Équations aux dérivées partielles/*Partial differential equations*

On the perturbation of eigenvalues for the p-Laplacian

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Abstract. In this note the differentiability with respect to the domain of the first Dirichlet eigenvalue of the minus p-Laplacian is shown for the first time. An explicit formula for the first variation is obtained. The proof, based on the variational formulation and $C^{1,\alpha}$- type estimates, even simplifies the corresponding to the linear case $p = 2$. An application to the profile behaviour of positive solutions to a class of logistic problem is given.

Sur la perturbation des valeurs propres pour le p-Laplacien

Resumé. Dans cette note nous démontrons pour la première fois la derivabilité de la première valeur propre du p-Laplacien par rapport au domaine. On obtient une formule explicite de la première variation. La démostration, basée sur la formulation variationnelle et les estimations $C^{1,\alpha}$, simplifie en particulier le cas linéaire $p = 2$. On donne une application à l'étude du comportement des solutions positives d'une classe de problèmes logistiques.

Version française abrégée. Il est bien connu que le problème des valeurs propres non linèaires

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{dans } G \\ u = 0 & \text{sur } \partial G \ , \end{cases} \qquad (P)$$

où $\Delta_p u = \text{div}\ (|\nabla u|^{p-2} \nabla u)$, $p > 1$, et où G est un domaine borné de classe $C^{2,\alpha}$ de $\mathbb{R}^n$, admet une seule valeur propre $\lambda_1(G)$ associée à une fonction propre positive qui soit simple et isolée (voir [1]). De plus, $\lambda_1(G)$ peut être caractérisée variationnellement.

L'objectif de cette note est de montrer que la valeur propre $\lambda_1(G)$ varie différentiablement quand on considère des perturbations différentiables du domaine G, de la form $G_\delta = T_\delta(G)$,

$$T_\delta(x) = x + \delta R(x) + S(x, \delta), \tag{1}$$

où $|\delta| < \varepsilon$, $R, S(\cdot, \delta) \in C^1(\overline{G}, \mathbb{R}^n)$, et où $S(x, \delta) = o(\delta)$ lorsque $\delta \to 0$ dans $C^1(\overline{G}, \mathbb{R}^n)$.

Le problème de la perturbation des valeurs propres consiste à étudier la dépendence de $\lambda(\delta) := \lambda_1(G_\delta)$, la première valeur propre du domain G_δ, par rapport à δ. En termes variationnels,

$$\lambda(\delta) = \inf_{u \in W_0^{1,p}(G)} \frac{\int_G |D(x,\delta)\nabla u|^p C(x,\delta)\; dx}{\int_G |u|^p C(x,\delta)\; dx},$$

où $D(x, \delta) = (D_x T_\delta(x))^{-1}$ et $C(x, \delta) = |\det\; (D_x T_\delta(x))|$.

Contrairement au cas linéaire, où on utilise les familles holomorphes du type-A de Kato (cf. [5]), nous procèderons directement avec la formulation variationnelle. De cette manière, le cas linéaire lui-même est considérablement simplifié.

Nos résultats sont les suivants:

Théorème 1 *Soient $G \subset \mathbb{R}^n$ un domaine borné $C^{2,\alpha}$ et $\lambda(\delta)$ la première valeur propre du problème (P) dans le domaine $G_\delta = T_\delta(G)$, où T_δ est une famille de difféomorphismes C^1 qui vérifient (1). La fonction $\lambda = \lambda(\delta)$ est alors continue en $\delta = 0$. De plus, si ϕ_δ est la fonction propre positive dans G_δ telle que $|\phi_\delta|_\infty = 1$, alors*

$$\phi_\delta \to \phi$$

dans $C_0^{1,\beta}(G)$ quand $\delta \to 0$ pour un certain β, $0 < \beta < 1$, où ϕ est la fonction propre positive normalisée pour $\delta = 0$.

