QUASILINEAR EQUATIONS WITH BOUNDARY BLOW-UP AND EXPONENTIAL REACTION

JORGE GARCÍA-MELIÁN

Abstract. We consider the quasilinear elliptic problem $\Delta_p u = a(x)e^u$ in a smooth bounded domain Ω of $\mathbb{R}^N$, with $u = +\infty$ on $\partial\Omega$. Here $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator with $p > 1$, and $a(x)$ is a continuous positive weight function which can be singular on $\partial\Omega$. Our results include existence, uniqueness and exact boundary behavior of solutions.

2000 Mathematics Subject Classification. Primary 35J25; Secondary 35J60.

Keywords: p-Laplacian, boundary blow-up, existence and uniqueness, boundary behavior.

1. Introduction

In this paper we consider the quasilinear boundary blow-up elliptic problem

$$(1.1) \qquad \begin{cases} \Delta_p u = a(x)e^u & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$ ($N \geq 2$) and Δ_p stands for the p-Laplacian operator defined by $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$. The weight function $a(x)$ is assumed to be continuous and positive in Ω but we allow the possibility that a is singular on $\partial\Omega$.

By a solution of (1.1) we mean a function $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ verifying $\Delta_p u = a(x)e^u$ in Ω in the weak sense and taking the boundary condition $u = +\infty$ on $\partial\Omega$ in the sense that $u(x) \to +\infty$ when $d(x) \to 0+$, where $d(x)$ denotes the distance function $\mathrm{dist}(x, \partial\Omega)$. We remark that since u is locally bounded, standard regularity gives $u \in C^{1,\eta}_{\mathrm{loc}}(\Omega)$ for some $\eta \in (0, 1)$ (cf. [12], [26], [37]), and thus the boundary condition makes full sense.

Quasilinear problems like (1.1), of the general form

$$(1.2) \qquad \begin{cases} \Delta_p u = a(x)f(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega \end{cases}$$

for general nonlinearities $f(u)$ have been considered very often in the recent literature. We quote [11] as a starting point for problems with

Supported by MEC and FEDER under grant MTM2005-06480.

p-Laplacian: the questions of existence, uniqueness and boundary behavior of solutions when $f(u) = u^q$, $q > p - 1$ were studied there. After [11], some other authors have also considered boundary blow-up problems for equations with the p-Laplacian. We mention for instance [22], [23], [31], [32], [13], [14] and [18].

But it has been the particular case $p = 2$ the one which has attracted more attention. See the pioneering works [3], [24] and [35], and also [1], [2], [27], [25], [39], [30], [10], [19], [15], [16] and [17] for different types of nonlinearities in the semilinear case.

Also in the semilinear case, the stress has been often put in the model problems where $f(u) = u^q$, $q > 1$ or $f(u) = e^u$, focusing the attention on the weight $a(x)$. It has been allowed to vanish on $\partial\Omega$ (cf. [7], [8], [28], [29], [15], [19], [17]) or to be singular on $\partial\Omega$ (see [33], [40], [4], [5], [34]).

We finally mention the work [18] which, at the best of our knowledge, is the first one to perform a systematic study for an equation involving p-Laplacian and singular weights.

In this paper, we consider problem (1.1) under the hypothesis that the weight function $a(x)$ is a positive continuous function in Ω, which verifies:

$$a(x) \sim Q(x_0) d(x)^{-\gamma}$$

as $x \to x_0$, for every $x_0 \in \partial\Omega$, where $\gamma < p$ and Q is a continuous function in $\overline{\Omega}$ which is strictly positive on $\partial\Omega$. However, we observe that the case $\gamma < 0$ is included in our assumption, being $a = 0$ on $\partial\Omega$ in that case. Let us remark at this point that the growth rate of $a(x)$ needs not be constant. It is possible to let γ vary on $\partial\Omega$, provided that it is a Hölder continuous function which is bounded away from p on $\partial\Omega$, as in [18]. But we are making the simplifying assumption that γ is constant.

