LIMIT CASES IN AN ELLIPTIC PROBLEM WITH A PARAMETER-DEPENDENT BOUNDARY CONDITION

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ABSTRACT. In this work we discuss existence, uniqueness and asymptotic profiles of positive solutions to the quasilinear problem

$$\begin{cases} -\Delta_p u + a(x)u^{p-1} = -u^r & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda u^{p-1} & \text{on } \partial\Omega, \end{cases}$$

for $\lambda \in \mathbb{R}$, where $r > p-1 > 0$, $a \in L^\infty(\Omega)$. We analyze the existence of solutions in terms of a principal eigenvalue, and determine their asymptotic behavior both when $r \to p-1$ and when $r \to \infty$.

1. INTRODUCTION

The aim of the present paper is to analyze some qualitative features exhibited by the positive solutions to

$$\begin{cases} -\Delta_p u(x) + a(x)u^{p-1}(x) = -u^r(x) & x \in \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu}(x) = \lambda u^{p-1}(x) & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda \in \mathbb{R}$, $r > p-1 > 0$, Ω is a class $C^{2,\alpha}$ bounded smooth domain of $\mathbb{R}^N$ $(N \geq 2)$, $0 < \alpha \leq 1$, and ν stands for the outward unit normal field on $\partial\Omega$. The operator Δ_p is the standard p-Laplacian, which is defined in the usual weak sense of $W^{1,p}(\Omega)$ as $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$. In addition, it will be assumed throughout that $a \in L^\infty(\Omega)$. The main feature of problem (1.1) is its dependence on the parameter λ precisely in the boundary condition.

Problem (1.1) was studied in [4] when $p = 2$ (in this case Δ_p is the usual Laplacian) with fixed $r > 1$ and $a = 0$. Under these conditions, it was shown there that this problem admits a unique positive solution $u_{r,\lambda}$ for every $\lambda > 0$, and no positive solutions when $\lambda \leq 0$. It was further shown that $u_{r,\lambda}$ is continuous and increasing as a function of λ, and its asymptotic behavior when $\lambda \to 0$ and $\lambda \to \infty$ was also completely elucidated (see [4] for additional features). However, as far as we know, the dependence of $u_{r,\lambda}$ on r has not yet been clarified. Thus, one of the objectives of this work is to analyze the variation of $u_{r,\lambda}$ with respect to r, especially in the extreme cases where $r \to 1+$ or $r \to \infty$. This study will be indeed extended to cover the more general problem (1.1).

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To deal with the quasilinear problem (1.1), a number of auxiliary results must be developed. In particular, a study of the flux-type eigenvalue problem

$$\begin{cases} -\Delta_p u(x) + a(x)|u|^{p-2}u(x) = \mu|u|^{p-2}u(x), & x \in \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu}(x) = \lambda|u|^{p-2}u(x), & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where λ is regarded as a parameter and it is assumed that $a \in L^\infty(\Omega)$. A number $\mu \in \mathbb{R}$ is said to be an eigenvalue to (1.2) if there exists $\phi \in W^{1,p}(\Omega)$, not vanishing identically in Ω, so that

$$\int_\Omega (|\nabla\phi|^{p-2}\nabla\phi\nabla\varphi + a(x)|\phi|^{p-2}\phi\varphi)\ dx = \lambda\int_{\partial\Omega}|\phi|^{p-2}\phi\varphi\ dS + \mu\int_\Omega |\phi|^{p-2}\phi\varphi\ dx,$$

for all $\varphi \in W^{1,p}(\Omega)$. In that case, ϕ is called an eigenfunction associated to μ.

Problem (1.2) has been studied in detail in [5] when $p = 2$, in which case it becomes

$$\begin{cases} -\Delta u(x) + a(x)u(x) = \mu u(x), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu}(x) = \lambda u(x), & x \in \partial\Omega. \end{cases} \tag{1.3}$$

The next statement is the extension to problem (1.2) of the corresponding results obtained for (1.3) contained in [5] (a slightly more general version of (1.3) was in fact considered there).

Theorem 1. *Problem* (1.2) *admits, for every $\lambda \in \mathbb{R}$, a unique* principal eigenvalue $\mu = \mu_{1,p}$, *i.e. an eigenvalue with a nonnegative associated eigenfunction $\phi \in W^{1,p}(\Omega)$. It is given by the variational expression*

$$\mu_{1,p} = \inf_{u\in W^{1,p}(\Omega),\ u\neq 0} \frac{\displaystyle\int_\Omega (|\nabla u|^p + a|u|^p)\ dx - \lambda\int_{\partial\Omega}|u|^p\ dS}{\displaystyle\int_\Omega |u|^p\ dx}.$$

In addition, the following properties hold true.

i) *$\mu_{1,p}$ is the unique principal eigenvalue.*

ii) *$\mu_{1,p}$ is isolated and simple.*

iii) *Every associated eigenfunction $\phi_1 \in W^{1,p}(\Omega)$ to $\mu_{1,p}$ satisfies $\phi \in L^\infty(\Omega)$ and furthermore $\phi \in C^{1,\beta}(\overline{\Omega}) \cap C^{2,\alpha}(U_\eta)$ for certain $\beta \in (0,1)$, $\eta > 0$, with $U_\eta = \{x \in \Omega : dist(x, \partial\Omega) < \eta\}$.*

iv) *As a function of λ, $\mu_{1,p}$ is concave, decreasing and verifies*

$$\lim_{\lambda\to-\infty}\mu_{1,p} = \lambda_{1,p}(a), \qquad \lim_{\lambda\to\infty}\mu_{1,p} = -\infty,$$

where $\lambda_{1,p}(a)$ is the first Dirichlet eigenvalue of $-\Delta_p u + a(x)|u|^{p-2}u$ in Ω.

Another auxiliary eigenvalue problem we will need is

$$\begin{cases} -\Delta_p u(x) + a(x)|u|^{p-2}u(x) = 0, & x \in \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu}(x) = \sigma|u|^{p-2}u(x), & x \in \partial\Omega, \end{cases} \tag{1.4}$$

which constitutes an extension to the p-Laplacian setting of the well-known Steklov problem (see [10] for a detailed analysis of the case $a = 0$). As a direct consequence of Theorem 1 the following statement holds true.

Theorem 2. *Problem* (1.4) *possesses a principal eigenvalue if and only if*

$$\lambda_{1,p}(a) > 0. \tag{1.5}$$

Furthermore,

i) *Provided that* (1.5) *is satisfied,* (1.4) *admits a unique principal eigenvalue* $\sigma_{1,p}$ *which is isolated and simple. In addition,*

$$\text{sign } \sigma_{1,p} = \text{sign } \lambda_{1,p}^{\mathcal{N}}(a) \tag{1.6}$$

where $\lambda_{1,p}^{\mathcal{N}}(a)$ *stands for the first Neumann eigenvalue of* $-\Delta_p u + a(x)|u|^{p-2}u$ *in* Ω.

ii) *Any eigenfunction* $\psi \in W^{1,p}(\Omega)$ *associated to* $\sigma_{1,p}$ *satisfies* $\psi \in C^{1,\beta}(\overline{\Omega}) \cap C^{2,\alpha}(U_\eta)$ *for certain* $\beta \in (0,1)$, $\eta > 0$, *with* $U_\eta = \{x : dist(x \in \Omega, \partial\Omega) < \eta\}$.

Remark 1. We will set $\sigma_{1,p} = -\infty$ when $\lambda_{1,p}(a) \le 0$, for reasons that will become clear later on (see (1.8) in Theorem 4 and Remark 3).

The well-known sub and supersolutions method is another tool that must be properly adapted to problem (1.1). A function $\overline{u} \in W^{1,p}(\Omega)$ is said to be a supersolution to problem

$$\begin{cases} -\Delta_p u(x) + a(x)|u|^{p-2}u(x) = f(x,u), & x \in \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu}(x) = g(x,u), & x \in \partial\Omega, \end{cases} \tag{1.7}$$

if

$$\int_\Omega (|\nabla \overline{u}|^{p-2}\nabla\overline{u}\nabla\varphi + a(x)|\overline{u}|^{p-2}\overline{u}\varphi)\, dx \ge \int_{\partial\Omega} g(x,\overline{u})\varphi\, dS + \int_\Omega f(x,\overline{u})\varphi\, dx,$$

holds for all nonnegative $\varphi \in W^{1,p}(\Omega)$. Subsolutions are defined in a symmetric way. Of course, the existence of the integrals involving f and g is implicitly assumed.