Théorème 2 *Soient $G \subset \mathbb{R}^n$ un domaine borné de classe $C^{2,\alpha}$, $A = A(x, \delta, \xi) \in C^1(\overline{G} \times \mathbb{R} \times \mathbb{R}^n)$, et $B = B(x, \delta, z) \in C^1(\overline{G} \times \mathbb{R} \times \mathbb{R})$. Suppossons que le problème variationnel*

$$\lambda(\delta) = \inf_{u \in W_0^{1,p}(G)} J_\delta(u) := \inf_{u \in W_0^{1,p}(G)} \frac{\int_G A(x, \delta, \nabla u)\ dx}{\int_G B(x, \delta, u)\ dx},$$

admet, pour $|\delta| < \varepsilon$, une seule solution normalisée *$u = u_\delta$ telle que $u_\delta \to u_0$ dans $C^1(\overline{G})$, où u_0 est la solution du problème pour $\delta = 0$. La fonction $\lambda = \lambda(\delta)$ est alors dérivable en $\delta = 0$, et on a*

$$\lambda'(0) = \frac{\int_G \{A_1(x, 0, \nabla u_0) - \lambda_0 B_1(x, 0, u_0)\}\ dx}{\int_G B(x, 0, u_0)\ dx},$$

où $A_1 = \partial_\delta A$, $B_1 = \partial_\delta B$ et $\lambda_0 = \lambda(0)$.

Théorème 3 *Soient $G \subset \mathbb{R}^n$ un domaine borné de classe $C^{2,\alpha}$, $G_\delta = T_\delta(G)$ la perturbation de G associée à une famille de difféomorphismes de classe C^1, $T = T_\delta(x)$, qui vérifient (1). La première valeur propre $\lambda(\delta) = \lambda_1(G_\delta)$ de $-\Delta_p$ dans G_δ est alors dérivable par rapport à δ en $\delta = 0$, et on a*

$$\lambda'(0) = -(p-1) \int_{\partial G} \langle R, \nu \rangle \left| \frac{\partial \phi}{\partial \nu} \right|^p\ d\sigma,$$

où ν est la normale unitaire extérieure et ϕ est la fonction propre positive dans G normalisée par:

$$\int_G |\phi|^p\ dx = 1.$$

Finalement, ces résultats sont appliqués à l'étude d'une classe de problèmes de type logistique singuliers où l'opérateur de diffusion est le p-Laplacien.

I. Results. It is well known that the nonlinear eigenvalue problem

$$(2) \qquad \begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } G \\ u = 0 & \text{on } \partial G\ , \end{cases}$$

where $\Delta_p u = \text{div}\,(|\nabla u|^{p-2}\nabla u)$, $p > 1$ and G is a bounded $C^{2,\alpha}$ domain of $\mathbb{R}^n$, admits a unique eigenvalue $\lambda_1(G)$ associated to a positive eigenfunction, which is simple and isolated (see [1]). Moreover, $\lambda_1(G)$ can be variationally characterized as

$$\lambda_1(G) = \inf_{u \in W_0^{1,p}(G)} \frac{\int_G |\nabla u|^p \, dx}{\int_G |u|^p \, dx} \; .$$

The objective of this note is to show that $\lambda_1(G)$ is differentiable when differentiable perturbations of the domain G are considered.

To this aim we will study the perturbation properties of $\lambda_1(G)$ with respect to G following the so-called "method of inner variations" of Hadamard. So, consider a family of diffeomorphisms $T = T_\delta(x) \in C^1(\overline{G}, \mathbb{R}^n)$, $\delta \in \mathbb{R}$, $|\delta| < \varepsilon$. We will assume

$$T_\delta(x) = x + \delta R(x) + S(x,\delta), \tag{3}$$

where $R, S(\cdot,\delta) \in C^1(\overline{G}, \mathbb{R}^n)$, $S(x,\delta) = o(\delta)$ as $\delta \to 0$ in $C^1(\overline{G}, \mathbb{R}^n)$.

The problem of perturbation of eigenvalues consists in analyzing the dependence of $\lambda(\delta) := \lambda_1(G_\delta)$, the principal eigenvalue of G_δ, with respect to δ. In variational terms,

$$\lambda(\delta) = \inf_{u \in W_0^{1,p}(G)} \frac{\int_G |D(x,\delta)\nabla u|^p C(x,\delta) \, dx}{\int_G |u|^p C(x,\delta) \, dx}, \tag{PV}$$

where $D(x,\delta) = (D_x T_\delta(x))^{-1}$, $C(x,\delta) = |\text{det}\,(D_x T_\delta(x))|$.

In the linear case $p = 2$ ($\Delta_p \equiv \Delta$), the by no means trivial question of differentiability of λ_1 is usually handled by using the theory of Kato's holomorphic families of type-A (cf. [5]). Our perturbation problem here is out of the scope of this approach, and differentiability has to be proved directly. However, it should be stressed that our procedure greatly simplifies also the proof of the linear case.

