observed that the uniqueness assertion in Theorem 2.1 of [19] can be obtained from their hypothesis (2.3) without further requiring the additional condition (2.7) in that paper. Indeed, the uniqueness follows from the monotonicity of the corresponding integral playing there the rôle of I in the context of Lemma 5 above. In the same way, a sort of uniqueness result in Theorem 2.2 there can be obtained if one is restricted, say, to the class of solutions with equally spaced "flat cores."

We are next dealing with the case $\Omega = B_R$, B_R the ball $\{x \in \mathbb{R}^N : |x| < R\}$, in order to get an upper estimate of the limits (3.2) and (3.5).

In the general case when Ω is a smooth domain, an upper estimate of both quantities $\lambda^{-\frac{1}{p}}\frac{\partial u_\lambda}{\partial \nu}(x_0)$ and $\lambda^{\frac{1}{p}}\text{dist}\,(x_0,\mathcal{O}_\lambda)$, $x_0 \in \partial\Omega$, will be obtained with a convenient choice of a ball $B_R(y_0) = B_R + y_0 \subset \Omega$, tangent to $\partial\Omega$ at x_0.

Before introducing the statement of our next proposition, let us note some features of the radial case $\Omega = B_R$. If $u = u(x)$ is, for $\lambda > \frac{\sigma_{1,p}(R)}{m}$, the positive solution to (P), then u must be radial; i.e., $u(x) = u(r)$ with $r = |x|$. In fact, u is of class $C^{1,\beta}$ while the weak version of $-\Delta_p u = \lambda f(u)$ is invariant under rotation. Thus, the uniqueness implies the assertion. In particular $u \in C^1([0,R])$, $u'(0) = u(R) = 0$. On the other hand,

$$\int_0^R |u'|^{p-2}u'\varphi'\ r^{N-1}\ dr = \lambda\int_0^R f(u)\varphi\ r^{N-1}\ dr\,,$$

for every $\varphi \in C^1([0,R])$, $\varphi'(0) = \varphi(R) = 0$. Thus, standard regularity arguments allow us to ensure that $|u'|^{p-2}u' \in C^1([0,R])$ and that

$$-(r^{N-1}|u'|^{p-2}u')' = \lambda r^{N-1} f(u)$$

holds pointwise in $0 < r < R$. In this sense we can assert that $u = u(r)$ is a classical solution of the problem

$$\begin{cases} -(r^{N-1}|u'|^{p-2}u')' = \lambda r^{N-1} f(u) & 0 < r < R \\ u'(0) = u(R) = 0\,. \end{cases} \tag{3.18}$$

Lemma 6. *Let $f = f(u)$ be a C^α function satisfying (Hf), and, for $\lambda > \frac{\sigma_{1,p}(R)}{m}$, let $u = u_\lambda(r)$, $0 < r < R$, be the unique positive solution to* (P) *in the ball $\Omega = B_R$. Then $u_\lambda \to u_0$ uniformly on compact subsets of B_R as $\lambda \to +\infty$, and the outer normal derivative $u_\lambda'(R)$ satisfies the estimate*

$$\limsup_{\lambda\to+\infty} \lambda^{-\frac{1}{p}} u_\lambda'(R) \le -(p'F(u_0))^{\frac{1}{p}}\,. \tag{3.19}$$

Moreover, if $k < p-1$, then the set $\mathcal{O}_\lambda = \{x :\ u_\lambda(x) = u_0\}$ is nonempty at least for $\lambda \ge \lambda^*(R) := (C_\theta/2R)^p C(f,p)$ and

$$\limsup_{\lambda\to+\infty} \lambda^{1/p}\,dist\,(\mathcal{O}_\lambda,\partial B) \le \frac{C(f,p)^{1/p}}{2}\,, \tag{3.20}$$

where θ, C_θ and $C(f,p)$ are the constants in the statement of Theorem 4.

Proof. Let us fix $\lambda > \frac{\sigma_{1,p}(R)}{m}$, and let $u = u_\lambda(r)$, $0 < r < R$, be the positive solution to (3.18). In order to furnish a suitable lower estimate of u_λ it is quite convenient to remove the dependence on r in the left-hand side of (3.18). To perform this, observe that such an equation can be written as

$$-r^{\theta}(|r^{\theta}u'|^{p-2}r^{\theta}u')' = \lambda\, r^{p\theta}f(u)\,, \tag{3.21}$$

where $\theta = \frac{N-1}{p-1}$. Thus, (3.21) suggests introducing a change of variable $\rho = \rho(r)$ such that $\frac{d\rho}{dr} = \pm r^{-\theta}$. If we take, say, the minus-sign option and normalize so that $\rho(R) = 0$, we arrive at the precise expression for this change, namely,

$$\rho = g(r) = \begin{cases} \frac{1}{1-\theta}\left(R^{1-\theta} - r^{1-\theta}\right) & p \neq N \\ \log\left(\frac{R}{r}\right) & p = N\,. \end{cases}$$

Observe that $0 < \rho < T$ if $0 < r < R$, where $T = +\infty$ if $p \leq N$ while $T = \frac{R^{1-\theta}}{1-\theta}$ when $p > N$.

Setting $v(\rho) = u(g^{-1}(\rho))$ in (3.18) leads to the problem

$$\begin{cases} -(|v'|^{p-2}v')' = \lambda(g^{-1}(\rho))^{p\theta}f(v) & 0 < \rho < T \\ v(0) = v'(T) = 0\,, \end{cases} \tag{3.22}$$

where $' = \frac{d}{d\rho}$. If we fix $0 < b < T$ and for $\lambda > \frac{\sigma_{1,p}(R)}{m}$, $v = v_\lambda(\rho) = u_\lambda(g^{-1}(\rho))$ stands for the unique positive solution to (3.22), then

$$-(|v'|^{p-2}v')' \geq \lambda(g^{-1}(b))^{p\theta}f(v)\,,$$

provided that $0 < \rho < b$. Let us introduce now the auxiliary problem

$$\begin{cases} -(|v'|^{p-2}v')' = \lambda(g^{-1}(b))^{p\theta}f(v) & 0 < \rho < b \\ v(0) = v'(b) = 0\,. \end{cases} \tag{3.23}$$