On the other hand, let us mention that it could be possible that a vanishes at points inside Ω, requiring only the positivity in a neighborhood of $\partial\Omega$. We again refer to [18] for this situation.

The tools employed in this paper are an adaptation of the corresponding ones for the semilinear case. There is however a notable exception: for the purposes of boundary estimates of solutions, we need a uniqueness result for problem (1.1) in a half-space. When $p = 2$, this result was obtained in [5], but we remark that the proof is essentially linear, and thus there is no hope in adapting it for $p \neq 2$. This problem was also present in [18], where a power nonlinearity was considered instead, but the proof does not seem to generalize, since uniqueness of solutions for (1.1) in a half-space does not hold in general, unless a growth condition is imposed on the solutions (see Section 2). Thus our proof here is completely different and relies on the strong comparison principle. One of the advantages of this proof is that it can be generalized to deal with some more general situations.

We now state our main result. We assume for simplicity that the function $Q(x)$ giving the boundary behavior of $a(x)$ is defined throughout $\overline{\Omega}$.

Theorem 1. *Let $a \in C(\Omega)$ be a positive function in Ω and assume there exists a function $Q \in C(\overline{\Omega})$ with $Q(x) > 0$ on $\partial\Omega$ and a real number $\gamma < p$ such that*

$$\lim_{x \to x_0} d(x)^\gamma a(x) = Q(x_0)$$

for every $x_0 \in \partial\Omega$. Then problem (1.1) admits a unique weak solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C^{1,\eta}_{\mathrm{loc}}(\Omega)$ for some $\eta \in (0,1)$, which in addition verifies

$$u(x) + (p-\gamma)\log d(x) \to \log\left(\frac{(p-1)(p-\gamma)^{p-1}}{Q(x_0)}\right) \tag{1.3}$$

as $x \to x_0$, for every $x_0 \in \partial\Omega$.

Remark 1. It can be shown that condition $\gamma < p$ is necessary for existence of blow-up solutions, since the upper inequality in (3.5) of Lemma 4 still holds in this case, and then solutions to the equation are bounded when $\gamma = p$ or verify $u(x) \to -\infty$ on $\partial\Omega$ when $\gamma > p$.

The paper is organized as follows: in Section 2 we consider the uniqueness result for a version of problem (1.1) in a half-space, while Section 3 is dedicated to prove Theorem 1.

2. A uniqueness theorem in a half-space

We dedicate this section to prove a uniqueness theorem for a problem related to (1.1) in a half-space $D = \{x \in \mathbb{R}^N : \ x_1 > 0\}$, where for $x \in \mathbb{R}^N$ we write $x = (x_1, x')$. That is, we consider

$$\begin{cases} \Delta_p u = Q_0 x_1^{-\gamma} e^u & \text{in } D, \\ u = +\infty & \text{on } \partial D. \end{cases} \tag{2.1}$$

where $Q_0 > 0$ and $\gamma < p$. The semilinear version $p = 2$ of this problem was analyzed in [5], but with a proof that does not seem to generalize to $p \neq 2$, since linearity was an essential ingredient. There are also corresponding versions when the nonlinearity e^u is replaced by u^q, $q > p-1$ (see [18] when $p \neq 2$ and [5] for $p = 2$).

There is however a fundamental difference between the problem with an exponential and the one with a power. In the latter, every solution (positive by definition) verifies a growth condition which is inherent to the nonlinearity, and this leads to uniqueness, while in the former this is not so. Namely, it is easily seen that when $p = 2$, $Q_0 = 1$ and $\gamma = 0$

$$u_\lambda(x) = \log(2\lambda^2) + \lambda x_1 - 2\log(e^{\lambda x_1} - 1),$$

is a solution to (2.1) for arbitrary $\lambda > 0$, and thus uniqueness is lost.