In order to avoid the use of comparison, which is certainly a delicate issue when dealing with the p-Laplacian, the next statement furnishes a variational version of the method of sub a supersolutions for problem (1.7) (cf. also [14]). Recall that a function $h : X \times \mathbb{R} \to \mathbb{R}$, (X,μ) a measure space, is a Carathéodory function if $h(\cdot,u)$ is measurable in X for all $u \in \mathbb{R}$ while $h(x,\cdot)$ is continuous in $\mathbb{R}$ for almost all $x \in X$.

Theorem 3. *Let* $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ *be Carathéodory functions satisfying* $|f(x,u)| \le M$ *and* $|g(x,u)| \le M$ *if* $(x,u) \in \Omega \times (-R,R)$ *and* $(x,u) \in \partial\Omega \times (-R,R)$, *respectively, for arbitrary* R, *where* $M = M(R)$. *Suppose* $\underline{u}$, $\overline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \cap L^\infty(\partial\Omega)$ *are a sub and a supersolution to* (1.7) *so that* $\underline{u} \le \overline{u}$ *a. e. in* Ω. *Then* (1.7) *admits a solution* $u \in W^{1,p}(\Omega)$ *verifying*

$$\underline{u} \le u \le \overline{u},$$

a. e. in Ω.

After these preliminary tools have been introduced, we can state a first group of results concerning problem (1.1).

Theorem 4. *Assume $\Omega \subset \mathbb{R}^N$ is a class $C^{2,\alpha}$ bounded domain and $r > p-1 > 0$. Then the following properties hold:*

i) *Problem* (1.1) *admits a positive solution if and only if*

$$\lambda > \sigma_{1,p}, \tag{1.8}$$

where the value $\sigma_{1,p} = -\infty$ is allowed. When (1.8) *holds, the positive solution is unique, and it will be denoted by $u_{r,\lambda} \in W^{1,p}(\Omega)$.*

ii) *The function $u_{r,\lambda}$ belongs to $C^{1,\beta}(\overline{\Omega}) \cap C^{2,\alpha}(U_\eta)$ for a certain $\beta \in (0,1)$ and $\eta > 0$ small enough, where $U_\eta = \{x \in \Omega : dist(x, \partial\Omega) < \eta\}$.*

iii) *The mapping $\lambda \to u_{r,\lambda}$ is increasing and continuous with values in $C^1(\overline{\Omega})$. Moreover,*

$$\lim_{\lambda \to \sigma_{1,p}+} u_{r,\lambda} = 0 \tag{1.9}$$

in $C^{1,\beta}(\overline{\Omega})$ provided $\sigma_{1,p} > -\infty$. If $\sigma_{1,p} = -\infty$ then

$$\lim_{\lambda \to \sigma_{1,p}+} u_{r,\lambda} = \begin{cases} 0 & \text{if} \quad \lambda_{1,p}(a) = 0 \\ w & \text{if} \quad \lambda_{1,p}(a) < 0, \end{cases} \tag{1.10}$$

where $u = w(x)$ stands for the unique positive solution to

$$\begin{cases} -\Delta_p u(x) + a(x)|u|^{p-2}u(x) = -u^r(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1.11}$$

when $\lambda_{1,p}(a) < 0$.

iv) *Let $u = U(x)$ be the minimal solution to the singular boundary value problem*

$$\begin{cases} -\Delta_p u(x) + a(x)|u|^{p-2}u(x) = -u^r(x), & x \in \Omega, \\ u = \infty & x \in \partial\Omega. \end{cases} \tag{1.12}$$

Then,

$$\lim_{\lambda \to \infty} u_{r,\lambda} = U, \tag{1.13}$$

in $C^1(\Omega)$.

We turn now to study the asymptotic behavior of the positive solution $u_{r,\lambda}$ to (1.1) both as $r \to (p-1)+$ and when $r \to \infty$. Let us begin with the former case and to this purpose notice that Theorem 1-iv) implies the existence of a value λ^* such that

$$\mu_{1,p}(\lambda^*) = -1,$$

provided that $\lambda_{1,p}(a) > -1$ ($\lambda_{1,p}(a)$ being the principal Dirichlet eigenvalue of $-\Delta_p u + a(x)|u|^{p-2}u$ in Ω). Observe that

$$\sigma_{1,p} < \lambda^*,$$

even in the case when $\sigma_{1,p} = -\infty$, while

$$0 < -\mu_{1,p}(\lambda) < 1 \qquad \text{for} \qquad \sigma_{1,p} < \lambda < \lambda^*.$$

Of course, the inequality holds for all $\lambda < \lambda^*$ if $\sigma_{1,p} = -\infty$. Similarly,

$$-\mu_{1,p}(\lambda) > 1 \qquad \text{if} \qquad \lambda > \lambda^*.$$

On the other hand,

$$-\mu_{1,p}(\lambda) > 1 \qquad \text{for all} \qquad \lambda$$

in the complementary case $\lambda_{1,p}(a) \le -1$ where the value λ^* does not exist.

Then we have:

Theorem 5. *For $\lambda > \sigma_{1,p} \ge -\infty$, let $u = u_{r,\lambda}$ be the unique positive solution to problem* (1.1) *for $r > p-1$. Then,*

$$(\sup_\Omega u_{r,\lambda})^{r-p+1} = -\mu_{1,p}(\lambda) + o(1)$$

as $r \to p-1+$ while

$$u_{r,\lambda} = (\sup_\Omega u_{r,\lambda})\{\phi_1(\lambda) + o(1)\}$$

in $C^1(\overline{\Omega})$ as $r \to p-1+$, where $\phi_1(\lambda)$ stands for the positive eigenfunction associated to $\mu_{1,p}(\lambda)$ normalized so as $\sup_\Omega \phi_1(\lambda) = 1$. In particular

a) *$u_{r,\lambda} \to 0$ uniformly in $\overline{\Omega}$ as $r \to (p-1)+$ if $\lambda < \lambda^*$ provided that $\lambda_{1,p}(a) > -1$.*

   Moreover, for $\lambda = \lambda^*$ and $p = 2$ in problem (1.1) *then*

$$u_{r,\lambda} \to A\ \phi_1(\lambda^*)$$

   uniformly in Ω as $r \to p-1+$ where A is given by

$$A = \exp\left(-\frac{\displaystyle\int_\Omega \phi_1^2 \log \phi_1\ dx}{\displaystyle\int_\Omega \phi_1^2\ dx} \right). \tag{1.14}$$

b) *$u_{r,\lambda} \to \infty$ uniformly in $\overline{\Omega}$ as $r \to (p-1)+$ either when $\lambda > \lambda^*$ if $\lambda_{1,p}(a) > -1$ or either for all $\lambda \in \mathbb{R}$ provided $\lambda_{1,p}(a) \le -1$.*

Note that in the previous theorem the case $\lambda = \lambda^*$ with $p \ne 2$ is left open.

As for the behavior of the solution $u_{r,\lambda}$ to (1.1) when $r \to \infty$ the first interesting conclusion is that for every $\lambda > \sigma_{1,p}$, $u_{r,\lambda}$ keeps uniformly bounded in Ω as $r \to \infty$. On the other hand, provided that coefficient $a = 0$ in (1.1) we achieve a better result. Namely, solutions become flat throughout the domain Ω as r increases.

Theorem 6. *Assume that $a = 0$ in problem* (1.1)*. Then, for any $\lambda > \sigma_{1,p}$ we have $u_{r,\lambda} \to 1$ uniformly in $\overline{\Omega}$ as $r \to \infty$.*

It should be mentioned that a similar analysis for the logistic problem

$$\begin{cases} -\Delta u(x) = \lambda u(x) - b(x) u^r(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.15}$$

which is somehow related to (1.1), was performed in [3], [2]. However, the situation was substantially different there when $r \to \infty$, since the limit problem so obtained is of a free boundary type, mainly due to the Dirichlet condition. On the other hand, if $u = \tilde{u}_{r,\lambda}$ stands for the unique positive solution to (1.15) for $\lambda > \lambda_1^{\mathcal{D}}$ (the first Dirichlet eigenvalue of $-\Delta$ in Ω), an important feature in the analysis in [3] is the fact that

$$(\sup_\Omega \tilde{u}_{r,\lambda})^{r-1}$$

remains bounded as $r \to \infty$. This follows easily from the boundary condition when $b > 0$ in $\overline{\Omega}$. This fact is in strong contrast with the next result.

