ON THE BEHAVIOUR OF THE FIRST EIGENFUNCTION OF THE p-LAPLACIAN NEAR ITS CRITICAL POINTS *

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Abstract

In this work we study the behaviour of the positive eigenfunction ϕ of $-\Delta_p u = \lambda |u|^{p-2}u$ in Ω, $u_{|\partial\Omega} = 0$, $p > 1$, near its critical points. Under some convexity and symmetry assumptions on Ω, we obtain that ϕ has a unique critical point at $x = 0$, and determine the behaviour of both ϕ and $\nabla\phi$ nearby. We also consider positive solutions u to some general problems $-\Delta_p u = f(u)$ in Ω, $u_{|\partial\Omega} = 0$, with some convexity restrictions on u.

1. Introduction

In the recent work [6], the problem of local bifurcation for the p-Laplacian was studied. The authors considered

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + g(\lambda, u) & \text{in} \quad B \\ \quad u = 0 & \text{on} \quad \partial B, \end{cases}$$

where $\Delta_p u = \text{div}\,(|\nabla u|^{p-2}\nabla u)$, $p > 1$, $\lambda \in \mathbb{R}$, $g(\lambda, u) = o(|u|^{p-1})$ as $u \to 0$ and B is the unit ball in $\mathbb{R}^N$. They obtained the existence of branches of nontrivial solutions bifurcating from the trivial branch $(\lambda, 0)$ at $(\lambda_n, 0)$, where $\{\lambda_n\}$ stands for the sequence of radial eigenvalues of the p-Laplacian in B. Moreover, the local uniqueness of these branches was also proved. Their approach relies on linearization of $-\Delta_p$ around the eigenfunctions ϕ_n, and it is essential in this approach to determine the behaviour of the eigenfunctions near their critical points.

A natural question to ask is whether this result will still be valid for general smooth domains Ω of $\mathbb{R}^N$. It turns out that a complete knowledge of the critical points of the eigenfunctions is needed, and this will be the aim of the present paper. We will come to the general bifurcation problem in a future work (see [4]).

At the best of our knowledge, very few attention has been paid to this question in the literature until now, even in the linear case. The main stress has been put in the plane, where complex variable arguments are often used. To mention some results let us quote [7] for eigenfunctions, where the authors obtain an upper estimate of the number of critical points in some very special domains, and [3], [8], [10] for p-harmonic functions. In the first of these three works the question of the behaviour of a p-harmonic function near its critical

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points is studied, while the second gives the exact number of critical points for a p-harmonic function with polynomial datum on the boundary of the unit circle in $\mathbb{R}^2$. [10] is dedicated to study the isolation of critical points (see also [1] for problems of the form div $(A(x)\nabla u) = 0$, with boundary conditions $A(x)\nabla u \cdot \nu = pu$).

In this work we will focus our attention on the eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \ . \end{cases} \tag{E}$$

It is well known that problem (E) admits an eigenvalue $\lambda_1 > 0$ which is the only one with an associated positive eigenfunction $\phi \in C^{1,\alpha}(\overline{\Omega})$. Moreover, λ_1 is simple and isolated (see [2] and [9]). Our objective will be to study the critical points of the eigenfunction ϕ (the normalization chosen for this eigenfunction will be immaterial to us as long as it is taken to be positive). The reason for restricting ourselves to the first eigenfunction will be clear below.

In the case of a ball B, the exact behaviour of the eigenfunction is obtained by means of a more general result (see Lemma 3 in [6] and Theorem 2.1 in [5]): if u is the solution to

$$\begin{cases} -\Delta_p u = f & \text{in} \quad B \\ u = 0 & \text{on} \quad \partial B \end{cases}$$

where $f \in C(\overline{B})$ is radially symmetric, and $f(0) \neq 0$, then

$$\lim_{x\to 0} \frac{u(0) - u(x)}{|x|^{p'}} = \frac{1}{p'} \left(\frac{f(0)}{N} \right)^{\frac{1}{p-1}} \ . \tag{1}$$

This is no longer true in a general domain Ω, as the following simple example shows: the function $u(x,y) = 1 - x^4 - y^2$ is the solution to

$$\begin{cases} -\Delta u = 12x^2 + 2 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \ , \end{cases}$$

where $\Omega = \{(x,y) \in \mathbb{R}^2 : \ x^4 + y^2 < 1\}$ is a convex smooth domain. Moreover:

$$\liminf_{(x,y)\to(0,0)} \frac{1 - u(x,y)}{x^2 + y^2} = 0 \ ,$$

and the limit does not even exist. Thus we have to proceed differently in order to obtain the estimates.