We can however recover the uniqueness of the solution if we impose an additional growth condition, which is exactly the one needed for proving boundary estimates in the next section. Thus we are only considering solutions which verify

$$-(p-\gamma)\log x_1 + C_1 \le u \le -(p-\gamma)\log x_1 + C_2 \qquad \text{in } D, \tag{2.2}$$

for some constants C_1, C_2. With this further requirement, we show that the solution coincides with the only one-dimensional solution which has the growth specified in (2.2). We mention in passing that the one-dimensional version of (2.1) can be dealt with as in Lemma 4 of [18].

Theorem 2. *Let $Q_0 > 0$, $\gamma < p$ and $u \in W^{1,p}_{\rm loc}(D)$ a weak solution to* (2.1) *verifying* (2.2) *for some constants C_1, C_2. Then*

$$u(x) = -(p-\gamma)\log x_1 + \log\left(\frac{(p-1)(p-\gamma)^{p-1}}{Q_0}\right). \tag{2.3}$$

Proof. We only prove that $u(x) \le -(p-\gamma)\log x_1 + B$, where $B = \log((p-1)(p-\gamma)^{p-1}/Q_0)$, since the other inequality can be shown in the same way. Let

$$K = \sup_{x\in D}(u(x) + (p-\gamma)\log x_1),$$

which is finite thanks to (2.2). We may assume $K \ge B$, since otherwise there is nothing to prove. In that case, it is not hard to see that $\overline{u}(x) = -(p-\gamma)\log x_1 + K$ is a supersolution to $\Delta_p u = Q_0 x_1^{-\gamma} e^u$ in D.

According to the definition of K, there exists a sequence $\{x_n\} \subset D$ such that $u(x_n) + (p-\gamma)\log(x_n)_1 \to K$, where $(x_n)_1$ stands for the first component of x_n. Let ξ_n be the projection of x_n onto $\partial\Omega$, and define

$$U_n(y) = u(\xi_n + (x_n)_1 y) + (p-\gamma)\log(x_n)_1, \quad y \in D.$$

Then $\Delta_p U_n = Q_0 y_1^{-\gamma} e^{U_n}$ in D, while thanks to (2.2) we have $U_n(y) \le \overline{u}(y)$. Notice that also $U_n(y) \ge -(p-\gamma)\log y_1 + C_1$, so that the sequence $\{U_n\}$ is locally uniformly bounded in D. Thanks to the $C^{1,\alpha}$ interior estimates for the p-Laplacian provided by [12], [26] or [37], we obtain that $\{U_n\}$ is precompact in $C^1_{\rm loc}(D)$, and passing to a subsequence we may assume that $U_n \to U$ in $C^1_{\rm loc}(D)$, which will be a weak solution to $\Delta_p U = Q_0 y_1^{-\gamma} e^U$ in D, verifying

$$U \le \overline{u}, \quad U(e_1) = K = \overline{u}(e_1).$$

Since U is a solution and $\overline{u}$ a supersolution, we obtain thanks to the strong comparison principle (see Proposition 3.3.2 in [36] or Theorem 1.4 in [9]) that $U \equiv \overline{u}$ in a neighborhood of e_1. Observe that $\nabla\overline{u} \ne 0$, so that the comparison principle can be applied.

Thus $\overline{u}$ is not only a supersolution but a solution in a neighborhood of e_1, and this implies that actually $K = B$, that is

$$u(x) \le -(p-\gamma)\log x_1 + B.$$

As has been already observed, the lower inequality is proved similarly. This concludes the proof. ∎

3. Proof of Theorem 1

In this section, we are proving the three aspects covered by Theorem 1: existence, boundary behavior and uniqueness of solutions to (1.1). We begin by recalling a result which is contained in [18], and is a generalization of an existence theorem in [21]. It will turn out to be important in the proof of existence of solutions, and also when obtaining lower bounds for the solutions near the boundary.