Theorem 7. *Let $a \in L^\infty(\Omega)$. Then, for fixed $\lambda > \sigma_{1,p}$*

$$\phi_1(\lambda) \le \varliminf_{r\to\infty} u_{r,\lambda} \le \varlimsup_{r\to\infty} u_{r,\lambda} \le 1,$$

where $\phi_1(\lambda)$ is the positive eigenfunction associated to $\mu_{1,p}(\lambda)$ normalized so that $\sup \phi_1(\lambda) = 1$. *In particular,*

$$\lim_{r\to\infty} \sup_\Omega u_{r,\lambda} = 1,$$

However, if either $a = 0$ or $a \in L^\infty(\Omega)$ is arbitrary but $\lambda > \sigma_1(|a|_\infty)$ in (1.1) *then*

$$\lim_{r\to\infty} \sup_\Omega (u_{r,\lambda})^{r-p+1} = \infty.$$

The rest of the paper is organized as follows: in Section 2 we analyze the eigenvalue problems (1.2) and (1.4). Section 3 is dedicated to develop the method of sub and supersolutions for problem (1.7), that will be used here for the proof of Theorem 4. Finally, in Section 4 the asymptotic behavior of the positive solution to (1.1) as $r \to p-1$ and $r \to +\infty$ is considered.

2. Eigenvalue problems

In this section we perform the analysis of the eigenvalue problems (1.2) and (1.4). We begin with a fundamental result concerning the boundedness of eigenfunctions.

Lemma 8. *Let $\phi \in W^{1,p}(\Omega)$ be an eigenfunction associated to an arbitrary eigenvalue μ of* (1.2). *Then $\phi \in L^\infty(\Omega)$.*

Proof. Notice that we may assume $1 < p \le N$, since otherwise $W^{1,p}(\Omega) \subset L^\infty(\Omega)$. Also, for the sake of simplicity we will only consider $p < N$, the case $p = N$ being handled in a similar way.

For $k > 0$ set $v = (\phi - k)^+$, $A_k = \{x \in \Omega :\ \phi(x) > k\}$. We show an estimate of the form

$$|v|_1 \le C k^\delta |A_k|^{1+\varepsilon}, \tag{2.1}$$

for every $k \ge k_0$ and certain positive constants k_0, C, δ, ε with $\delta \le 1 + \varepsilon$, where $|v|_1 = |v|_{L^1(\Omega)}$.

By using v as a test function in the equation for ϕ we obtain

$$\begin{aligned} \int_\Omega (|\nabla v|^p + \varphi_p(\phi) v)\ dx &\le \lambda \int_{\partial\Omega} \varphi_p(\phi) v\ dS + (\mu + |a|_\infty + 1) \int_\Omega \varphi_p(\phi) v\ dx \\ &\le C \left\{ \int_{\partial\Omega} \varphi_p(\phi) v\ dS + \int_\Omega \varphi_p(\phi) v\ dx \right\}, \end{aligned} \tag{2.2}$$

where $\varphi_p(\phi) = |\phi|^{p-2}\phi$ and C will stand in the sequel for a positive constant independent of ϕ and k, not necessarily the same everywhere.

Next notice that $0 < v < \phi$ in the support of v and $\phi \le v + k$, hence $\varphi_p(\phi) \le C(v^{p-1} + k^{p-1})$. Thus (2.2) implies

$$|v|_{1,p}^p \le C \left\{ \int_{\partial\Omega} v^p\ dS + k^{p-1} \int_{\partial\Omega} v\ dS + \int_\Omega v^p\ dx + k^{p-1} \int_\Omega v\ dx \right\}, \tag{2.3}$$

for all $k > 0$, where $|v|_{1,p} = |v|_{W^{1,p}(\Omega)}$.

On the other hand, we notice that, thanks to Hölder's and Sobolev's inequalities:

$$\int_\Omega v^p\, dx \le |A_k|^{\frac{p}{N}} \left(\int_\Omega v^{p^*}\, dx\right)^{\frac{p}{p^*}} \le C|A_k|^{\frac{p}{N}} \left(\int_\Omega |\nabla v|^p\, dx + \int_\Omega v^p\, dx\right)$$

where $p^* = \frac{Np}{N-p}$, and, since $|A_k| \to 0$,

$$\int_\Omega v^p\, dx \le C|A_k|^{\frac{p}{N}} \int_\Omega |\nabla v|^p\, dx, \tag{2.4}$$

for $k \ge k_0$ and certain positive k_0.

Furthermore, it is useful to recall that for every $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that

$$\int_{\partial\Omega} |u|^p\, dS \le \varepsilon \int_\Omega |\nabla u|^p\, dx + C(\varepsilon) \int_\Omega |u|^p\, dx, \tag{2.5}$$

for every $u \in W^{1,p}(\Omega)$ (see for instance Lemma 6 in [5] for a proof when $p = 2$). This inequality combined with (2.4) yields

$$\int_{\partial\Omega} v^p\, dS \le (\varepsilon + C(\varepsilon)|A_k|^{\frac{p}{N}}) \int_\Omega |\nabla v|^p\, dx, \tag{2.6}$$

for $k \ge k_0$. Inequalities (2.3), (2.4) and (2.6) imply, taking ε sufficiently small,

$$|v|_{1,p}^p \le Ck^{p-1}\{|v|_{1,\partial\Omega} + |v|_1\}, \tag{2.7}$$

for $k \ge k_0$, where $|v|_{1,\partial\Omega} = |v|_{L^1(\partial\Omega)}$.

Observe now that, thanks to the immersion $W^{1,1}(\Omega) \subset L^1(\partial\Omega)$ and Hölder's inequality

$$|v|_{1,\partial\Omega} \le C|v|_{W^{1,1}(\Omega)} \le C|A_k|^{1-\frac{1}{p}}|v|_{1,p},$$

while the Sobolev immersion gives

$$|v|_1 \le C|A_k|^{1-\frac{1}{p^*}}|v|_{1,p}. \tag{2.8}$$

Thus, from (2.7) we get

$$|v|_{1,p} \le Ck\{|A_k|^{\frac{1}{p}} + |A_k|^{\frac{1}{p-1}(1-\frac{1}{p^*})}\} \le Ck|A_k|^{\frac{1}{p}}$$

for all $k \ge k_0$, since $\frac{1}{p} < \frac{1}{p-1}(1 - \frac{1}{p^*})$ and $|A_k| \to 0$. This inequality allows us to conclude, thanks to (2.8), that

$$|v|_1 \le Ck|A_k|^{1+\frac{1}{N}}, \tag{2.9}$$

for large k, which is the desired inequality.

Finally, when (2.9) is combined with Lemma 5.1 in Chapter 2 in [9] we obtain $\phi^+ \in L^\infty(\Omega)$, and since $-\phi$ is also an eigenfunction, the preceding argument also says that $\phi \in L^\infty(\Omega)$. ☐

Remark 2. Lemma 8 can be also shown by means of a Moser's iteration procedure following the ideas in [5] (see Lemma 5 there).