A first approximation result is given in the next theorem.

Theorem 1 *Let $G \subset \mathbb{R}^n$ be a bounded $C^{2,\alpha}$ domain, $\lambda(\delta)$ the principal eigenvalue of (2) in the domain $G_\delta = T_\delta(G)$, where T_δ is a family of C^1 diffeomorphisms verifying (3). Then the function $\lambda = \lambda(\delta)$ is continuous at $\delta = 0$, and if ϕ_δ denotes the positive eigenfunction in G_δ normalized as $|\phi_\delta|_\infty = 1$ we have:*

$$\phi_\delta \to \phi,$$

in $C_0^{1,\beta}(G)$ as $\delta \to 0$ for some $0 < \beta < 1$, where ϕ is the positive normalized eigenfunction of (2) in $\delta = 0$.

Our first fundamental result is the following abstract theorem concerning differentiability of variational problems.

Theorem 2 *Let $G \subset \mathbb{R}^n$ be a bounded $C^{2,\alpha}$ domain, $A = A(x, \delta, \xi) \in C^1(\overline{G} \times \mathbb{R} \times \mathbb{R}^n)$, $B = B(x, \delta, z) \in C^1(\overline{G} \times \mathbb{R} \times \mathbb{R})$. Assume that the variational problem*

$$\lambda(\delta) = \inf_{u \in W_0^{1,p}(G)} J_\delta(u) := \inf_{u \in W_0^{1,p}(G)} \frac{\int_G A(x, \delta, \nabla u)\ dx}{\int_G B(x, \delta, u)\ dx}, \tag{4}$$

admits, for $|\delta| < \varepsilon$, a unique normalized *solution $u = u_\delta$ with the property that $u_\delta \to u_0$ in $C^1(\overline{G})$, where u_0 is the solution of the problem for $\delta = 0$. Then the function $\lambda = \lambda(\delta)$ turns out to be differentiable at $\delta = 0$ and, moreover*

$$\lambda'(0) = \frac{\int_G \{A_1(x, 0, \nabla u_0) - \lambda_0 B_1(x, 0, u_0)\}\ dx}{\int_G B(x, 0, u_0)\ dx}, \tag{5}$$

where $A_1 = \partial_\delta A$, $B_1 = \partial_\delta B$ and $\lambda_0 = \lambda(0)$.

Theorem 2 can be applied to our perturbation problem to obtain an explicit expression of the first variation of $\lambda_1(G_\delta)$ with respect to δ. The following result extends to the context of the p-Laplacian the formula of variation of the first eigenvalue of $-\Delta$ contained in [7].

Theorem 3 *Let $G \subset \mathbb{R}^n$ be a bounded $C^{2,\alpha}$ domain, $G_\delta = T_\delta(G)$ the perturbation of G associated to a family of C^1 diffeomorphisms $T = T_\delta(x)$, $|\delta| < \varepsilon$, verifying (3). Then the principal eigenvalue $\lambda(\delta) = \lambda_1(G_\delta)$ of $-\Delta_p$ in G_δ is differentiable with respect to δ at $\delta = 0$. Moreover*

$$\lambda'(0) = -(p-1)\int_{\partial G} \langle R, \nu\rangle \left|\frac{\partial \phi}{\partial \nu}\right|^p \, d\sigma, \tag{6}$$

where ν is the outward unit normal and ϕ the positive eigenfunction in G normalized so that

$$\int_G |\phi|^p \, dx = 1.$$

II. Sketch of the proof of Theorems 1,2 and 3.

Proof of Theorem 1. The basic ingredients are the boundedness of $\lambda_1(G_\delta)$ and the $C^{1,\beta}$ estimates in [6]. This allows us to conclude in a standard way that $\lambda(\delta) \to \mu$ and ϕ_δ converges in $C^{1,\beta}(G)$ to a positive eigenfunction ψ as $\delta \to 0$. The properties of $\lambda_1(G)$ permit then to conclude $\mu = \lambda_1(G)$ and $\psi = \phi$. □

Proof of Theorem 2. In virtue of characterization (4), we have

$$J_\delta(u_\delta) - J_\delta(u_0) \le \lambda(\delta) - \lambda(0) \le J_\delta(u_0) - J_0(u_0)\ .$$