Lemma 3. *Let $f \in L^\infty_{\rm loc}(\Omega)$ be such that $|f| \le Md^{-\gamma}$ for some $M > 0$ and $\gamma \in (1, p)$. Then the problem*

$$\left\{ \begin{array}{ll} -\Delta_p u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{array} \right. \tag{3.1}$$

has a unique weak solution $u \in W^{1,p}_{\rm loc}(\Omega) \cap C^{1,\eta}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$ for some $\eta \in (0, 1)$. Moreover, there exists a positive constant C not depending on f such that

$$|u| \le CM^{\frac{1}{p-1}} d^{\frac{p-\gamma}{p-1}} \qquad \text{in } \Omega. \tag{3.2}$$

We observe that the existence and uniqueness issues in Lemma 3 are still valid even if $0 < \gamma \le 1$, since $|f| \le Md^{-\gamma}$ implies $|f| \le M'd^{-\theta}$ for every $\theta > \gamma$. In that case, the growth control (3.2) is no longer the optimal one, but this will suffice for our purposes here.

Proof of existence. We claim that the problem with finite datum

$$\left\{ \begin{array}{ll} \Delta_p u = a(x)e^u & \text{in } \Omega, \\ u = n & \text{on } \partial\Omega, \end{array} \right. \tag{3.3}$$

has a unique solution for every positive integer n. We first assume that the weight $a(x)$ is bounded. In that case, it is easily seen that $\overline{u} = n$ is a supersolution, while a subsolution is given by the function $\underline{u} = C(|x|^{\frac{p}{p-1}} - R^{\frac{p}{p-1}})$, where $C > 0$ is large enough and R is chosen to have $|x| < R$ in Ω ($C \ge (p-1)|a|_\infty^{\frac{1}{p-1}}/(pN^{\frac{1}{p-1}})$ will suffice). Then the existence of solution follows. The uniqueness is a consequence of the comparison principle.

The above shows that problem (3.3) has a unique solution when $\gamma \le 0$. If $\gamma > 0$, then a is unbounded, and we truncate the weight as follows: define

$$a_k(x) = \left\{ \begin{array}{ll} \left(\dfrac{1}{a(x)} + \dfrac{1}{k} \right)^{-1} & x \in \Omega \\ k, & x \in \partial\Omega. \end{array} \right.$$

Then $a_k \in C(\overline{\Omega})$, while a_k is increasing in k and $a_k(x) \le a(x) \le Cd(x)^{-\gamma}$, for some positive constant C. Moreover, $a_k \to a$ uniformly in compacts of Ω.

Now we consider the truncated problem

$$\begin{cases} \Delta_p u = a_k(x)e^u & \text{in } \Omega, \\ u = n & \text{on } \partial\Omega, \end{cases}$$

which has a unique solution by the preceding discussion. Let us denote the solution by $u_{k,n}$. According to standard regularity (cf. [12], [37], [26]), $u_{k,n} \in C^{1,\eta}(\overline{\Omega})$ for some $\eta \in (0,1)$. Moreover, notice that since a_k is increasing then $u_{n,k}$ is decreasing in k for every fixed n. We now prove that $u_{k,n}$ is bounded from below for fixed n.

Let ϕ be the unique solution to

$$\begin{cases} -\Delta_p \phi = Ce^n d(x)^{-\gamma} & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

which exists thanks to Lemma 3 and the remarks after it (notice that ϕ depends on n, but we are omitting this dependence, which will be of no importance in what follows). The function ϕ is positive by the maximum principle.

Since $n-\phi$ satisfies

$$\Delta_p(n-\phi) = Ce^n d(x)^{-\gamma} \geq a_k(x)e^{n-\phi}$$

in Ω while $n-\phi = n$ on $\partial\Omega$, we obtain thanks to the comparison principle that $u_{k,n} \geq n-\phi$ in Ω. It is thus standard to conclude that $u_{k,n} \to u_n$ in $C^1_{\text{loc}}(\Omega)$, where u_n is a weak solution to (3.3).