We observe now that (3.23) admits a unique positive solution $v = \underline{v}_\lambda(\rho, b)$ provided $\lambda > \lambda_{\min}(2b)/(g^{-1}(b))^{p\theta}$. In fact, it suffices with restricting to $0 < \rho < b$ the solution to

$$\begin{cases} -(|v'|^{p-2}v')' = \lambda(g^{-1}(b))^{p\theta}f(v) & 0 < \rho < 2b \\ v(0) = v(2b) = 0\,. \end{cases} \tag{3.24}$$

On the other hand, if the restriction to $0 < \rho < b$ of the solution $v_\lambda(\rho)$ to (3.22) is extended as an even function with respect to b to the whole interval

$0 < \rho < 2b$, we get a weak supersolution $\bar{v}$ to (3.24). Therefore, we can conclude

$$0 < \underline{v}_\lambda(\rho, b) \le v_\lambda(\rho), \qquad \text{for } 0 < \rho < b\,. \tag{3.25}$$

In particular, since u_λ is decreasing this implies $u_\lambda(r) \ge \underline{v}_\lambda(\frac{b}{2}, b)$, $0 \le r \le g^{-1}(\frac{b}{2})$. Now Lemma 5 yields $\underline{v}_\lambda(\frac{b}{2}, b) \to u_0$ as $\lambda \to +\infty$, and since $g^{-1}(\frac{b}{2}) \to R$ as $b \to 0+$, the first assertion is shown.

On the other hand, equation (3.25) also implies $0 < \underline{v}_\lambda'(0, b) \le v_\lambda'(0)$, provided $\lambda > \max\{\frac{\sigma_{1,p}(R)}{m}, \frac{\lambda_{\min}(2b)}{(g^{-1}(b))^{p\theta}}\}$. That inequality can be read as

$$0 < \frac{d\underline{v}_\lambda}{d\rho}(0, b) \le -R^\theta \, \frac{du_\lambda}{dr}(R)\,,$$

which implies

$$0 < \frac{\lambda^{-\frac{1}{p}}}{R^\theta} \, \frac{d\underline{v}_\lambda}{d\rho}(0, b) \le -\lambda^{-\frac{1}{p}} \frac{du_\lambda}{dr}(R)\,. \tag{3.26}$$

Following Lemma 5, $\lambda^{-\frac{1}{p}} \frac{d\underline{v}_\lambda}{d\rho}(0, b) \sim (g^{-1}(b))^\theta (p' F(u_0))^{\frac{1}{p}}$ as $\lambda \to +\infty$. Therefore, taking inferior limits in (3.26) leads to

$$\Big(\frac{g^{-1}(b)}{R}\Big)^\theta (p' F(u_0))^{\frac{1}{p}} \le \liminf_{\lambda \to +\infty} -\lambda^{-\frac{1}{p}} \frac{du_\lambda}{dr}(R)$$

or, equivalently,

$$\limsup_{\lambda \to +\infty} \lambda^{-\frac{1}{p}} \frac{du_\lambda}{dr}(R) \le -\Big(\frac{g^{-1}(b)}{R}\Big)^\theta (p' F(u_0))^{\frac{1}{p}}\,.$$

Finally, to arrive at the desired inequality (3.19) it suffices to let $b \to 0+$ in the last inequality.

Now notice that $v = \underline{v}_\lambda(\rho, b)$ develops a dead core for each

$$\lambda \ge C(f, p)/(2b(g^{-1}(b))^\theta)^p.$$

Thus, we can assert that dead cores arise at least when λ exceeds the value

$$\lambda^*(R) = \inf \; \frac{C(f, p)}{(2b(g^{-1}(b))^\theta)^p} = \frac{1}{2^p} \frac{C(f, p)}{\sup\,(b(g^{-1}(b))^\theta)^p}\,,$$

both the infimum and the supremum being taken over $0 < b < T$ (recall that g depends on the radius R). It is readily seen that the maximum value of $b(g^{-1}(b))^\theta$ is R/C_θ.

Let us proceed now to prove (3.20). By the decreasing character of $u_\lambda(r)$ and the fact that $\underline{v}_\lambda(\rho, b)$ exhibits a dead core for $\lambda > (2bg^{-1}(b)^\theta)^{-p}C(f,p)$ we arrive at $u_\lambda(x) = u_0$ for all $|x| \le g^{-1}(\frac{C(f,p)^{\frac{1}{p}}}{2g^{-1}(b)^\theta \lambda^{\frac{1}{p}}})$. Thus,

$$0 < \text{dist}\,(\mathcal{O}_\lambda, \partial B) \le R - g^{-1}\Big(\frac{C(f,p)^{\frac{1}{p}}}{2g^{-1}(b)^\theta \lambda^{\frac{1}{p}}}\Big)\,.$$

If we put $\hat{d}(\lambda) := R - g^{-1}(C(f,p)^{\frac{1}{p}}/(2g^{-1}(b)^\theta\lambda^{\frac{1}{p}}))$, it follows that

$$\hat{d}(\lambda) = \frac{C(f,p)^{1/p}}{2}\Big(\frac{R}{g^{-1}(b)}\Big)^\theta \lambda^{-1/p} + o(\lambda^{-1/p}) \text{ as } \lambda \to +\infty.$$

Since

$$\limsup_{\lambda\to+\infty} \lambda^{1/p}\text{dist}\,(\mathcal{O}_\lambda, \partial B) \le \lim_{\lambda\to+\infty} \lambda^{1/p}\hat{d}(\lambda) = \frac{C(f,p)^{1/p}}{2}\Big(\frac{R}{g^{-1}(b)}\Big)^\theta, \quad (3.27)$$

it is obtained from (3.27), after passing to the limit as $b \to 0+$, that

$$\limsup_{\lambda\to+\infty} \lambda^{1/p}\text{dist}\,(\mathcal{O}_\lambda, \partial B) \le \frac{C(f,p)^{1/p}}{2}\,. \quad (3.28)$$

This finishes the proof of Lemma 6. □

Next we confine ourselves to obtaining a lower estimate of the limits (3.2) and (3.5) in the case of the annulus $A(a,R) = \{x \in \mathbb{R}^N : 0 < a < |x| < R\}$ and at points located in the part $|x| = a$ of its boundary ∂A. This will allow us, when dealing with the case of a general $C^{2,\alpha}$ domain $\Omega \subset \mathbb{R}^N$, to produce lower estimates of both the inferior limit $\liminf_{\lambda\to+\infty} \frac{\partial u}{\partial \nu}(x_0)$ and the distance $\text{dist}(\mathcal{O}_\lambda, x_0)$ at any $x_0 \in \partial\Omega$. To proceed in that general situation, a suitable annulus $A(a,R;y_0) := A(a,R) + y_0$, $\Omega \subset A(a,R;y_0)$, will be chosen to be tangent to $\partial\Omega$ at x_0.