A first important restriction to be imposed on the domain Ω to ensure (1) is convexity. Indeed, when Ω is an annulus $\{r < |x| < R\}$, the set of critical points of ϕ is $\{|x| = r_0\}$, for some $r < r_0 < R$, and the limit in (1) need not exist (observe that the inferior limit is clearly zero in this case). Notice that, even in the radial case, the eigenfunctions associated to eigenvalues $\lambda_n \neq \lambda_1$ cannot satisfy (1), since the set of critical points also contains at least a circle $\{|x| = r_0\}$, for some $0 < r_0 < 1$. Thus, we have to consider only the first eigenfunction in a convex domain.

Also, in the context of this work, the following symmetry hypothesis will be important. Let T_k, $k = 1, \ldots, N-1$ be (two-dimensional) planes of the form $\{x_i = 0 : \ i \neq i_k, i_{k+1}\}$, where $i_k \neq i_{k+1}$, the i_k's range the whole set $\{1, 2, \ldots, N\}$ and every index appears at most twice. With no loss of generality we could assume that $T_k = \{(x_1, \ldots, x_n) : x_k = x_{k+1} = 0\}$ (e. g. when $N = 2$, we only have the XY plane). We say that Ω is *symmetric* if it is invariant by a rotation of $\theta_k \neq 0, \pi$ radians in the planes T_k. Simple examples of symmetric domains are the balls and cubes in $\mathbb{R}^N$. This symmetry is stated with respect to $x = 0$, but the arguments below are valid when the origin is replaced by any point.

With this definition in mind we can now state our main results.

Theorem 1 *Assume $p > 1$, and let Ω be a bounded, convex, symmetric smooth domain of $\mathbb{R}^N$. Then the principal eigenfunction ϕ has a unique critical point at $x = 0$ and*

$$\liminf_{x \to 0} \frac{\phi(0) - \phi(x)}{|x|^{p'}} > 0 \ , \tag{2}$$

where $p' = p/(p-1)$. Moreover there exist positive constants c, C such that

$$c \leq \frac{|\nabla \phi(x)|^{p-1}}{|x|} \leq C, \ \ x \in \overline{\Omega} \ . \tag{3}$$

Remarks 1 a) As a consequence of the proof of Theorem 1 (see §§2 and 3) it follows that

$$\limsup_{x \to 0} \frac{\phi(0) - \phi(x)}{|x|^{p'}} < +\infty \ .$$

This estimate does *not* need the convexity or symmetry of Ω.

b) From the limit (2) and a) we can also derive some optimal regularity results for ϕ. Namely $\phi \in C^2(\overline{\Omega})$ when $1 < p \leq 2$ (irrespectively of the symmetry of Ω), and $\phi \in C^{1,\alpha}(\overline{\Omega}) \cap C^2(\overline{\Omega} \setminus \{0\})$ if $p > 2$ (symmetry is needed here). A careful examination of the proofs in §§2 and 3 shows that for $1 < p \leq 2$, this conclusion remains valid if we assume that every critical point of ϕ is a maximum or a minimum, so that no convexity or symmetry is needed to obtain $\phi \in C^2(\overline{\Omega})$.

c) No smoothness is needed to produce estimate (2), since this is a local result. However, if Ω is C^2, say, then $\phi \in C^1(\overline{\Omega})$ and (3) makes full sense. Similar remarks are in order for the theorems below.

In the linear case $p = 2$, more can be said about (2). Namely, this limit exists and can be evaluated.