Proof of Theorem 1. To show the existence of a principal eigenvalue we borrow ideas from Lemma 7 in [5]. Thus, consider $\mathcal{M} := \{u \in W^{1,p}(\Omega) : \int_\Omega |u|^p = 1\}$, and the functional

$$J(u) = \int_\Omega (|\nabla u|^p + a(x)|u|^p)\ dx - \lambda \int_{\partial\Omega} |u|^p\ dS.$$

Inequality (2.5) implies that

$$J(u) \geq (1 - \varepsilon|\lambda|) \int_\Omega |\nabla u|^p\ dx - (|a|_\infty + C(\varepsilon)|\lambda|) \int_\Omega |u|^p\ dx,$$

for all $u \in W^{1,p}(\Omega)$. This means that J is coercive in $\mathcal{M}$ and the direct method in the calculus of variations (see Theorem 1.2 in [14]) implies the finiteness of

$$\mu_{1,p} = \inf_{u\in W^{1,p}(\Omega),\ u\neq 0} \frac{\displaystyle\int_\Omega (|\nabla u|^p + a(x)|u|^p)\ dx - \lambda \int_{\partial\Omega} |u|^p\ dS}{\displaystyle\int_\Omega |u|^p\ dx},$$

and the existence of $\phi \in W^{1,p}(\Omega)$ such that the infimum is achieved at $u = \phi$. Since the infimum is also attained at $|\phi|$, it is easily checked that $|\phi|$ defines an eigenfunction associated to $\mu_{1,p}$, hence $\mu_{1,p}$ is a principal eigenvalue.

Next, let $\phi \in W^{1,p}(\Omega)$ be a nonnegative eigenfunction associated to $\mu_{1,p}$. Lemma 8 and Lieberman's regularity results ([12]) imply that $\phi \in C^{1,\beta}(\overline{\Omega})$ for a certain $0 < \beta < 1$ while the Strong Maximum Principle in [15] implies that $\phi > 0$ throughout $\overline{\Omega}$ together with $|\nabla\phi| > 0$ in some strip $U_\eta = \{x \in \Omega :\ \mathrm{dist}(x, \partial\Omega) < \eta\}$. Then, the equation for ϕ becomes strictly elliptic in U_η and standard theory of quasilinear equations yields $\phi \in C^{2,\alpha}(U_\eta)$ (cf. [9]).

As a consequence of the preceding assertions it follows that every eigenfunction ϕ associated to $\mu_{1,p}$ is either positive or negative in Ω. In fact, if $\phi^+ \neq 0$ then, since ϕ^+ is also an eigenfunction associated to $\mu_{1,p}$, we get $\phi^+ > 0$ in $\overline{\Omega}$. Thus, $\phi^- = 0$ and ϕ is positive.

We show now the simplicity of $\mu_{1,p}$. To this purpose, for two positive eigenfunctions ϕ, ψ associated to $\mu_{1,p}$ consider the integral

$$I := \int_\Omega \left\{ |\nabla\phi|^{p-2}\nabla\phi\nabla\left(\frac{\phi^p - \psi^p}{\phi^{p-1}}\right) - |\nabla\psi|^{p-2}\nabla\psi\nabla\left(\frac{\phi^p - \psi^p}{\psi^{p-1}}\right) \right\} dx.$$

Under the sole assumption that both $\phi, \psi \in W^{1,p}(\Omega)$ are positive and bounded in $\overline{\Omega}$ it follows that $I \geq 0$, and $I = 0$ only when $\psi = c\phi$ for a positive constant c. This is a consequence of the analysis in [13]. For the reader's benefit we sketch the argument. In fact,

$$\begin{aligned} I = &\int_\Omega (\phi^p - \psi^p)(|\nabla\log\phi|^p - |\nabla\log\psi|^p)\ dx - \\ &\int_\Omega p\psi^p|\nabla\log\phi|^{p-2}(\nabla\log\phi)(\nabla\log\psi - \nabla\log\phi)\ dx - \\ &\int_\Omega p\phi^p|\nabla\log\psi|^{p-2}(\nabla\log\psi)(\nabla\log\phi - \nabla\log\psi)\ dx. \end{aligned}$$

Hence, by using the convexity of function $|x|^p$ with $p > 1$ (inequality (4.1) in [13]) we infer that $I \geq 0$, and moreover $I = 0$ only when $\psi = c\phi$ for a positive constant c. Thus the simplicity of $\mu_{1,p}$ is proved.

The same argument implies that $\mu_{1,p}$ is the unique principal eigenvalue. In fact, suppose that ϕ is a positive eigenfunction associated to $\mu_{1,p}$ while $\mu \neq \mu_{1,p}$ is another eigenvalue which possesses a positive eigenfunction ψ. In this case, if we use $(\phi^p - \psi^p)/\phi^{p-1}$ as a test function in the equation for ϕ as an eigenfunction associated to $\mu_{1,p}$ and similarly employ $(\phi^p - \psi^p)/\psi^{p-1}$ in the equation for ψ then, after subtracting the resulting equalities, we arrive to

$$I = (\mu_{1,p} - \mu) \int_\Omega (\phi^p - \psi^p)\ dx \geq 0.$$

However $\mu > \mu_{1,p}$ and ϕ can be chosen greater than ψ in Ω. Since this contradicts the inequality, such an eigenvalue μ cannot exist.

To show the isolation of $\mu_{1,p}$ we follow the spirit of the corresponding statement in [1] (see also [10] for the case of the principal eigenvalue of (1.4) and $a = 1$), which we simplify in view of Lemma 8. Thus, assume on the contrary that there exists a sequence of eigenvalues $\mu_n \neq \mu_{1,p}$ with associated eigenfunction ϕ_n normalized by $\int_\Omega |\phi_n|^p = 1$ for all n, verifying $\mu_n \to \mu_{1,p}$. Notice that $\phi_n^\pm \neq 0$ for all n. Then, from the weak formulation of (1.2), we obtain

$$\int_\Omega (|\nabla\phi_n|^p + a|\phi_n|^p)\ dx - \lambda \int_{\partial\Omega} |\phi_n|^p\ dS = \mu_n.$$

By means of (2.5) we see that $|\phi_n|_{1,p}$ is bounded and so, passing to a subsequence, $\phi_n \rightharpoonup \phi_1$ weakly in $W^{1,p}(\Omega)$. It follows that ϕ_1 is a principal eigenfunction which can be assumed to be positive.

On the other hand, from the weak formulation of the equation satisfied by ϕ_n and by using ϕ_n^- as a test function, arguments similar as the ones employed in Lemma 8 show that

$$|\phi_n^-|_{1,p}^p \leq C \int_\Omega |\phi_n^-|^p\ dx,$$

for a positive constant C, not depending on n. Hence

$$|\{\phi_n < 0\}| \geq k > 0 \tag{2.10}$$

for some $k > 0$ and all n. However, since modulus a subsequence, $\phi_n \to \phi_1$ in $L^p(\Omega)$ and ϕ_1 is positive, Egorov's theorem implies that the uniform estimate (2.10) is not possible. Therefore, $\mu_{1,p}$ is isolated.

Finally, the features and asymptotic behavior of $\mu_{1,p}(\lambda)$ contained in statement iv) can be shown by following the corresponding proof of Lemma 8 in [5]. □

Proof of Theorem 2. By using the terminology of Theorem 1, the key point is that σ is a principal eigenvalue of (1.4) if and only if

$$\mu_{1,p}(\sigma) = 0.$$

In view of property iv) in Theorem 1 it is clear that (1.5) characterizes the existence of a zero of $\mu_{1,p}$ and so it characterizes the existence of a unique principal eigenvalue $\sigma := \sigma_{1,p}$ of (1.4) as well.

In addition

$$\int_\Omega (|\nabla\psi|^p + a|\psi|^p)\ dx - \sigma \int_{\partial\Omega} |\psi|^p\ dS = 0,$$

if σ is a principal eigenvalue. Since $\lambda_{1,p}(a) > 0$ it follows that $\psi \neq 0$ on $\partial\Omega$ and so

$$\sigma_{1,p} = \frac{\displaystyle\int_\Omega (|\nabla\psi|^p + a|\psi|^p)\ dx}{\displaystyle\int_{\partial\Omega} |\psi|^p\ dS} \leq \frac{\displaystyle\int_\Omega (|\nabla u|^p + a|u|^p)\ dx}{\displaystyle\int_{\partial\Omega} |u|^p\ dS}, \tag{2.11}$$

for all $u \in W^{1,p}(\Omega)$, $u \neq 0$ on $\partial\Omega$. Thus, $\sigma = \sigma_{1,p}$ also defines the first eigenvalue to (1.4). Relation (1.6) follows from the decreasing character of $\mu_{1,p}$ and the fact that $\lambda_{1,p}^{\mathcal{N}} = \mu_{1,p}(0)$.