As in the case of the ball (see the observations before Lemma 6), the positive solution to (P) in $\Omega = A(a,R)$ corresponding to $\lambda > \frac{\sigma_{1,p}(A)}{m}$ must be radially symmetric; i.e., $u = u_\lambda(r)$, $a < r < R$, $r = |x|$. Moreover, both u_λ and $|u_\lambda'|^{p-2}u_\lambda'$ are C^1 in $a \le r \le R$ and u_λ solves the problem

$$\begin{cases} -(r^{N-1}|u'|^{p-2}u')' = \lambda r^{N-1} f(u) & a < r < R \\ u(a) = u(R) = 0\,. \end{cases} \quad (3.29)$$

Let us now state our next result.

Lemma 7. *Let us consider the problem* (P) *in the annulus* $A(a,R)=\{x\in\mathbb{R}^N : 0<a<|x|<R\}$, $f=f(u)$ *being a* C^α *function satisfying* (Hf). *For* $\lambda>\frac{\sigma_{1,p}(A)}{m}$ *let* $u=u_\lambda(r)$ *be the unique positive solution to* (P) *in such an annulus. Then,*

$$\liminf_{\lambda\to+\infty} -\lambda^{-\frac{1}{p}}u_\lambda'(a)\geq -(p'F(u_0))^{\frac{1}{p}}\,. \tag{3.30}$$

Moreover, if $k<p-1$ *and* $\mathcal{O}_\lambda=\{x : u_\lambda(x)=u_0\}$, *then*

$$\liminf_{\lambda\to+\infty} \lambda^{\frac{1}{p}}\,dist\,(\mathcal{O}_\lambda,\Gamma)\geq \frac{C(f,p)^{\frac{1}{p}}}{2}\,, \tag{3.31}$$

where $\Gamma=\{x:\ |x|=a\}$.

Proof. Following the idea used in the proof of Lemma 6, the variable r can be removed from the left-hand side of (3.29) by introducing a new variable $\rho=\rho(r)$ by means of the expression

$$\rho=g(r)=\begin{cases}\frac{1}{1-\theta}[r^{1-\theta}-a^{1-\theta}] & p\neq N\\ \log\left(\frac{r}{a}\right) & p=N\,,\end{cases}$$

where $\theta=\frac{N-1}{p-1}$. Observe that now $\frac{d\rho}{dr}=r^{-\theta}$ and that $0<\rho<T$ as $a<r<R$, where $T=g(R)$. By setting $v(\rho)=u(g^{-1}(\rho))$ the problem (3.29) is rewritten as

$$\begin{cases}-(|v'|^{p-2}v')'=\lambda(g^{-1}(\rho))^{p\theta}f(v) & 0<\rho<T\\ v(0)=v(T)=0\,,\end{cases} \tag{3.32}$$

where $'=\frac{d}{d\rho}$. If v_λ designates the unique positive solution to (3.32) and $\lambda>a^{-p\theta}\lambda_{\min}(T)$, then v_λ is a supersolution of the problem

$$\begin{cases}-(|v'|^{p-2}v')'=\lambda a^{p\theta}f(v) & 0<\rho<T\\ v(0)=v(T)=0\,.\end{cases} \tag{3.33}$$

We can conclude then that for $\lambda>a^{-p\theta}\lambda_{\min}(T)$,

$$0<v_\lambda^-(\rho)\leq v_\lambda(\rho)\qquad 0<\rho<T\,, \tag{3.34}$$

where v_λ^- is the unique positive solution to (3.33). From Lemma 5 it follows that $v_\lambda\to u_0$ as $\lambda\to+\infty$, uniformly on compact subsets of $0<\rho<T$.

To achieve (3.30) we will proceed in different ways according to the relative values of k and $p-1$. Let us begin with the case $k \geq p-1$. Choose a positive number $0 < b < \frac{T}{2}$, and consider the auxiliary boundary value problem in the interval $0 < \rho < b$,

$$\begin{cases} -(|w'|^{p-2}w')' = \lambda(g^{-1}(b))^{p\theta} f(w) & 0 < \rho < b \\ w(0) = 0\,, \quad w(b) = v_\lambda(b)\,, \end{cases} \tag{3.35}$$

where v_λ is the positive solution to (3.32) corresponding to $\lambda > \frac{\sigma_{1,p}(A)}{m}$. Then observe that (3.35) can exhibit at most one positive solution. In fact, it suffices to repeat the argument used to achieve the uniqueness assertion in Theorem 1. However, one needs now a version of inequality (2.6) allowing u_1, $u_2 \in W^{1,p}(\Omega)$ but having $u_1 = u_2$ on $\partial\Omega$. A careful checking of the proof of (2.6) in [2] reveals that the inequality also holds under such assumptions.