Theorem 2 *Assume $p = 2$ and let Ω be a bounded, convex, symmetric smooth domain of $\mathbb{R}^N$. Then the principal eigenfunction ϕ has a unique critical point at $x = 0$ and*

$$\lim_{x \to 0} \frac{\phi(0) - \phi(x)}{|x|^2} = \frac{\lambda_1}{2N} \phi(0) \ . \tag{4}$$

Moreover

$$\lim_{x\to 0}\frac{|\nabla\phi(x)|}{|x|}=\frac{\lambda_1}{N}\phi(0)\ , \tag{5}$$

and there exist positive constants c, C such that

$$c\le\frac{|\nabla\phi(x)|}{|x|}\le C,\ \ x\in\overline{\Omega}\ .$$

Further, we have for the second derivatives $\partial_{ij}\phi(0)=-\delta_{ij}\lambda_1\phi(0)/2N$, where δ_{ij} stands for Kronecker's delta.

Remarks 2 a) The symmetry condition on Ω is essential for the existence of the limit (4). In the simple case of a cylinder $\Omega=[-a,a]\times B_R$ in $\mathbb{R}^N$, $a\neq R$, we only have

$$\liminf_{x\to 0}\frac{\phi(0)-\phi(x)}{|x|^2}>0\ ,$$

(this is easily achieved by separation of variables) but the limit does not exist.

b) Although limit (5) exists in this case, the corresponding limits

$$\lim_{x\to 0}\frac{\partial_i\phi(x)}{|x|}$$

$i=1,\ldots,N$, do not exist, and the inferior limit is zero.

We are finally providing an adaptation of the results on Theorem 1 to a nonlinear situation. We recall that a function u is said to be ψ–concave (respectively ψ–convex) if $\psi(u)$ is concave (resp. convex) (two important examples are $\psi(s)=s$ and $\psi(s)=\log s$).

Theorem 3 *Assume $p>1$, and let Ω be a bounded, convex, symmetric smooth domain of $\mathbb{R}^N$. Let $u\in C^1(\overline{\Omega})$ be a positive solution to the problem*

$$\begin{cases} -\Delta_p u=f(u) & in \quad \Omega\\ \quad u=0 & on \quad \partial\Omega\ ,\end{cases}$$

where f is continuous and $f(u(0))>0$. Suppose in addition that u is ψ–concave (respectively ψ–convex) and $\psi'(u(0))>0$ (resp. $\psi'(u(0))<0$) for some $\psi\in C^1(0,\infty)$. Then u has a unique critical point at $x=0$, and

$$\liminf_{x\to 0}\frac{u(0)-u(x)}{|x|^{p'}}>0\ .$$

Moreover, there exist positive constants c, C such that

$$c\le\frac{|\nabla\phi(x)|^{p-1}}{|x|}\le C,\ \ x\in\overline{\Omega}\ .$$

Remark 3 For the semilinear case $p = 2$, the conclusion is the same as in Theorem 2. The limits (4) and (5) corresponding to both the function and the gradient exist and equal respectively $f(u(0))/2N$ and $f(u(0))/N$.

The paper is organized as follows: in Section 2 we study the tools needed to analyze the behaviour of the eigenfunction. This material can be used to produce similar estimates in domains with some other symmetry conditions. Section 3 is devoted to the proof of Theorems 1, 2 and 3.

2. A general framework

In this section we introduce the essential tools to study the behaviour of the eigenfunction near its critical points. Since these results are local in nature and do not depend on the symmetry of Ω, they can be used to analyze the principal eigenfunction in domains with some other kind of symmetries. We begin with a simple lemma.

Lemma 4 *Let Ω be a convex domain, and $u \in C^1(\overline{\Omega})$ be a convex (respectively concave) function. Then the set of critical points C of u is convex, and u attains its minimum (resp. maximum) in every point of C.*

Proof. It is an easy matter to show that the function $v(t) = u(tx_0+(1-t)y_0)$ is convex (resp. concave) in $[0, 1]$ for every $x_0, y_0 \in \Omega$. Thus, v' is nondecreasing (resp. nonincreasing). Now if $x_0, y_0 \in C$, we obtain

$$v'(t) = \nabla u(tx_0 + (1-t)y_0)(x_0 - y_0) = 0, \ t \in [0, 1]. \tag{6}$$

In particular, choose x_0 such that u attains its minimum (resp. maximum). Then (6) shows that u equals its minimum (resp. maximum) in $[x_0, y_0]$, and $[x_0, y_0] \subset C$. This proves both assertions. □