The remaining assertions in Theorem 2 are consequences of Theorem 1. □

Remark 3. Inequality (2.11) states

$$\sigma_{1,p} = \inf_{u\in W^{1,p}(\Omega),\ u\neq 0} \frac{\displaystyle\int_\Omega (|\nabla u|^p + a|u|^p)\ dx}{\displaystyle\int_{\partial\Omega} |u|^p\ dS}. \tag{2.12}$$

As already seen, such infimum is finite when $\lambda_{1,p}(a) > 0$. However, it can be checked that the infimum is $-\infty$ when $\lambda_{1,p}(a) \leq 0$ (details are omitted for brevity). This suggests setting $\sigma_{1,p} = -\infty$ in that case.

3. Existence and uniqueness

Our first objective is to prove the variational version of the method of sub and supersolutions. For $p > 1$ we recall the notation $\varphi_p(t) = |t|^{p-2}t$.

Proof of Theorem 3. Following the ideas in [14] (see Theorem 2.4) we introduce the functional

$$J(u) = \int_\Omega \left\{\frac{1}{p}|\nabla u|^p + \frac{a(x)}{p}|u|^p - F(x,u)\right\} dx - \int_{\partial\Omega} G(x,u)\ dS,$$

with $F(x,u) = \int_0^u f(x,t)\ dt$ for $x \in \Omega$, $G(x,u) = \int_0^u g(x,t)\ dt$ for $x \in \partial\Omega$, which we consider in the convex set

$$\mathcal{M} = \{u \in W^{1,p}(\Omega) : \underline{u} \leq u \leq \overline{u} \ \text{ a.e.in } \ \Omega\}.$$

Notice that $\mathcal{M}$ defines a weakly closed subset of $W^{1,p}(\Omega)$. The functional J is sequentially lower semicontinuous and since both $\underline{u}$, $\overline{u}$ are bounded it is coercive in $\mathcal{M}$. Thus J achieves its infimum at some $u \in \mathcal{M}$ and we are showing that u is a weak solution to (1.7). For this, it is enough to show that $DJ[u](\varphi) \geq 0$ for every $\varphi \in C^1(\overline{\Omega})$.

To such proposal, for $\varepsilon > 0$ and arbitrary $\varphi \in C^1(\overline{\Omega})$ we set

$$\varphi_{\varepsilon,+} = (u + \varepsilon\varphi - \overline{u})^+ \qquad \varphi_{\varepsilon,-} = (\underline{u} - u - \varepsilon\varphi)^+,$$

and observe that

$$u_\varepsilon := u + \varepsilon\varphi - \varphi_{\varepsilon,+} + \varphi_{\varepsilon,-} \in \mathcal{M},$$

for all $0 < \varepsilon < \varepsilon_0$. By taking the derivative of J at u in the direction of $u_\varepsilon - u$ we get

$$DJ(u)[u_\varepsilon - u] \geq 0.$$

This implies that,

$$\varepsilon DJ(u)[\varphi] \geq DJ(u)[\varphi_{\varepsilon,+}] - DJ(u)[\varphi_{\varepsilon,-}], \tag{3.1}$$

and we are showing next that

$$DJ(u)[\varphi_{\varepsilon,+}] \geq \rho(\varepsilon),$$

where $\rho(\varepsilon) = o(\varepsilon)$ as $\varepsilon \to 0+$. In fact, since $DJ(\overline{u})[\varphi_{\varepsilon,+}] \geq 0$,

$$DJ(u)[\varphi_{\varepsilon,+}] \geq (DJ(u) - DJ(\overline{u}))[\varphi_{\varepsilon,+}],$$

and,

$$\begin{aligned} &(DJ(u) - DJ(\overline{u}))[\varphi_{\varepsilon,+}] = \tag{3.2} \\ &\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla \overline{u}|^{p-2}\nabla \overline{u})\nabla \varphi_{\varepsilon,+}\, dx + \int_\Omega a(x)(\varphi_p(u) - \varphi_p(\overline{u}))\varphi_{\varepsilon,+}\, dx \\ &\quad - \int_\Omega (f(x,u) - f(x,\overline{u}))\varphi_{\varepsilon,+}\, dx - \int_{\partial\Omega} (g(x,u) - g(x,\overline{u}))\varphi_{\varepsilon,+}\, dS. \end{aligned}$$

By using the monotonicity of the p-Laplacian,

$$\begin{aligned} &\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla \overline{u}|^{p-2}\nabla \overline{u})\nabla \varphi_{\varepsilon,+}\, dx \tag{3.3} \\ &\quad \geq \varepsilon \int_{\{\varphi_{\varepsilon,+} > 0\}} (|\nabla u|^{p-2}\nabla u - |\nabla \overline{u}|^{p-2}\nabla \overline{u})\nabla \varphi\, dx \\ &\quad \geq \varepsilon \int_{\{\varphi_{\varepsilon,+} > 0\} \cap \{\overline{u} > u\}} (|\nabla u|^{p-2}\nabla u - |\nabla \overline{u}|^{p-2}\nabla \overline{u})\nabla \varphi\, dx, \end{aligned}$$

since $\nabla u = \nabla \overline{u}$ almost everywhere in $\{u = \overline{u}\}$ ([8]). Observe now that $|\{\varphi_{\varepsilon,+} > 0\} \cap \{\overline{u} > u\}| \to 0$ as $\varepsilon \to 0+$ and so the latter integral in (3.3) is $o(\varepsilon)$ as $\varepsilon \to 0+$.

On the other hand, $|\varphi_{\varepsilon,+}| < \varepsilon|\varphi|$ in $\{\varphi_{\varepsilon,+} > 0\} \cap \{\overline{u} > u\}$. Hence,

$$\begin{aligned} &\left| \int_\Omega (f(x,u) - f(x,\overline{u}))\varphi_{\varepsilon,+}\, dx \right| \tag{3.4} \\ &\quad \leq \varepsilon \int_{\{\varphi_{\varepsilon,+} > 0\} \cap \{\overline{u} > u\}} |f(x,u) - f(x,\overline{u})||\varphi|\, dx = o(\varepsilon), \end{aligned}$$

as $\varepsilon \to 0+$. The remaining terms in (3.2) can be treated in the same way and so we achieve that,

$$DJ(u)[\varphi_{\varepsilon,+}] \geq o(\varepsilon) \qquad \varepsilon \to 0+\,.$$

A complementary argument shows that $DJ(u)[\varphi_{\varepsilon,-}] \leq o(\varepsilon)$ as $\varepsilon \to 0+$. Therefore, (3.1) implies that

$$DJ(u)[\varphi] \geq 0,$$

for arbitrary $\varphi \in C^1(\overline{\Omega})$. This means that u is a solution to (1.7). □

Remark 4. Theorem 3 can be extended to cover slightly more general settings. Namely, suppose that $\Omega \subset \mathbb{R}^N$ is smooth and $\partial\Omega = \Gamma_1 \cup \Gamma_2$ with Γ_1, Γ_2 disjoint $(N-1)-$dimensional closed manifolds. Consider the mixed problem

$$\begin{cases} -\Delta_p u(x) + a(x)|u|^{p-2}u(x) = f(x,u), & x \in \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu}(x) = g(x,u), & x \in \Gamma_1, \\ u(x) = h(x) & x \in \Gamma_2, \end{cases} \tag{3.5}$$

with $h \in L^\infty(\Gamma_2)$. Then, under the extra condition

$$\underline{u} \le h \le \overline{u} \qquad \text{on } \Gamma_2$$

and the hypotheses of Theorem 3 we achieve again a solution $u \in W^{1,p}(\Omega)$ to (3.5) lying between $\underline{u}$ and $\overline{u}$. The proof runs by the same lines of Theorem 3. As minor modifications, we have to take care of the condition $u = h$ on Γ_2 that must be incorporated to the definition of $\mathcal{M}$ and testing must be performed with functions $\varphi \in W^{1,p}(\Omega)$ vanishing on Γ_2.

As an immediate application of Theorem 3 we undertake the proof of Theorem 4.