On the other hand, the positive solution $v_\lambda(\rho)$ of (3.33) is a subsolution to (3.35) while $w^+ = u_0$ defines a supersolution. Thus, (3.35) exhibits, for each $\lambda > \frac{\sigma_{1,p}(A)}{m}$, a unique positive solution $w = w_\lambda(\rho)$ such that

$$0 < v_\lambda(\rho) \leq w_\lambda(\rho) \qquad 0 < \rho < b\,, \tag{3.36}$$

which implies

$$v'_\lambda(0) \leq w'_\lambda(0) = \left\{(w'_\lambda(b))^p + \lambda p'(g^{-1}(b))^{p\theta} F(v_\lambda(b))\right\}^{\frac{1}{p}}\,. \tag{3.37}$$

Phase-space analysis on the equation in (3.35) permits us to conclude that $\frac{(w'_\lambda(b))^p}{\lambda}$ remains bounded as $\lambda \to +\infty$. This information is then easily used to show that $\frac{(w'_\lambda(b))^p}{\lambda} \to 0$ as $\lambda \to +\infty$. This fact together with the convergence $\lim_{\lambda\to+\infty} v_\lambda(b) = u_0$ and (3.37) lead us to

$$\limsup_{\lambda\to+\infty} \lambda^{-\frac{1}{p}} v'_\lambda(0) \leq (g^{-1}(b))^\theta (p' F(u_0))^{\frac{1}{p}}\,.$$

Therefore,

$$\limsup_{\lambda\to+\infty} \lambda^{-\frac{1}{p}} u'_\lambda(a) \leq \frac{(g^{-1}(b))^\theta}{a^\theta} (p' F(u_0))^{\frac{1}{p}}\,. \tag{3.38}$$

To get the desired estimate (3.30) it suffices to let $b \to 0+$ in (3.38).

Let us undertake now the degenerate case $k < p-1$ under a different strategy. Consider the auxiliary problem

$$\begin{cases} -(|w'|^{p-2}w')' = \lambda(g^{-1}(b))^{p\theta} f(w) & 0<\rho<b \\ w(0)=0\,, \quad w(b)=u_0\,, \end{cases} \tag{3.39}$$

where b is again an arbitrary fixed number so that $0 < b < T/2$. It can be checked that (3.39) exhibits a unique positive solution $w = \underline{w}_\lambda(\rho, b)$ for λ large enough. Moreover, if $\rho(\lambda) := \frac{1}{2}\frac{C(f,p)^{1/p}}{(g^{-1}(b))^\theta}\lambda^{-1/p}$, then $\underline{w}_\lambda = u_0$ for $\rho(\lambda) \le \rho \le b$, while $0 < \underline{w}_\lambda < u_0$ in $0 < \rho < \rho(\lambda)$.

On the other hand, since v_λ solves (3.32) in $0 < \rho < T$, it defines a subsolution to (3.39) in $0 < \rho < b$. Thus, $0 < v_\lambda(\rho) \le \underline{w}_\lambda(\rho, b) \le u_0$ for $0 < \rho < b$. Two conclusions are obtained from this inequality. The first one is $v_\lambda'(0) \le \underline{w}_\lambda'(0, b)$, and so, if $u = u_\lambda(r)$ is the positive solution to (P) in $\Omega = A(a, R)$, we find that

$$\frac{u_\lambda'(a)}{\lambda^{\frac{1}{p}}} \le (p'F(u_0))^{\frac{1}{p}} \Big(\frac{g^{-1}(b)}{a}\Big)^\theta, \tag{3.40}$$

and the desired estimate (3.30) follows then from (3.40) by letting $\lambda \to +\infty$ and $b \to 0+$. A second conclusion is that $u_\lambda(x) \le \underline{w}_\lambda(g(r), b) < u_0$, provided that $x \in \Omega$ and $a < r < g^{-1}(\rho(\lambda))$, with $r = |x|$. This means that

$$\text{dist }(\mathcal{O}_\lambda, \Gamma) \ge g^{-1}(\rho(\lambda)) - a\,. \tag{3.41}$$

Observing that $(g^{-1}(\rho(\lambda)) - a) \sim \frac{C(f,p)^{1/p}}{2}\,(\frac{a}{g^{-1}(b)})^\theta \lambda^{-1/p}$ as $\lambda \to +\infty$, we conclude from (3.41) that

$$\liminf_{\lambda\to+\infty} \lambda^{1/p}\text{dist }(\mathcal{O}_\lambda, \Gamma) \ge \frac{C(f,p)^{\frac{1}{p}}}{2}\Big(\frac{a}{g^{-1}(b)}\Big)^\theta.$$

Such an estimate rapidly leads, by letting $b \to 0+$, to the desired result, namely,

$$\liminf_{\lambda\to+\infty} \lambda^{1/p}\text{dist }(\mathcal{O}_\lambda, \Gamma) \ge \frac{C(f,p)^{1/p}}{2}\,.$$

This concludes the proof of Lemma 7. □

Proof of Theorem 3. Let us begin with the proof of (i). Consider any ball $B \subset \Omega$, and for $\lambda > \frac{\sigma_{1,p}}{m}$ let $u_{\lambda,B}$ be the solution to

$$\begin{cases} -\Delta_p u = \lambda f(u) & x \in B \\ u = 0 & x \in \partial B. \end{cases} \tag{3.42}$$

Notice that u_λ is a supersolution to problem (3.42). Then since $\delta\phi_{1,B}$ (where $\delta > 0$ is small enough and $\phi_{1,B}$ is the principal eigenfunction in B) can always be taken as a subsolution provided that $\lambda > \sigma_{1,p}(R)/m > \sigma_{1,p}/m$ (see Theorem 1), we can conclude that the solution u_λ to (P) satisfies, via (2.5),

$$0 < u_{\lambda,B}(x) \le u_\lambda(x) \le u_0\,, \qquad \text{for each } x \in B. \tag{3.43}$$

It should be remarked that (3.43) holds true no matter what the location of the ball $B = B_R(x_0)$ in Ω is. Thus, if $K \subset \Omega$ is any compact set with $d = \text{dist }(K, \partial\Omega)$ and we set $R = \frac{d}{2}$, then $B_R(x_0) \subset \Omega$ for every $x_0 \in K$. Thus we find from Lemma 6 that $u_\lambda \to u_0$ uniformly in K as $\lambda \to +\infty$.