In order to study the behaviour of the eigenfunction near its critical points, we are assuming henceforth for simplicity that ϕ attains its maximum at $x = 0$, and introduce, for $\lambda > 0$, the functions

$$u_\lambda(x) = \frac{\phi(0) - \phi(\lambda x)}{\lambda^{p'}}, \ x \in \Omega_\lambda \ , \tag{7}$$

where $\Omega_\lambda = \{x \in \mathbb{R}^N : \ \lambda x \in \Omega\}$, and $p' = p/(p-1)$. The essential property of these functions is contained in the following lemma.

Lemma 5 *Let $\lambda_n \to 0$ and u_{λ_n} as before. Then there exists a subsequence (denoted again $\{\lambda_n\}$) such that $u_{\lambda_n} \to \bar{u}$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$, where $\bar{u}$ is a nonnegative solution to*

$$\Delta_p u = \lambda_1 \phi(0)^{p-1} \quad in \quad \mathbb{R}^N, \tag{8}$$

and $\bar{u}(0) = 0$.

Proof. The functions u_λ are nonnegative, and solve the problem

$$\Delta_p u = \lambda_1 \phi(\lambda x)^{p-1} \text{ in } \Omega_\lambda \ . \tag{9}$$

Thus the Harnack inequality in [12] (Theorems 5, 6 and 9) implies the existence of constants $C, k > 0$ such that $\max_{B_R} u_\lambda \le C(\inf_{B_R} u_\lambda + k) = Ck$, for every ball B_R centered at $x = 0$ such that $B_{3R} \subset \Omega_\lambda$. Let $K \subset R^N$ be an arbitrary compact. Since $\Omega_\lambda \to \mathbb{R}^N$ as $\lambda \to 0$, there exists a constant $C > 0$ with

$$0 \le \max_K u_\lambda \le C, \ 0 < \lambda < \lambda_0.$$

Now we use the local $C^{1,\alpha}$ estimates of Tolksdorf (see [13]). We can find a constant $C' > 0$ such that

$$\|u_\lambda\|_{C^{1,\alpha}(K)} \le C', \ 0 < \lambda < \lambda_0,$$

and by Ascoli-Arzelá's theorem and a standard diagonal procedure we obtain, for every sequence $\{\lambda_n\}$ converging to zero, a subsequence $\{\lambda_n\}$ such that $u_{\lambda_n} \to \bar{u}$ in $C^1_{\text{loc}}(\mathbb{R}^N)$.

Passing to the limit in the weak formulation of (9) we see that $\bar{u}$ verifies (8). The other assertions are immediate, since $u_{\lambda_n} \ge 0$ and $u_{\lambda_n}(0) = 0$. □

Of special importance will be the unique radially symmetric solution to (8) which verifies $u(0) = 0$, given by

$$u(x) = \frac{1}{p'}\phi(0)\left(\frac{\lambda_1}{N}\right)^{\frac{1}{p-1}} |x|^{p'}, \ x \in \mathbb{R}^N \ .$$

However, problem (8) has infinitely many solutions satisfying $u(0) = 0$ (see Lemma 9 below).

When the domain Ω is convex, thanks to convexity properties of the eigenfunction, we obtain an important feature of the limit function $\bar{u}$.

Lemma 6 *Let ϕ be the principal eigenfunction in a convex bounded smooth domain Ω, and for $\lambda_n \to 0$, let u_{λ_n} be defined by (7). Then the function $\bar{u}$ given by Lemma 5 is convex.*

Proof. By a theorem of Sakaguchi ([11]), the principal eigenfunction ϕ is log-concave. This implies that for every $t \in [0, 1]$:

$$\phi(t\lambda x + (1-t)\lambda y) \ge \phi(\lambda x)^t \phi(\lambda y)^{1-t},$$

whenever $x, y \in \Omega_\lambda$ (observe that Ω_λ is clearly convex). By the definition of u_λ, this in turn leads to the following

$$u_\lambda(tx + (1-t)y) \le \frac{1}{\lambda^{p'}}\left(\phi(0) - (\phi(0) - \lambda^{p'} u_\lambda(x))^t (\phi(0) - \lambda^{p'} u_\lambda(y))^{1-t}\right).$$