Proof of Theorem 4. To prove the necessity of (1.8) we only consider, obviously, the case $\sigma_{1,p} > -\infty$. If a positive solution u to (1.1) exists then $u \neq 0$ on $\partial\Omega$. Otherwise,

$$-\Delta_p u + a\varphi_p(u) \le 0$$

implies $u \le 0$ in Ω if $u_{\partial\Omega} = 0$ (notice that $\sigma_{1,p}$ is finite if and only if $\lambda_{1,p}(a) > 0$). Thus, since $u \neq 0$ on $\partial\Omega$ we conclude that

$$\sigma_{1,p} \le \frac{\displaystyle\int_\Omega (|\nabla u|^p + a|u|^p)\ dx}{\displaystyle\int_{\partial\Omega} |u|^p\ dS} < \lambda.$$

Assume now that $\lambda > \sigma_{1,p} \ge -\infty$. Let $\phi_1(\lambda)$ denote the principal positive eigenfunction satisfying $\sup_\Omega \phi_1(\lambda) = 1$. Then it can checked that $\underline{u} = A\phi_1(\lambda)$, $\overline{u} = B\phi_1(\lambda)$ define a subsolution and a supersolution to (1.1) provided that

$$0 < A \le (-\mu_{1,p})^{\frac{1}{r-p+1}} \qquad B \ge \frac{(-\mu_{1,p})^{\frac{1}{r-p+1}}}{\inf \phi_1(\lambda)}.$$

Notice that a choice of A and B for all values of λ is possible when $\sigma_{1,p} = -\infty$. Thus, for suitable values of A and B we obtain, via Theorem 3, a positive solution to (1.1).

As for the uniqueness of a positive solution to (1.1) we first assert that all positive solutions $u \in W^{1,p}(\Omega)$ lie in $L^\infty(\Omega)$. In fact, observe that by setting $v = (u-k)^+$, $k > 0$, and employing v as a test function in the equation for u we arrive at

$$\int_\Omega (|\nabla v|^p + a(x)\varphi_p(u)v)\ dx \le |\lambda| \int_{\partial\Omega} \varphi_p(u)v\ dS.$$

By adding to both sides of the inequality a term $M\int_\Omega \varphi_p(u)v$ with large enough M we get

$$|v|_{1,p}^p \leq C\Big\{\int_\Omega \varphi_p(u)v\ dx + \int_{\partial\Omega} \varphi_p(u)v\ dS\Big\}.$$

But such an estimate (see (2.2), (2.3)) is just the starting point that leads to the boundedness of u if one proceeds as in Lemma 8. Thus $u \in L^\infty(\Omega)$. Notice in passing that the same argument works for the mixed problem (3.5) with $f = -u^r$, $g = \lambda\varphi_p(u)$ since the test function $v = (u-k)^+$ vanishes on Γ_2 provided that $k \geq |h|_\infty$.

Since a positive solution $u \in W^{1,p}(\Omega)$ is bounded, then $u \in C^{1,\beta}(\overline{\Omega}) \cap C^{2,\alpha}(U_\eta)$ by the same reasons as those providing the smoothness of the eigenfunction ϕ_1 in Theorem 1. Hence, for two positive solutions u_1, u_2 to (1.1) we can consider the test functions $\varphi_1 = (u_1^p - u_2^p)/u_1^{p-1}$, $\varphi_2 = (u_1^p - u_2^p)/u_2^{p-1}$. With them we obtain the inequality (see [13])

$$I = \int_\Omega (|\nabla u_1|^{p-2}\nabla u_1 \nabla\varphi_1 - |\nabla u_2|^{p-2}\nabla u_2\nabla\varphi_2)\ dx \geq 0.$$

However, since

$$I = -\int_\Omega (u_1^{r-p+1} - u_2^{r-p+1})(u_1^p - u_2^p)\ dx,$$

then $u_1 = u_2$ is the unique option permitted by the former inequality. Thus, (1.1) admits a unique positive solution.

Regarding iii), that $u_{r,\lambda}$ increases with λ is implied by the fact that $u_{r,\lambda}$ is subsolution to (1.1) with λ replaced by $\lambda' \geq \lambda$. The uniqueness of positive solutions together with the existence, via [12], of uniform $C^{1,\beta}$ bounds of $u_{r,\lambda}$ when λ varies in bounded intervals, yield the continuous dependence of $u_{r,\lambda}$ with values in, say, $C^1(\overline{\Omega})$. Moreover, such continuity and the nonexistence of positive solutions for $\lambda = \sigma_{1,p}$ entail (1.9) when $\sigma_{1,p} > -\infty$.

To show (1.10), assume $\sigma_{1,p} = -\infty$, take $\lambda_n \to -\infty$ and set $u_n = u_{r,\lambda_n}$. From the equality

$$\int_\Omega (|\nabla u_n|^p + a u_n^p)\ dx + (-\lambda_n)\int_{\partial\Omega} u_n^p\ dS + \int_\Omega u_n^{r+1}\ dx = 0,$$

together with the fact $0 < u_n \leq u_{n_0} \in L^\infty(\Omega)$ for $n \geq n_0$ we conclude, passing to a subsequence, that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$, with $u \geq 0$. Since

$$(-\lambda_n)\int_{\partial\Omega} u_n^p\ dS = O(1)$$

we have $u = 0$ on $\partial\Omega$. By using test functions in $W_0^{1,p}(\Omega)$ in the weak formulation of the equation for u_n and passing to the limit, we see that u defines a solution to

$$-\Delta_p u + a\varphi_p(u) = -u^r$$

in Ω. When $\lambda_{1,p}(a) = 0$, this yields $u = 0$, so that $u_{r,\lambda} \to 0$ in $W^{1,p}(\Omega)$ as $\lambda \to -\infty$.

On the other hand, when $\lambda_{1,p}(a) < 0$ we obtain that $u > 0$ in Ω. In fact, let ϕ_n be the positive eigenfunction associated to $\mu_{1,p}(\lambda_n)$, normalized by

$\sup_\Omega \phi_n = 1$. Then we have

$$\{-\mu_{1,p}(\lambda_n)\}^{\frac{1}{r-p+1}}\phi_n \le u_n \quad \text{in } \Omega. \tag{3.6}$$

Next take α_n such that $\hat{\phi}_n = \alpha_n\phi_n$ verifies $|\hat{\phi}_n|_p = 1$ and observe that $\alpha_n \ge |\Omega|^{-1}$. We find that $\hat{\phi}_n \rightharpoonup \hat{\phi}$ weakly in $W^{1,p}(\Omega)$, where $\hat{\phi} > 0$ (indeed $|\hat{\phi}|_p = 1$). On the other hand, a careful analysis of the proof of Lemma 8 reveals that

$$\sup \alpha_n < \infty.$$

Hence we achieve, by passing to a subsequence if necessary,

$$\phi_n \rightharpoonup \frac{1}{\theta}\hat{\phi},$$

weakly in $W^{1,p}(\Omega)$, where $\theta := \overline{\lim}\,\alpha_n > 0$. Passing to the limit in (3.6), we finally obtain

$$\theta^{-1}(-\lambda_{1,p}(a))^{\frac{1}{r-p+1}}\hat{\phi} \le u$$

in Ω. Thus, $u > 0$, and it defines the unique positive solution to (1.11) when $\lambda_{1,p}(a) < 0$. By uniqueness, we obtain $u_n \to u$ weakly in $W^{1,p}(\Omega)$. This concludes the proof of iii).

The proof of part iv) will be included in the next section. □

4. Limit profiles

To prove Theorem 5 our first ingredient is a property on the maximum of solutions to (1.1) with varying r. The proof is based on a simple comparison argument.

Lemma 9. *For $r > p-1$ let $M_{r,\lambda} := \sup_\Omega u_{r,\lambda}$. Then $M_{r,\lambda}^{r-p+1}$ is an increasing function of r.*

Proof. Assume $r > q > p-1 > 0$. Then we clearly have

$$-\Delta_p u_{r,\lambda} + a\varphi_p(u_{r,\lambda}) = -{u_{r,\lambda}}^r \ge -M_{r,\lambda}^{r-q}{u_{r,\lambda}}^q \quad \text{in } \Omega,$$

while the boundary condition rests unchanged. It follows that the function

$$\overline{u} = M_{r,\lambda}^{\frac{r-q}{q-p+1}}u_{r,\lambda}$$

is a supersolution to problem (1.1) with r replaced by q. Since $\underline{u} = \varepsilon u_{q,\lambda}$ is a small enough subsolution (for small ε) we obtain by uniqueness $\overline{u} \ge u_{q,\lambda}$. Thus $M_{r,\lambda}^{\frac{r-p+1}{q-p+1}} \ge M_{q,\lambda}$, which is the desired inequality. □

We can now proceed to prove Theorem 5.