Let us next show assertion (ii). To do so, choose an arbitrary $x_0 \in \partial\Omega$ with associated outward unit normal $\nu = \nu(x_0)$. Since $\partial\Omega$ is a class $C^{2,\alpha}$ surface (cf. [18]) it is possible to find points $y_i \in \Omega$, $y_e \in \mathbb{R}^N \setminus \overline{\Omega}$ and positive numbers R, a, R_e such that the ball $B_i := B_R(y_i) \subset \Omega$, the annulus $A_e := A(a, R_e, y_e) = \{a < |x - y_e| < R_e\}$ contains Ω, i.e., $\Omega \subset A_e$, while finally both B_i and A_e are tangent to $\partial\Omega$ at x_0.

If, following the previous notation, we introduce the function u_{λ,B_i} it is found that $u_{\lambda,B_i}(x) \le u_\lambda(x)$ for $x \in B_i$. Hence $\frac{\partial u_\lambda}{\partial\nu}(x_0) \le \frac{\partial u_{\lambda,B_i}}{\partial\nu}(x_0)$ for $\lambda > \frac{\sigma_{1,p}(R_i)}{m}$. Lemma 6 then implies that

$$\limsup_{\lambda\to+\infty} \lambda^{-\frac{1}{p}} \frac{\partial u_\lambda}{\partial\nu}(x_0) \le \limsup_{\lambda\to+\infty} \lambda^{-\frac{1}{p}} \frac{\partial u_{\lambda,B_i}}{\partial\nu}(x_0) \le -(p'F(u_0))^{\frac{1}{p}}\,. \tag{3.44}$$

In a symmetric way, consider problem (P) taking the annulus A_e as the domain Ω, and let u_{λ,A_e} be its unique positive solution for $\lambda > \frac{\sigma_{1,p}(A_e)}{m}$. Then, since u_{λ,A_e} is a supersolution to (P), it follows as before that $u_\lambda(x) \le u_{\lambda,A_e}(x)$ for every $x \in \Omega$. So, $\frac{\partial u_{\lambda,A_e}}{\partial\nu}(x_0) \le \frac{\partial u_\lambda}{\partial\nu}(x_0)$. On the other hand, if $r = |x - y_e|$, then $u_{\lambda,A_e} = u_{\lambda,A_e}(r)$, while $\frac{\partial u_{\lambda,A_e}}{\partial\nu}(x_0) = -u'_{\lambda,A_e}(a)$ $(' = \frac{d}{dr})$. Thus Lemma 7 permits us to conclude that

$$-(p'F(u_0))^{\frac{-1}{p}} \le \liminf_{\lambda\to+\infty} \lambda^{\frac{-1}{p}} \frac{\partial u_{\lambda,A_e}}{\partial\nu}(x_0) \le \liminf_{\lambda\to+\infty} \lambda^{\frac{1}{p}} \frac{\partial u_\lambda}{\partial\nu}(x_0). \tag{3.45}$$

Finally, combining (3.44) and (3.45) we obtain the desired identity (3.2) at $x_0 \in \partial\Omega$. On the other hand, the compactness of $\partial\Omega$ allows us to perform a choice of a, R_i, R_e not depending on the position of x_0 in $\partial\Omega$. Therefore, the limit (3.2) is indeed uniform. This concludes the proof of Theorem 3. $\square$

Proof of Theorem 4. We introduce λ^* as the first (in principle possibly infinity) value of $\lambda > \frac{\sigma_{1,p}}{m}$ such that $\mathcal{O}_\lambda \neq \emptyset$. Notice that $\mathcal{O}_\lambda = \emptyset$ for $\lambda > \frac{\sigma_{1,p}}{m}$ and is close enough to $\frac{\sigma_{1,p}}{m}$. In fact (see Theorem 1-(iv)) u_λ bifurcates from zero at $\lambda = \frac{\sigma_{1,p}}{m}$. Thus $\lambda^* > \frac{\sigma_{1,p}}{m}$. Using also Theorem 1-(iv) note that $\lambda^* < +\infty$ implies $\mathcal{O}_\lambda \neq \emptyset$ for all $\lambda \geq \lambda^*$, since the family of dead cores $\mathcal{O}_\lambda$ is continuous and increasing in λ.

Let us estimate λ^* from below. Following the terminology preceding the statement, choose any $\eta \in S^{N-1}$ and some associated $x_\eta \in \partial\Omega$ and $l_\eta > 0$ so that $\Omega \subset \{x : 0 < (x - x_\eta)\eta < l_\eta\}$. Define $u = u_\lambda^+(x) = w_\lambda(\xi)$, where $\xi = (x - x_\eta)\eta$ and w_λ designates the unique positive solution to

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda f(u) & 0 < \xi < l_\eta \\ u(0) = u(l_\eta) = 0 \end{cases}$$

($' = \frac{d}{d\xi}$) corresponding to $\lambda > \lambda_{\min}(l_\eta)$. It turns out that u_λ^+ provides a supersolution to (P) in Ω. On the other hand, w_λ exhibits a dead core for $\lambda \geq C(f,p)/l_\eta^p$ (cf. Lemma 5). Thus, necessarily $\lambda^* \geq C(f,p)/l_\eta^p$. Therefore, $\lambda^* \geq \frac{C(f,p)}{\inf_{\eta \in S^{N-1}} l_\eta^p}$, which implies the lower estimate in (3.4).

To get the upper estimate we consider a ball B with radius R contained in Ω. As before let $u_{\lambda,B}$ be the unique solution to (P) in B for λ large. Then $u_\lambda \geq u_{\lambda,B}$ in B, and thus u_λ develops a dead core for $\lambda \geq \lambda^*(R)$ (Lemma 6). This implies $\lambda^* \leq \inf \lambda^*(R) = \lambda^*(R_\Omega)$, the infimum being extended to every $R > 0$ such that $B_R(x_0) \subset \Omega$ for some $x_0 \in \Omega$, and (3.4) is proved.