For fixed $x, y \in \mathbb{R}^N$ and λ small enough, we have $\phi(0) - \lambda^{p'} u_\lambda(x), \phi(0) - \lambda^{p'} u_\lambda(y) > 0$ so we can apply the mean value theorem to obtain

$$u_\lambda(tx + (1-t)y) \le$$

$$tu_\lambda(x)\left(\frac{\phi(0)}{\xi_\lambda}\right)^{1-t} + (1-t)u_\lambda(y)\left(\frac{\phi(0)}{\eta_\lambda}\right)^t - t(1-t)\lambda^{p'} u_\lambda(x)u_\lambda(y)\xi_\lambda^{t-1}\eta_\lambda^{-t},$$

where $\xi_\lambda, \eta_\lambda \to \phi(0)$ as $\lambda \to 0+$. Setting $\lambda = \lambda_n$ and letting $n \to +\infty$, we obtain that $\bar{u}$ is convex, as was to be shown. □

It will be aparent in what follows that the behaviour of the eigenfunction near $x = 0$ strongly depends on the nature of the solution $\bar{u}$. More precisely, we obtain

Theorem 7 *Let Ω be a convex bounded smooth domain, and assume that for every sequence $\{\lambda_n\}$ tending to zero, the limit function $\bar{u}$ given by Lemma 5 verifies $\bar{u} > 0$ in $\mathbb{R}^N \setminus \{0\}$. Then the principal eigenfunction ϕ has a unique critical point at 0, and*

$$\liminf_{x\to 0} \frac{\phi(0)-\phi(x)}{|x|^{p'}} > 0. \tag{10}$$

Moreover, there exist positive constants c and C such that

$$c \le \frac{|\nabla \phi|^{p-1}}{|x|} \le C, \ \ x \in \Omega. \tag{11}$$

Proof. Assume on the contrary that there exists a sequence $\{x_n\}$ in Ω such that $x_n \to 0$ and

$$\lim_{n\to+\infty} \frac{\phi(0)-\phi(x_n)}{|x_n|^{p'}} = 0. \tag{12}$$

Let $\lambda_n = |x_n|$, $y_n = x_n/|x_n|$. Passing to a subsequence, we may assume $y_n \to y_0$, $|y_0| = 1$ and $u_{\lambda_n} \to \bar{u}$ in $C^1_{\text{loc}}(\mathbb{R}^N)$. Then (12) implies $u_{\lambda_n}(y_n) \to 0$, contradicting the fact that $u_{\lambda_n}(y_n) \to \bar{u}(y_0) > 0$. Estimate (11) is proved in the same way. Finally, (10) implies that ϕ has a unique critical point at zero. Indeed, if $x_0 \neq 0$ is another critical point, Lemma 4 implies that $\phi = \phi(0)$ in $[0, x_0]$. This contradicts (10) and the theorem gets proved. □

In view of the inferior limit in (10), a natural question to ask is whether the limit

$$\lim_{x\to 0} \frac{\phi(0)-\phi(x)}{|x|^{p'}} \tag{13}$$

exists or not. The only possibility for this to happen is that the function $\bar{u}$ is radially symmetric.

Theorem 8 *Let ϕ be the principal eigenfunction in a bounded smooth domain Ω. Then the limit (13) exists if and only if the limit function $\bar{u}$ is radially symmetric for every sequence $\lambda_n \to 0$. In that case,*

$$\lim_{x\to 0} \frac{\phi(0)-\phi(x)}{|x|^{p'}} = \frac{1}{p'}\phi(0)\left(\frac{\lambda_1}{N}\right)^{\frac{1}{p-1}},$$

and

$$\lim_{x\to 0} \frac{|\nabla \phi(x)|^{p-1}}{|x|} = \frac{\lambda_1}{N}\phi(0). \tag{14}$$