Proof of Theorem 5. Let $v_r = u_{r,\lambda}/M_{r,\lambda}$. This function verifies

$$\begin{cases} -\Delta_p v(x) + av^{p-1}(x) = -M_{r,\lambda}^{r-p+1}v^r(x), & x \in \Omega, \\ |\nabla v|^{p-2}\dfrac{\partial v}{\partial \nu}(x) = \lambda v^{p-1}(x), & x \in \partial\Omega, \end{cases} \tag{4.1}$$

and $|v_r|_\infty = 1$. Thanks to Lemma 9 we have $0 < M_{r,\lambda}^{r-p+1} \le M_{p,\lambda}$, when $p-1 < r < p$, so that by the estimates in [12] we obtain that v_r is bounded in $C^{1,\beta}(\overline{\Omega})$ for certain $\beta \in (0,1)$. Thus for every sequence $r_n \to p-1+$ we may extract a subsequence, which will be relabeled as v_n, such that

$$v_n \to v$$

in $C^1(\overline{\Omega})$. We may also assume that

$$M_{r_n,\lambda}^{r_n-p+1} \to \theta$$

for some real number θ. Passing to the limit in the weak formulation of (4.1) we arrive at

$$\begin{cases} -\Delta_p v(x) + av^{p-1}(x) = -\theta v^{p-1}(x), & x \in \Omega, \\ |\nabla v|^{p-2} \dfrac{\partial v}{\partial \nu}(x) = \lambda v^{p-1}(x), & x \in \partial\Omega, \end{cases}$$

with $v \geq 0$, $|v|_\infty = 1$ and thus $v > 0$ in $\overline{\Omega}$. Hence, thanks to the uniqueness assertion in Theorem 1 we have that

$$\theta = -\mu_{1,p}(\lambda),$$

while

$$v = \phi_1(\lambda)$$

where $\phi_1(\lambda)$ stands for the positive eigenfunction associated to $\mu_{1,p}$ with $\sup_\Omega \phi_1(\lambda) = 1$. It follows that $v_n \to \phi_1(\lambda)$ in $C^1(\overline{\Omega})$.

By writing

$$u_n = M_{r_n,\lambda} v_n = (-\mu_{1,p}(\lambda) + o(1))^{\frac{1}{r_n-p+1}}(\phi_1(\lambda) + o(1)),$$

it is clear that assertions a) and b) follow immediately from the fact that $0 < -\mu_{1,p}(\lambda) < 1$ if $\lambda < \lambda^*$, provided λ^* exists (i. e. $\lambda_{1,p}(a) > -1$) while $-\mu_{1,p}(\lambda) > 1$ either if $\lambda > \lambda^*$ ($\lambda_{1,p}(a) > -1$) or either for all λ ($\lambda_{1,p}(a) \leq -1$).

When $\lambda = \lambda^*$, we have $\mu_{1,p} = -1$, so that $M_{r,\lambda}^{r-p+1} \to 1$ as $r \to p-1+$. However, no further information on $M_{r,\lambda}$ is available from this convergence and a more subtle analysis is required.

Now, for technical reasons we restrict ourselves to the case of linear diffusion, that is, we consider $p = 2$. Multiplying (4.1) by ϕ_1 and integrating in Ω leads to

$$\int_\Omega \phi_1 (M_{r,\lambda}^{r-1} v_r^r - v_r)\ dx = 0.$$

We may rewrite this equality as

$$\frac{M_{r,\lambda}^{r-1} - 1}{r-1} \int_\Omega \phi_1 v_r^r\ dx = \int_\Omega \phi_1 v_r \frac{1 - v_r^{r-1}}{r-1}\ dx. \tag{4.2}$$

Taking into account that $v_r \to \phi_1$ uniformly in $\overline{\Omega}$, and since $\phi_1 > 0$ in $\overline{\Omega}$, we obtain

$$v_r \frac{1 - v_r^{r-1}}{r-1} \to -\phi_1 \log \phi_1$$

uniformly in $\overline{\Omega}$ and hence, from (4.2),

$$\lim_{r\to 1+} \frac{M_{r,\lambda}^{r-1} - 1}{r-1} = -\frac{\displaystyle\int_\Omega \phi_1^2 \log \phi_1\ dx}{\displaystyle\int_\Omega \phi_1^2\ dx} = \log A, \tag{4.3}$$

where A is given by (1.14). Now, since from (4.3) we have

$$M_{r,\lambda} = \exp\left\{\frac{1}{r-1}\log\left(1+(\log A)(r-1)+o(r-1)\right)\right\}$$

then we obtain

$$\lim_{r\to 1+} M_{r,\lambda} = A,$$

as was to be shown. The proof is finished. □

Now we deal with the limit as $r \to \infty$.

Proof of Theorem 6. Since $a = 0$ we consider the problem

$$\begin{cases} \Delta_p u(x) = u^r(x), & x \in \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu}(x) = \lambda u^{p-1}(x), & x \in \partial\Omega. \end{cases} \tag{4.4}$$

To obtain the asymptotic behavior of $u_{r,\lambda}$ as $r \to \infty$ we construct suitable sub and supersolutions. To get a subsolution we pick $\psi \in W^{1,p}(\Omega)\cap C^{1,\beta}(\overline{\Omega})$ the solution to

$$\begin{cases} -\Delta_p u(x) = 1, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{4.5}$$

The strong maximum principle ([15]) yields $\psi > 0$ in Ω while

$$c_1 \le -|\nabla\psi|^{p-2}\frac{\partial\psi}{\partial\nu} \le c_2 \quad \text{on } \partial\Omega,$$

for some positive constants c_1, c_2.

We look for a subsolution $\underline{u}$ under the form

$$\underline{u} = A(\psi+\gamma)^{-\alpha} \qquad \alpha = \frac{p}{r-p+1}, \tag{4.6}$$

where positive constants A, γ must be found. The condition

$$|\nabla \underline{u}|^{p-2}\frac{\partial \underline{u}}{\partial\nu} \le \lambda \underline{u}^{p-1}$$

on $\partial\Omega$ is furnished by the choice $\gamma = \gamma_-$ with

$$\gamma_- = \left(\frac{c_2}{\lambda}\right)^{\frac{1}{p-1}}\alpha.$$

On the other hand, in order that $\underline{u}$ be a subsolution it is required that

$$\alpha^{p-1}\{(p-1)(\alpha+1)|\nabla\psi|^p + (\psi+\gamma)\} \ge A^{r-p+1}$$

in Ω. Setting

$$\Phi = (p-1)|\nabla\psi|^p + \psi,$$

such inequality is satisfied if $A = A_-$ with

$$A_- = \alpha^{\frac{p-1}{r-p+1}}(\inf_{\Omega}\Phi)^{\frac{1}{r-p+1}}.$$

A supersolution of the form

$$\overline{u} = A_+(\psi+\gamma_+)^{-\alpha},$$

satisfying

$$\underline{u} \le \overline{u}$$

in Ω is found by choosing the values:

$$\gamma_+ = \left(\frac{c_1}{\lambda}\right)^{\frac{1}{p-1}}\alpha \qquad A_+ = \alpha^{\frac{p-1}{r-p+1}}(2\sup_\Omega \Phi)^{\frac{1}{r-p+1}},$$

provided that r is conveniently large (notice that $\gamma_+ \to 0$ as $r \to \infty$).

Finally, since

$$A_-(\psi(x)+\gamma_-)^{-\alpha} \le u_{r,\lambda}(x) \le A_+(\psi(x)+\gamma_+)^{-\alpha} \tag{4.7}$$

in Ω for large r we conclude that $u_{r,\lambda} \to 1$ uniformly in Ω as $r \to \infty$. □

Now we use the previous construction to conclude the proof of Theorem 4.