On the other hand, if $x_0 \in \partial\Omega$ and B is chosen to be tangent to $\partial\Omega$ at x_0 and $B \subset \Omega$, we have dist $(x_0, \mathcal{O}_\lambda) \leq$ dist $(x_0, \mathcal{O}_{\lambda,B})$, where $\mathcal{O}_{\lambda,B} = \{x \in B : u_{\lambda,B}(x) = u_0\}$, and Lemma 6 implies

$$\limsup_{\lambda \to +\infty} \lambda^{\frac{1}{p}} \text{dist } (\mathcal{O}_\lambda, \partial\Omega) \leq \frac{C(f,p)^{\frac{1}{p}}}{2}. \tag{3.46}$$

To get the lower estimate in (3.5), we use the annulus $A_e = \{x : a < |x - y_e| < R_e\}$, $\Omega \subset A_e$, A_e tangent to $\partial\Omega$ at x_0, introduced in the proof of Theorem 3. Recall that $u_\lambda \leq u_{\lambda,A_e}$ in Ω, where $u = u_{\lambda,A_e}(r)$ is the positive solution to (P) in the annulus A_e, $r = |x - y_e|$. Thus if $\mathcal{O}_{\lambda,A_e} = \{x : \ u_{\lambda,A_e}(x) = u_0\}$, we have

$$\liminf_{\lambda\to+\infty} \lambda^{\frac{1}{p}} \text{dist } (x_0, \mathcal{O}_{\lambda,A_e}) \le \liminf_{\lambda\to+\infty} \lambda^{\frac{1}{p}} \text{dist } (x_0, \mathcal{O}_{\lambda}),$$

and Lemma 7, together with (3.46), gives (3.5). The proof is concluded. □

4. Applications: logistic-type problems and certain perturbations. We are now confining ourselves to the analysis of a particular kind of boundary value problem. The first one is the p-Laplacian version of the well-known logistic equation with diffusion (Section 1), namely,

$$\begin{cases} -\Delta_p u = m\,\lambda |u|^{p-2}u - |u|^{q-1}u & x \in \Omega \\ \qquad u = 0 & x \in \partial\Omega, \end{cases} \tag{L}$$

where m, p, q are positive constants, $p > 1$, $q > p - 1$ and λ is a positive parameter. Along with (L) we also will be interested in the following kind of "lower order" perturbations at infinity of (L), i.e.,

$$\begin{cases} -\Delta_p u = m\,\lambda |u|^{p-2}u - |u|^{q-1}u + g(u) & x \in \Omega \\ \qquad u = 0 & x \in \partial\Omega, \end{cases} \tag{LP}$$

where the perturbation term $g = g(u)$ is assumed to satisfy the following growth conditions near zero and infinity:

(Hg) $g = g(u) \in C^\alpha(\mathbb{R}^+)$ satisfies $\lim_{u\to 0+} \frac{g(u)}{u^{p-1}} = \lim_{u\to+\infty} \frac{g(u)}{u^q} = 0$.

As a first conclusion, it will be seen that the logistic problem (L) falls into the scope of the results of Sections 2 and 3. In fact, by setting the scale change $u = (m\lambda)^{1/(q-p+1)} w$ problem (L) becomes $-\Delta_p w = (m\,\lambda)(w^{p-1} - w^q)$ in Ω, $w|_{\partial\Omega} = 0$ (notation has been cleared out since we are everywhere considering positive solutions). To deal with that problem it is convenient to introduce the following normalized version,

$$\begin{cases} -\Delta_p v = \lambda\,(v^{p-1} - v^q) & x \in \Omega \\ \qquad v = 0 & x \in \partial\Omega\,, \end{cases} \tag{4.1}$$

which falls directly in the layout of Sections 2 and 3 with $m = 1$, $u_0 = 1$, $k = 1$ and $C_0 = q - p + 1$. Therefore, (4.1) admits for each $\lambda > \sigma_{1,p}$ a unique positive solution $v = v_\lambda(x) \in C_0^{1,\beta}(\overline{\Omega})$ for some $0 < \beta < 1$, which satisfies $0 < v_\lambda(x) \le 1$ in Ω, together with $\lim_{\lambda\to\sigma_{1,p}+} v_\lambda = 0$ in $C_0^{1,\beta_0}(\overline{\Omega})$ for any $0 < \beta_0 < \beta$. As a consequence, (L) has a positive solution only when $\lambda > \sigma_{1,p}/m$. For λ in that range, its unique positive solution u_λ is given by the expression $u_\lambda = (m\lambda)^{1/(q-p+1)} v_{m\lambda}(x)$.

Let us focus next our attention on the perturbed problem (LP). Both for searching for constant supersolutions and achieving upper estimates, it is convenient to examine first the positive zeros of $G(\lambda, u) := m\lambda u^{p-1} - u^q + g(u)$, which can be written as $G = (m\lambda - h(u))u^{p-1}$ with $h(u) = \frac{u^q - g(u)}{u^{p-1}}$.

Since $\lim_{u\to 0} h(u) = 0$ while $h(u) \sim u^{q-p+1}$ as $u \to +\infty$, for each $\lambda > 0$ there exists a maximum zero for G, $u_0(\lambda) := \sup\{u > 0 : h(u) = m\lambda\}$. In particular, $G(\lambda, u) < 0$ for every $u > u_0(\lambda)$. It can be proved in addition that $u_0(\lambda) \sim (m\lambda)^{\frac{1}{q-p+1}}$ as $\lambda \to +\infty$. In fact, $(1-\mu)u^{q-p+1} \le h(u) \le (1+\mu)u^{q-p+1}$ for a fixed $0 < \mu < 1$ and u greater than some $K > 0$. Hence, one has $u_0(\lambda) \ge (\frac{m\lambda}{\mu+1})^{\frac{1}{q-p+1}}$ for λ large enough, and so $u_0(\lambda) \to +\infty$. The desired assertion follows from the fact that $h(u_0(\lambda)) = m\lambda$ for each $\lambda > 0$.

Let us consider now the questions of existence and estimates of positive solutions to (LP). Firstly, observe that if we set $\underline{h} := \inf_{u\in\mathbb{R}^+} h(u)$, then $G(\lambda, u) \le 0$ in $u \ge 0$ provided that $\lambda \le \underline{h}/m$. Therefore, positive weak solutions u to (LP) can never exist when $\lambda \le \underline{h}/m$. Indeed $\int_\Omega |\nabla u|^p \, dx = \int_\Omega G(\lambda, u)u \, dx \le 0$, and so u must be zero.