Proof. Assume that the limit (13) exists. Fix a unit vector e and $r > 0$, and choose an arbitrary sequence $\lambda_n \to 0$. Passing to a subsequence, $u_{\lambda_n} \to \bar{u}$ in $C^1_{\rm loc}(\mathbb{R}^N)$. Then

$$\frac{u_{\lambda_n}(re)}{r^{p'}} = \frac{\phi(0) - \phi(\lambda_n re)}{(\lambda_n r)^{p'}} \to \frac{\bar{u}(re)}{r^{p'}},$$

and so $\bar{u}(re) = \ell r^{p'}$, where ℓ stands for the value of the limit in (13). This implies that $\bar{u}$ is radially symmetric. Since $\bar{u}$ solves (8), it follows that

$$\ell = \frac{1}{p'}\phi(0)\left(\frac{\lambda_1}{N}\right)^{\frac{1}{p-1}},$$

and the convergence of u_{λ_n} in $C^1_{\rm loc}(\mathbb{R}^N)$ also proves (14).

Conversely, assume $\bar{u}$ is radially symmetric for every sequence $\lambda_n \to 0$. Choose an arbitrary sequence $x_n \to 0$, and set as in Theorem 7 $\lambda_n = |x_n|$, $y_n = x_n/|x_n|$. With no loss of generality, $y_n \to y_0$ with $|y_0| = 1$, and $u_{\lambda_n} \to \bar{u}$ in $C^1_{\rm loc}(\mathbb{R}^N)$. Then

$$u_{\lambda_n}(y_n) = \frac{\phi(0) - \phi(x_n)}{|x_n|^{p'}} \to \bar{u}(y_0) = \frac{1}{p'}\phi(0)\left(\frac{\lambda_1}{N}\right)^{\frac{1}{p-1}}.$$

Since $\{x_n\}$ was arbitrary, it follows that the limit exists. A similar reasoning also shows (14), and the proof is complete. □

3. PROOF OF THEOREMS 1, 2 AND 3

In this final section we are showing how to use the results in §2 to prove our theorems. Our first step concerns the limit problem (9) in the semilinear case $p = 2$. It turns out that the only interesting solutions to us are quadratic forms.

Lemma 9 *Let $\bar{u} \in C^2(\mathbb{R}^N)$ be a nonnegative convex solution to the problem*

$$\Delta u = \gamma \quad in \quad \mathbb{R}^N,$$

where $\gamma > 0$, verifying $\bar{u}(0) = 0$. Then there exists a symmetric, positive semidefinite $N \times N$ matrix A such that

$$\bar{u}(x) = x^T A x, \quad x \in \mathbb{R}^N . \tag{15}$$

Moreover, trace A $= \gamma/2$.

Proof. Since $\bar{u}$ is convex, it follows that $\partial_{11}u, \partial_{22}u \geq 0$, $(\partial_{12}u)^2 \leq \partial_{11}u\partial_{22}u$, where $\partial_{ij}u$ stands for the second derivative of u with respect to x_i and x_j. But all these second derivatives are harmonic functions, so this implies they are constant. The same argument carries over to all the second derivatives. Thus, $\bar{u}$ is a polynomial of degree two, which can be written in the form

$$\bar{u}(x) = x^T A x + Bx + C ,$$

where A is a symmetric $N \times N$ matrix, B is a vector and C a constant. The conditions $\bar{u}(0) = 0$ and $\bar{u} \geq 0$ lead to $B = C = 0$, and A positive semidefinite. Finally, $\Delta\bar{u} = 2\,\text{trace}\,A$, and the lemma is proved. □

Remark 4 Notice that this proof is essentially linear, and there is no way to generalize it to the degenerate cases $p \neq 2$. Indeed, in view of Lemma 9 one could expect solutions of the form

$$\bar{u}(x) = C \left(\sum_{i=1}^{N} \alpha_i x_i^2 \right)^{p'/2} , \tag{16}$$

for some $C > 0$, $0 \leq \alpha_k \leq 1$. But if $p \neq 2$, these are solutions only if $\alpha_k = 0$ or 1 for every k. In particular, the only solution given by (16) which is strictly positive in $\mathbb{R}^N \setminus \{0\}$ is the radial one ($\alpha_k = 1$ for every k). This is in strong contrast with Lemma 9, since it is only needed there that A is positive definite for $\bar{u}$ to be strictly positive.