Proof of Theorem 4-iv). We first briefly discuss the existence of solutions to (1.12). Observe that the problem

$$\begin{cases} -\Delta_p u + a u^{p-1} = -u^r & \in \Omega \\ u = M & x \in \partial\Omega, \end{cases}$$

has a unique positive solution $u = u_M \in C^{1,\beta}(\Omega)$ for every $M > 0$. In fact $\underline{u} = 0$, $\overline{u} = B\phi_1(\lambda_0)$ with $B > 0$ large can be used as a sub and a supersolution provided $\mu_{1,p}(\lambda_0) < 0$. Uniqueness, which is achieved by the same ideas as in Theorem 1, implies that u_M is increasing with M.

On the other hand, local uniform $C^{1,\beta}$ bounds for u_M follow from the estimate

$$u_M \le v_B \qquad x \in B$$

for every ball $B \subset \overline{B} \subset \Omega$, where $v = v_B$ is the minimal solution to

$$\begin{cases} -\Delta_p v(x) = |a|_\infty v^{p-1}(x) - v^r(x), & x \in B, \\ v = \infty & x \in \partial B. \end{cases}$$

The existence of v_B is well documented (see for instance [11] and Theorem 3 in [7]). In conclusion,

$$u_M \to U$$

in $C^1(\Omega)$ where U defines a weak solution to (1.12) in the sense that $U \to \infty$ as $\operatorname{dist}(x,\partial\Omega) \to 0$.

We now claim that, for fixed $r > p-1$,

$$u_{r,\lambda} \to \infty$$

uniformly on $\partial\Omega$ as $\lambda \to \infty$. Since $u_M \le u_{r,\lambda} \le U$ in Ω for λ large we immediately achieve (1.13).

To show the claim we construct a suitable subsolution $\underline{u}_\lambda$ to the auxiliary problem

$$\begin{cases} -\Delta_p u(x) + a u^{p-1}(x) = -u^r(x), & x \in U_\eta, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu}(x) = \lambda u^{p-1}(x), & x \in \partial\Omega, \\ u(x) = u_{r,\lambda}(x), & \operatorname{dist}(x,\partial\Omega) = \eta, \end{cases} \tag{4.8}$$

where $U_\eta = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) < \eta\}$ and $\eta > 0$ is small. Notice that $u = u_{r,\lambda}$ is its unique solution (check once more the uniqueness proof in Theorem 1).

Following the preceding proof, a subsolution of the form

$$\underline{u}_\lambda = A(\psi+\gamma)^{-\alpha},$$

with ψ and α as before, can be found in U_η by choosing

$$\gamma = \alpha \left\{ \frac{\sup_{\partial\Omega} |\nabla\psi|^{p-2}(-\frac{\partial\psi}{\partial\nu})}{\lambda} \right\}^{\frac{1}{(p-1)}},$$

and taking $\lambda \geq \lambda_0$, $\eta \leq \eta_0$ and $0 < A \leq A_0$. Remark that

$$u_{r,\lambda} \geq u_{r,\lambda_0} \geq A\psi^{-\alpha} \geq \underline{u}_\lambda$$

on $\mathrm{dist}(x,\partial\Omega) = \eta$ for all $\lambda \geq \lambda_0$ provided $A < A_1$.

Now, by using $\overline{u}_\lambda = Bu_{r,\lambda}$, B large enough, as a supersolution, Theorem 3 (see Remark 4) implies in particular that

$$u_{r,\lambda} \geq \underline{u}_\lambda$$

for large λ. This shows the claim. □

Prof of Theorem 7. As observed in Theorem 4, sub and supersolutions to (1.1) of the form $\underline{u} = A\phi_1(\lambda)$, $\overline{u} = B\phi_1(\lambda)$ can be found. Thus one arrives at

$$(-\mu_{1,p}(\lambda))^{\frac{1}{r-p+1}}\phi_1(\lambda)(x) \leq u_{r,\lambda}(x) \leq (-\mu_{1,p}(\lambda))^{\frac{1}{r-p+1}} \frac{\phi_1(\lambda)(x)}{\inf_\Omega \phi_1(\lambda)}$$

for all $r > p-1$. This implies that

$$\underline{\lim}_{r\to\infty} u_{r,\lambda}(x) \geq \phi_1(\lambda)(x), \qquad x\in\Omega.$$

On the other hand, as in the proof of Theorem 6, a supersolution to (1.1) can be obtained in the form

$$\overline{u} = A(\psi(x)+\gamma)^{-\alpha},$$

with α, $\gamma = \gamma_+$ and ψ as in that proof, while A is chosen such that

$$A^{r-p+1} = 1 + |a|_\infty(\sup_\Omega \psi + 1)^p,$$

for sufficiently large r. From the inequality $u_{r,\lambda} \leq \overline{u}$ one easily gets,

$$\overline{\lim}_{r\to\infty} u_{r,\lambda}(x) \leq 1.$$

A combination of these inequalities also gives

$$\lim_{r\to\infty} \sup_\Omega u_{r,\lambda} = 1.$$

To study the behavior of $\sup u_{r,\lambda}^{r-p+1}$ we first consider $a=0$ in (1.1) but $p>1$ arbitrary. In this case, inequality (4.7) directly leads to

$$u_{r,\lambda}{}^{r-p+1}(x) \geq A_-^{r-p+1}\gamma_-^{-p}$$

on $\partial\Omega$. Since $\gamma_- \sim C\alpha$ as $r\to\infty$ such inequality says that

$$\lim_{r\to\infty} \sup_\Omega (u_{r,\lambda})^{r-p+1} = \infty. \tag{4.9}$$

To conclude with the case $a \in L^\infty(\Omega)$ arbitrary with λ large, we use an argument inspired in [3]. Let us begin assuming $a>0$ in Ω and assume, arguing by contradiction, that $\sup u_{r,\lambda}^{r-p+1}$ is bounded. Choose $r_n \to \infty$ and set $u_n = u_{r_n,\lambda}$, $t_n = \sup u_n$, $u_n = t_n v_n$. Then v_n solves

$$\begin{cases} -\Delta_p v_n(x) + a v_n^{p-1}(x) = -u_n^{r_n-p+1} v_n^{p-1}(x), & x \in \Omega, \\ |\nabla v_n|^{p-2} \dfrac{\partial v}{\partial \nu}(x) = \lambda v_n^{p-1}(x), & x \in \partial\Omega. \end{cases}$$

Now, passing to a subsequence, $v_n^{r_n-p+1} \rightharpoonup h$ in $L^q(\Omega)$ for a nonnegative $h \in L^\infty(\Omega)$ and a conveniently large chosen $q > 1$. On the other hand, the estimates in [12] permit us showing that $v_n \to v$ in $C^{1,\gamma}(\overline{\Omega})$ where v is positive, $|v|_\infty = 1$ and solves

$$\begin{cases} -\Delta_p v(x) + a v^{p-1}(x) = -h v^{p-1}(x), & x \in \Omega, \\ |\nabla v|^{p-2} \dfrac{\partial v}{\partial \nu}(x) = \lambda v^{p-1}(x), & x \in \partial\Omega. \end{cases}$$

Sice $0 < v(x) \le 1$ in Ω and v is p-subharmonic it follows that $v(x) < 1$ for all $x \in \Omega$. Otherwise, $v = 1$ and from the equation $a + h = 0$ in Ω what is impossible. However, $v < 1$ implies $h = 0$ in Ω. Hence, v solves

$$\begin{cases} -\Delta_p v(x) + a v^{p-1}(x) = 0, & x \in \Omega, \\ |\nabla v|^{p-2} \dfrac{\partial v}{\partial \nu}(x) = \lambda v^{p-1}(x), & x \in \partial\Omega. \end{cases}$$

But this implies $\mu_1(\lambda) = 0$ which contradicts the existence of a positive solution to (1.1) (Theorem 1).

For an arbitrary $a \in L^\infty(\Omega)$, not necessarily positive let $u = \tilde{u}_{r,\lambda}$ be the solution to (1.1) with a replaced by $|a|_\infty > 0$ and notice that

$$u_{r,\lambda} \ge \tilde{u}_{r,\lambda}.$$

The conclusion follows from the fact that $\tilde{u}_{r,\lambda}$ satisfies (4.9). □

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