As for positive solutions, the same arguments as in the proof of Theorem 1 furnish the existence of a positive weak solution $u \in C_0^{1,\beta}(\overline{\Omega})$ for some $0 < \beta < 1$ provided that $\lambda > \sigma_{1,p}/m$, together with the estimate

$$0 < u(x) \le u_0(\lambda) \qquad x \in \Omega\,, \tag{4.2}$$

for every positive solution to (LP).

Remark 4.1. Concerning uniqueness, the analysis in [12] for the case $p = 2$ shows that although uniqueness of positive solutions may fail if λ remains of the order of magnitude of $\sigma_{1,p}/m$, one finally finds a unique positive solution to (LP) as $\lambda \to +\infty$, provided that the perturbation term $g = g(u)$ satisfies the growth conditions (Hg). However, the study of the same question for the case $p \ne 2$ is much harder. In fact, the perturbation technique employed to show uniqueness in the case $p = 2$, which is essentially based on linearization around positive solutions, does not work, for instance, in the case $p > 2$, due to the degeneracy of Δ_p. Such a degeneracy would become critical if the positive solutions around which (LP) is linearized exhibit dead cores. In fact, it will be shown in [17] that this is just the case for all possible positive solutions to (LP) when λ is large and k and p fall in the range $0 < k < p-1$.

Nevertheless, we present a result which is "in spirit" close to uniqueness. It states that all possible positive solutions to (LP) are comprised between the positive solutions of nearby logistic equations (L) when λ is large enough (the larger λ is, the closer the logistic equations are). In fact, let u be any positive solution to (LP). Then,

$$-\Delta_p u = \lambda\big(m + \frac{1}{\lambda}\frac{g(u)}{u^{p-1}}\big)u^{p-1} - u^q \qquad \text{in } \Omega\,.$$

We are showing now that the term $\frac{1}{\lambda}\frac{g(u)}{u^{p-1}}$ is as small as desired provided λ is large. In fact notice that $|g(u)| \le \epsilon u^q$ if $u \ge u_\epsilon$. By using the estimate (4.2) we obtain

$$\sup_{x\in\Omega} \frac{1}{\lambda}\frac{|g(u(x))|}{u(x)^{p-1}} \le \frac{1}{\lambda} \sup_{0\le u\le u_\epsilon} \frac{|g(u)|}{u^{p-1}} + \frac{1}{\lambda} \sup_{u_\epsilon \le u \le u_0(\lambda)} \frac{|g(u)|}{u^{p-1}}\,.$$

Now observe that the first term is of the order of ϵ if λ is large. As for the second notice that $|\frac{g(u)}{u^{p-1}}| \le \epsilon u^{q-p+1} \le \epsilon u_0(\lambda)^{q-p+1}$. Since $u_0(\lambda)^{q-p+1} \sim m\lambda$ as $\lambda \to \infty$, this term is also of the order of ϵ as $\lambda \to \infty$. Therefore, $\sup_{0\le u\le u_0(\lambda)} \lambda^{-1} g(u)/u^{p-1} \to 0$ as $\lambda \to +\infty$. In particular, if $\epsilon > 0$ is small and fixed, then some $\lambda(\epsilon) > 0$ exists such that, for any positive solution $u = u(x)$ to (LP) with $\lambda \ge \lambda(\epsilon)$, the following inequalities hold:

$$\lambda(m-\epsilon)u^{p-1} - u^q \le -\Delta_p u \le \lambda(m+\epsilon)u^{p-1} - u^q\,.$$

If λ is so large that $\lambda \ge \frac{\sigma_{1,p}}{m-\epsilon} \ge \frac{\sigma_{1,p}}{m+\epsilon}$, the uniqueness of positive solutions to (L) leads to the desired estimates. Namely,

$$((m-\epsilon)\lambda)^{\frac{1}{q-p+1}}\, v_{(m-\epsilon)\lambda}(x) \le u(x) \le ((m+\epsilon)\lambda)^{\frac{1}{q-p+1}}\, v_{(m+\epsilon)\lambda}(x)\,,$$

for each $x \in \Omega$, where $v = v_\lambda(x)$ stands for the unique solution to the normalized problem (4.1) for $\lambda > \sigma_{1,p}$.

Thus, as a summary of all the results just discussed, we can state the following theorem.

Theorem 8. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{2,\alpha}$ for some $0 < \alpha < 1$, while $m > 0$, $p > 1$ and $q > p-1$. Assume also that the perturbation term $g = g(u)$ satisfies* (Hg). *Then,*

A) *The logistic problem* (L) *satisfies the following properties,*

i) *There exists a positive solution $u \in W_0^{1,p}(\Omega)$ only if $\lambda > \sigma_{1,p}/m$. Such a solution is unique in that range of λ. Moreover, it can be written as $u_\lambda = (m\lambda)^{\frac{1}{q-p+1}} v_{m\lambda}$ where v_λ designates the unique positive solution to* (4.1) *if $\lambda > \sigma_{1,p}$.*

ii) *For each $\lambda > \sigma_{1,p}/m$, u_λ satisfies the estimate*

$$0 < u_\lambda(x) \le (m\lambda)^{\frac{1}{q-p+1}} \qquad x \in \Omega.$$

Furthermore, there exists $0 < \beta < 1$ such that $u_\lambda \in C_0^{1,\beta}(\overline{\Omega}) \cap C^{2,\gamma}(\overline{U_\epsilon})$, where $U_\epsilon = \{x \in \Omega : \text{dist } (x, \partial\Omega) < \epsilon\}$, $\epsilon = \epsilon(\lambda)$ and $\gamma = \min\{\alpha, p-1\}$. In addition, $\lambda \to u_\lambda \in C_0^{1,\beta}(\overline{\Omega})$ is increasing and continuous.