Let us finally proceed to the proof of theorems 1, 2 and 3.

Proof of Theorem 1. Let us first show that $x = 0$ is the only critical point of ϕ. Let R denote the matrix associated to one of the rotations in the symmetry condition on Ω. It follows that $\phi(Rx)$ is again an eigenfunction to λ_1, and its simplicity implies $\phi(Rx) = \phi(x)$, $x \in \Omega$.

Let $x_0 \in \Omega \setminus \{0\}$ be a critical point of ϕ. The symmetry of ϕ with respect to the rotations R_i, $i = 1, 2, \ldots, N-1$, implies that $\phi(R_i x_0) = \phi(R_i^2 x_0) = \phi(x_0)$, and Lemma 4 (together with the log–convexity of ϕ) gives that $\phi = \phi(x_0)$ in the convex hull of the points x_0, $R_i x_0$, $R_i^2 x_0$, $i = 1, \ldots, N-1$. Since every rotation is about a different plane, and $R_i^2 \neq I$, it follows that the points x_0, $R_i x_0$, $R_i^2 x_0$, $i = 1, \ldots, N-1$ do not belong to the same hyperplane, and thus $\nabla\phi = 0$ in a set with nonempty interior, contradicting the equation in (E). Thus $\nabla\phi \neq 0$ in $\Omega \setminus \{0\}$.

To prove (2) and (3), we only have to show that the function $\bar{u}$ given by Lemma 5 verifies $\bar{u} > 0$ in $\mathbb{R}^N \setminus \{0\}$ (cf. Theorem 7). Assume on the contrary that $\bar{u}(x_0) = 0$ for some $x_0 \neq 0$. Notice that the invariance of ϕ with respect to the rotations R_i carries over to $\bar{u}$. The same argument as above (together with the convexity given by Lemma 6) gives that $\bar{u} = 0$ in a set with nonempty interior, and this contradicts (8). Thus, $\bar{u} > 0$ in $\mathbb{R}^N \setminus \{0\}$, and the proof is finished. □

Proof of Theorem 2. In virtue of Theorem 8, we need to prove that $\bar{u}$ is radially symmetric for every sequence $\{\lambda_n\}$ converging to zero.

By the symmetry of $\bar{u}$ with respect to the rotations R_i, $i = 1, 2, \ldots, N-1$, and Lemma 9, we have $(R_i x)^T A (R_i x) = x^T A x$ for $x \in \mathbb{R}^N$, and since $R_i^T A R_i$ is also symmetric, we have $R_i^T A R_i = A$.

With no loss of generality we can assume that the planes T_k appearing in the symmetry condition on Ω are the coordinate planes $x_1 x_2$, $x_2 x_3, \ldots, x_{N-1} x_N$. In this case, the matrix R_1 for the first rotation has the special form

$$R_1 = \begin{pmatrix} T & O & \ldots & O \\ O & O & \ldots & O \\ \ldots & \ldots & \ldots & \ldots \\ O & O & \ldots & O \end{pmatrix} ,$$

where O stands for a two–dimensional zero–matrix, and

$$T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

(strictly speaking, this is only valid for even N, but the case of odd N can be covered with a slight change of notation). The condition $R_1^T A R_1 = A$, together with $\sin\theta \neq 0$, which is a consequence of $\theta \neq 0, \pi$, leads directly to

$$a_{1i} = 0,\ a_{2j} = 0,\ i \neq 1, j \neq 2, \qquad a_{11} = a_{22}\ .$$

Using the same argument with the rest of the matrices R_i, $i = 2, \ldots, N-1$, we obtain that A is a matrix of the form $A = aI$, where $a = \lambda_1\phi(0)/2N$, and thus $\bar{u}(x) = \lambda_1\phi(0)/2N|x|^2$.