Since we can write $A(x,\delta,\xi) = A(x,0,\xi) + \delta A_1(x,0,\xi) + A_2(x,\delta,\xi)$, $B(x,\delta,z) = B(x,0,\delta) + \delta B_1(x,0,z) + B_2(x,\delta,z)$, where A_2, B_2 are $o(\delta)$ as $\delta \to 0+$, uniformly in $|\xi| \le M$, $|z| \le M$, we have the estimate

$$\lambda(\delta) - \lambda(0) \le \delta \frac{\int_G \{A_1(x,0,\nabla u_0) - \lambda_0 B_1(x,0,u_0)\}dx}{\int_G B(x,\delta,u_0)dx} + \frac{\int_G \{A_2(x,\delta,\nabla u_0) - \lambda_0 B_2(x,\delta,u_0)\}dx}{\int_G B(x,\delta,u_0)dx}\ .$$

We deduce, passing to the limit, that

$$\limsup_{\delta\to 0+} \frac{\lambda(\delta) - \lambda_0}{\delta} \le \frac{\int_G \{A_1(x,0,\nabla u_0) - \lambda_0 B_1(x,0,u_0)\}dx}{\int_G B(x,0,u_0)dx}\ ,$$

and

$$\liminf_{\delta\to 0-} \frac{\lambda(\delta)-\lambda_0}{\delta} \geq \frac{\int_G \{A_1(x,0,\nabla u_0) - \lambda_0 B_1(x,0,u_0)\}dx}{\int_G B(x,0,u_0)dx} \ .$$

With a similar procedure, we obtain an estimate for the inferior limit as $\delta \to 0+$ and the superior limit as $\delta \to 0-$, and we deduce that the limit exists and is given by (5). □

Proof of Theorem 3. The variational problem (PV) can be cast into the form (4), where

$$A_1(x,0,\xi) = \text{div } |\xi|^p R(x) - p|\xi|^{p-2} \langle \xi, R'(x)\xi \rangle \ , \quad B_1(x,0,z) = \text{div } |z|^p R(x)$$

($\langle , \rangle$ stands for the scalar product in $\mathbb{R}^n$). Thus,

$$\lambda'(0) = \int_G \{\text{div } |\nabla\phi|^p R - p|\nabla\phi|^{p-2} \langle \nabla\phi, R'\nabla\phi \rangle\} \ dx - \lambda_0 \int_G \text{div } |\phi|^p R \ dx.$$

Our intention is to perform an integration by parts in this expression. The lack of C^2 regularity of ϕ leads us to consider the problem

$$\begin{cases} -\text{div } ((\varepsilon + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u) = \lambda_0 (\phi+\varepsilon)^{p-1} & \text{in } G \\ u = 0 & \text{on } \partial G \ , \end{cases}$$

for $\varepsilon > 0$ small. As a consequence of the quasilinear theory in [4], this problem has a unique solution $u_\varepsilon \in C^{2,\alpha}(\overline{G})$. Moreover, the $C^{1,\beta}$ estimates in [6] imply $u_\varepsilon \to \phi$ in $C^1(\overline{G})$ as $\varepsilon \to 0$. Let

$$\lambda_\varepsilon = \int_G \{\text{div } (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p}{2}} R - p(\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \langle \nabla u_\varepsilon, R'\nabla u_\varepsilon \rangle - \lambda_0 \text{div } |\phi|^p R\} \ dx.$$

Integrating by parts in the three terms, we obtain

$$\lambda_\varepsilon = \int_{\partial G} \{\langle R, \nu\rangle (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p}{2}} - p(\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \langle R, \nabla u_\varepsilon\rangle \langle \nabla u_\varepsilon, \nu\rangle\} d\sigma$$

$$-p\lambda_0 \int_G \{\langle R, \nabla u_\varepsilon\rangle (\phi+\varepsilon)^{p-1} - \langle R, \nabla\phi\rangle \phi^{p-1}\} dx \ ,$$

where ν is the outward unit normal to ∂G. We finally arrive at (6) letting $\varepsilon \to 0+$, and noticing that $\nabla \phi = \frac{\partial \phi}{\partial \nu} \nu$. $\square$

III. An application to logistic problems. The class of problems

$$\begin{cases} -\Delta_p u &= \lambda u^{p-1} - a(x) u^q \quad \text{in } \Omega \\ \quad u &= 0 \qquad\qquad\qquad\quad \text{on } \partial\Omega \ , \end{cases} \tag{L}$$

where $q > p - 1$, Ω is a bounded $C^{2,\alpha}$ domain, $a \in C(\overline{\Omega})$ and $a \geq 0$, is the p-Laplacian version of the logistic equation in population dynamics.