B) *The perturbed logistic problem* (LP) *exhibits the following features:*

i) *There are no positive solutions if $\lambda m \le \inf_{u>0} \frac{u^q - g(u)}{u^{p-1}}$, while there exists at least one positive solution $u \in W_0^{1,p}(\Omega)$ if $\lambda > \sigma_{1,p}/m$.*

ii) *For $\lambda > 0$ let $u_0(\lambda) = \sup\{u > 0 : m\lambda u^{p-1} - u^q + g(u) = 0\}$. Then, every positive solution u satisfies*

$$0 < u(x) \le u_0(\lambda) \qquad x \in \Omega\,, \tag{4.2}$$

where in addition $u_0(\lambda) \sim (m\lambda)^{\frac{1}{q-p+1}}$ as $\lambda \to +\infty$. On the other hand $u \in C_0^{1,\beta}(\overline{\Omega}) \cap C^{2,\gamma}(\overline{U_\epsilon})$ with γ and U_ϵ as above.

iii) *For any $\epsilon > 0$ there exists a $\lambda(\epsilon)$ such that any positive solution u to* (LP) *corresponding to $\lambda \ge \lambda(\epsilon)$ satisfies*

$$((m-\epsilon)\lambda)^{1/\delta} v_{(m-\epsilon)\lambda} \le u \le ((m+\epsilon)\lambda)^{1/\delta} v_{(m+\epsilon)\lambda} \qquad \text{in } \Omega, \tag{4.3}$$

with $\delta = q-p+1$ and v_λ being the positive solution to (4.1), *$\lambda > \sigma_{1,p}$.*

The results in Section 3 will be used next to show that the behaviour of positive solutions to both problems (L) and (LP) as $\lambda \to +\infty$ is exactly the same, regarding homogenization and boundary layer estimates. In fact, let $\{u_\lambda\}$, $\lambda > \sigma_{1,p}/m$, be any family of solutions to (LP). Then, using estimate

(4.3) we find that both the inner and the boundary behaviour of $\{u_\lambda\}$ is determined by means of the corresponding behaviour of the family v_λ, $\lambda > \sigma_{1,p}$, of positive solutions to the normalized problem (4.1). A direct application of Theorem 3 to (4.1) implies that $v_\lambda \to 1$ uniformly over compact subsets of Ω as $\lambda \to +\infty$. Since $F(v) = \frac{1}{p}v^p - \frac{1}{q+1}v^{q+1}$ in (4.1) we also obtain the precise boundary behaviour of v_λ in terms of

$$\frac{\partial v}{\partial \nu} \sim -\lambda^{\frac{1}{p}}\Big[\frac{\delta}{(q+1)(p-1)}\Big]^{\frac{1}{p}}$$

uniformly on $\partial\Omega$ as $\lambda \to +\infty$, ν being the outward unit normal. Therefore we can readily state the following result.

Corollary 9. *Let Ω be a class $C^{2,\alpha}$ domain of $\mathbb{R}^N$, $0 < \alpha < 1$, and for $\lambda > \sigma_{1,p}/m$ let $\{u_\lambda\}$ be either the family of positive solutions to the logistic problem* (L) *or any family of positive solutions to (LP) where the perturbation term $g = g(u)$ satisfies hypothesis* (Hg)*. In both cases,*

$$\lim_{\lambda\to+\infty} \lambda^{-\frac{1}{\delta}} u_\lambda(x) = m^{\frac{1}{\delta}} \qquad \text{for each } x \in \Omega,$$

uniformly over compact sets of Ω, where $\delta = q - p + 1$. On the other hand,

$$\lim_{\lambda\to+\infty} \lambda^{-\frac{p+\delta}{\delta p}} \frac{\partial u_\lambda}{\partial \nu} = -\, m^{\frac{p+\delta}{\delta p}} \Big(\frac{\delta}{(p+\delta)(p-1)}\Big)^{\frac{1}{p}}$$

uniformly on $\partial\Omega$.

Remark 4.2. In light of the arguments leading to Corollary 9, one can still conceive of another class of problems whose positive solutions exhibit the asymptotic features described in Theorem 3. Indeed, consider

$$\begin{cases} -\Delta_p u = \lambda f(u) + g(u) & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases} \tag{4.4}$$

where f satisfies (Hf) and g is a C^α perturbation such that $g = o(u^{p-1})$ as $u \to 0+$, while $g = o(f)$ as $u \to +\infty$. As in the case of (LP), it can be proved that any family $\{u_\lambda\}$, $\lambda > \sigma_{1,p}/m$, of positive solutions to (4.4) satisfies the assertions of Theorem 3.

The analysis of dead cores for the logistic problem (L) coincides with that for the normalized one (4.1). In view of Theorem 4, (4.1) can only exhibit dead cores $\{x : v_\lambda = 1\}$ when $p > 2$. If the conclusions of Theorem 4 are applied to (4.1) then we achieve the next result.

Corollary 10. *Let $\Omega \subset \mathbb{R}^N$ be a $C^{2,\alpha}$ bounded domain with inner radius R_Ω. Then, the positive solution u_λ to* (L) *with $p > 2$ develops a dead core $\mathcal{O}_\lambda = \{x \in \Omega : u_\lambda = (m\lambda)^{1/p}\}$ provided*

$$\lambda \geq \frac{(p+\delta)(p-1)}{m}\Big\{\frac{C_\theta B(p,\delta)}{R_\Omega}\Big\}^p,$$

where $\delta = q+p-1$, C_θ is as in Section 3 *and $B(p,\delta)$ designates the convergent integral*

$$B(p,\delta) = \int_0^1 \frac{ds}{[\delta - (p+\delta)s^p + ps^{p+\delta}]^{\frac{1}{p}}}\,.$$

Moreover,

$$dist\,(\mathcal{O}_\lambda, \partial\Omega) \sim \frac{(p+\delta)^{1/p}(p-1)^{1/p}}{m^{1/p}}B(p,\delta)\lambda^{-1/p}\,,$$

as $\lambda \to +\infty$.

Remark 4.3. Observe that, in contrast with the case of nearly flat solutions ($k \geq p-1$), the conclusions of Theorem 4 can not be transferred, in principle, to the perturbations (LP) of the logistic problem (L) by means of the comparison arguments used in the present section. As a matter of fact, observe that in the estimate (4.3), both solutions at the extreme sides of the inequality develop flat cores when $p > 2$, for different values of u_0. Namely, $(m \pm \epsilon)^{1/\delta}$, $\delta = q - p + 1$. Nevertheless, a precise study of the flat pattern formation in problems such as (LP), being beyond of the scope of the present work, will be completely studied in [17].

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