To prove the assertion about the second derivatives, observe that a bootstrap argument allows us to improve the convergence in Lemma 5 to obtain $u_{\lambda_n} \to \bar{u}$ in $C^2_{\text{loc}}(\mathbb{R}^N)$. Then $\partial_{ij}\phi(0) = -\partial_{ij}u_\lambda(0) \to -\partial_{ij}\bar{u}(0) = -\delta_{ij}\lambda_1\phi(0)/N$. This concludes the proof. □

Proof of Theorem 3. The arguments in the proof of Theorem 3 are very much the same as in the previous ones. Define $u_\lambda(x)$ according to (7). Then

$$\Delta_p u_\lambda(x) = f(u(\lambda x)),\ x \in \Omega_\lambda,$$

where $\Omega_\lambda = \{x \in \mathbb{R}^N : \ \lambda x \in \Omega\}$. The same procedure as in Section 3 gives, for every sequence $\lambda_n \to 0$, a subsequence such that

$$u_{\lambda_n} \to \bar{u} \text{ in } C^1_{\text{loc}}(\mathbb{R}^N)\ ,$$

where $\bar{u}$ is a solution to $\Delta_p u = f(u(0))$, with $f(u(0)) > 0$. If we show that $\bar{u}$ is convex, then the rest of the proof will be exactly as in Theorems 1 or 2.

With no loss of generality, assume that u is ψ–concave and $\psi'(u(0)) > 0$. Then, for $\lambda > 0$, $x, y \in \Omega_\lambda$ and $t \in [0,1]$,

$$\psi(u(t\lambda x + (1-t)\lambda y)) \geq t\psi(u(\lambda x)) + (1-t)\psi(u(\lambda y))\ .$$

We note that $u(\lambda z) = u(0) - \lambda^{p'}u_\lambda(z)$ for every $z \in \Omega_\lambda$, so an application of the mean value theorem leads to

$$\psi'(\xi)u_\lambda(tx + (1-t)y) \leq \psi'(\eta)tu_\lambda(x) + \psi'(\chi)(1-t)u_\lambda(y)\ ,$$

where $\xi, \eta, \chi \to u(0)$ as $\lambda \to 0+$. Setting $\lambda = \lambda_n$, passing to the corresponding subsequence and then letting $n \to +\infty$, we arrive at $\bar{u}(tx + (1-t)y) \leq t\bar{u}(x) + (1-t)\bar{u}(y)$ (in this point we have to use $\psi'(u(0)) > 0$), proving the convexity of $\bar{u}$. □

REFERENCES

[1] ALESSANDRINI G., MAGNANINI R., *Elliptic equations in divergence form, geometric critical points of solutions and Stekloff eigenfunctions*, SIAM J. Math. Anal. **25** (5) (1994), 1259-1268.

[2] ANANE A., *Simplicité et isolation de la première valeur propre du p-Laplacien avec poids*, Comptes Rendus Acad. Sci. Paris Ser. I Math, **305** (1987), 725-728.

[3] Aronsson G., *Representation of a p-harmonic function near a critical point in the plane*, Manuscripta Math. **66** (1) (1989), 73-95.

[4] García-Melián J., *A local bifurcation theorem for the p-Laplacian in symmetric domains*, preprint.

[5] García-Melián J., Sabina de Lis J., *Uniqueness to quasilinear problems for the p-Laplacian in radially symmetric domains*, Nonlinear Anal. **43** (7) (2001), 803-835.

[6] García-Melián J., Sabina de Lis J., *A local bifurcation theorem for degenerate equations with radial symmetry,* to appear in J. Differential Equations.

[7] Jakobson D., Nadirashvili N., *Eigenfunctions with few critical points*, J. Differential Geom. **53** (1) (1999), 177-182.

[8] Lewis J., *On critical points of p-harmonic functions in the plane*, Electronic J. Differential Eqns. **3** (1994), 1-4.

[9] Lindqvist P., *On the equation* $\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2}u = 0$, Proc. Amer. Math. Soc. **109**, 1 (1990), 157-164.

[10] Manfredi J., *p-harmonic functions in the plane*, Proc. Amer. Math. Soc. **103**, 2 (1988), 473-479.

[11] Sakaguchi S., *Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet problems*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (IV) **14** (1987), 403-421.

[12] Serrin J., *Local behaviour of solutions of quasilinear elliptic equations*, Acta Math. **111** (1964), 247-302.

[13] Tolksdorf P., *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations **51** (1984), 126-150.