When $a > 0$ in $\overline{\Omega}$, sub and supersolutions, combined with the uniqueness result in [2] allow us to conclude that (L) has positive solutions $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ only when $\lambda > \lambda_1(\Omega)$, the solution being unique in this range.

In the so called "refuge case" we assume on the contrary that $D := \{x \in \Omega : \ a(x) > 0\}$ is a $C^{2,\alpha}$ domain, $\overline{D} \subset \Omega$ whose boundary ∂D consists of a finite number of connected components $\Gamma_1, \ldots, \Gamma_m$, so that $G := \Omega \setminus \overline{D}$ is connected. Since $a = 0$ in G now, G is called a refuge for the species u. Under these conditions, it is possible to show using again sub and supersolutions that (L) admits positive solutions $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ only if $\lambda_1(\Omega) < \lambda < \lambda_1(G)$. This positive solution is again unique, and will be denoted by u_λ. It can also be shown that u_λ is increasing in λ, while

$$\|u_\lambda\|_{\infty,\Omega} \to +\infty, \qquad \lambda \uparrow \lambda_1(G) \ .$$

In other words, u_λ bifurcates from infinity at $\lambda = \lambda_1(G)$. A natural question is to determine at which points of Ω the divergence to infinity of u_λ takes place as $\lambda \to \lambda_1(G)$. When $p = 2$ it was proved in [7] that $u_\lambda \to +\infty$ in G as $\lambda \to \lambda_1(G)$, using that u_λ is differentiable with respect to λ in that case. Following the same ideas as in [7] we will prove now that $u_\lambda \to +\infty$

uniformly on compacts of $G \cup \partial D$.

In fact, we can selectively enlarge G an amount δ in the direction of the outward normal to $\partial G \setminus \partial\Omega = \Gamma_1 \cup \cdots \cup \Gamma_m$ to obtain the perturbed domain $G_\delta = G \cup \{\cup_{j=1}^k \{x \in \mathbb{R}^n \setminus G : \text{dist }(x, \Gamma_j) < \delta\}\}$. It can be shown that this perturbation is of the form $G_\delta = T_\delta(G)$, where $T_\delta(x) = x + \delta R(x)$, $R = \nu_j$ on Γ_j (ν_j the outward unit normal to Γ_j), and $R = 0$ on $\partial\Omega$ (see [7]). By Theorem 3,

$$\lambda'(0) = -(p-1) \sum_{j=1}^{k} \int_{\Gamma_j} \left| \frac{\partial \phi}{\partial \nu_j} \right|^p \, d\sigma < 0 \ . \tag{7}$$

If $a(x) = o(\text{dist }(x, \partial D))$ as dist $(x, \partial D) \to 0+$, it is possible to find $C(\delta)$ such that

$$u_\lambda(x) \geq C(\delta)\phi_\delta(x), \qquad x \in G \cup \partial D$$

for $\lambda \geq \lambda_1(G_{\delta/2})$, in such a way that $C(\delta) \to +\infty$ as $\delta \downarrow 0$ (thus $\lambda \uparrow \lambda_1(G)$), while we even have $C(\delta)\phi_\delta(x) \to +\infty$ on ∂D.

Indeed, it is enough to define ϕ_δ by 0 outside G_δ to achieve that $\underline{u} = C\phi_\delta$ is a subsolution to (L) if $\lambda_1(G_{\delta/2}) < \lambda < \lambda_1(G)$. An admissible value for C is given by

$$C(\delta) = \left(\frac{\lambda_1(G_{\delta/2}) - \lambda_1(G_\delta)}{\sup_{\text{dist }(x,\partial D)<\delta} a(x)} \right)^{1/(q-p+1)} .$$

Since (7) implies that $\lambda_1(G_{\delta/2}) - \lambda_1(G_\delta) \sim -\lambda'(0)\delta/2$ as $\delta \to 0+$, we obtain that $C(\delta) \to +\infty$ as $\delta \to 0+$, and it verifies the required properties.

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