Thus (3.3) has a solution u_n in both cases $\gamma \leq 0$ and $\gamma > 0$. The uniqueness of u_n is a consequence once more of the comparison principle. Moreover, uniqueness also implies that the sequence $\{u_n\}$ is increasing.

Finally we need to obtain bounds for the solutions u_n. Since $a > 0$ in Ω, it follows that $a \geq a_0 > 0$ in Ω' for every smooth subset $\Omega' \subset\subset \Omega$. Thus

$$\Delta_p u_n \geq a_0 e^{u_n} \qquad \text{in } \Omega',$$

and it follows that $u_n \leq U$, where U is the minimal solution to $\Delta_p U = a_0 e^U$ in Ω' with $U = +\infty$ on $\partial\Omega'$ (cf. [31]). This shows that the sequence $\{u_n\}$ is locally uniformly bounded, and then it is standard to pass to the limit and obtain that $u_n \to u$ in $C^1_{\text{loc}}(\Omega)$, where the function u is a solution to $\Delta_p u = a(x)e^u$ in Ω, and $u = +\infty$ on $\partial\Omega$. Thus we have shown the existence of a solution u to (1.1), which by standard regularity verifies $u \in C^{1,\eta}_{\text{loc}}(\Omega)$ for some $\eta \in (0,1)$. □

Before obtaining the boundary behavior of solutions, we need some preliminary estimates, which confirm somehow that the growth of the solutions is actually the one stated in (1.3) (see [5], [17], [18] and [20] for related results).

Lemma 4. *Let u be a weak solution to* (1.1) *and $x_0 \in \partial\Omega$. Then there exists a neighborhood $\mathcal{U}$ of x_0 and constants C_1, C_2 such that*

$$-(p-\gamma)\log d(x) + C_1 \leq u(x) \leq -(p-\gamma)\log d(x) + C_2 \tag{3.5}$$

in $\mathcal{U}$.

Proof. For fixed $x \in \Omega$ we define $v(y) = u(x+d(x)y) + (p-\gamma)\log d(x)$, where $y \in B_1(0)$, the unit ball. Then v verifies

$$\Delta_p v = d(x)^\gamma a(x+d(x)y)e^v \qquad \text{in } B_1(0).$$

Since

$$a(x+d(x)y) \geq Cd(x+d(x)y)^{-\gamma} \geq Cd(x)^{-\gamma}$$

for x near x_0, we obtain $\Delta_p v \geq Ce^v$ in $B_1(0)$, and thus $v \leq V$, the unique solution to $\Delta_p V = Ce^V$ in $B_1(0)$ with $V = +\infty$ on $\partial B_1(0)$ (we are using the letter C to denote positive constants, not necessarily the same everywhere). Making $y = 0$ we arrive at

$$u(x) \leq -(p-\gamma)\log d(x) + V(0), \tag{3.6}$$

which is the upper inequality in (3.5).

The lower inequality can not be obtained as in [18], since the solution v needs not be positive. Thus we proceed differently, by arguing as in [5] with the aid of Lemma 3.

Let $u = -\alpha \log w$ for some $\alpha > 0$ to be chosen. Then the function w is a weak solution to

$$\begin{cases} -\Delta_p w + (p-1)\dfrac{|\nabla w|^p}{w} = \dfrac{1}{\alpha^{p-1}} a(x) w^{p-1-\alpha} & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

On the other hand, thanks to (3.6) we know that $w \geq Cd^{\frac{p-\gamma}{\alpha}}$ for some positive constant C. Thus if $\alpha > p-1$ we have

$$-\Delta_p w \leq Cd(x)^{-\gamma+\frac{(p-1-\alpha)(p-\gamma)}{\alpha}} = Cd(x)^{\frac{(p-1)(p-\gamma)}{\alpha}-p} \qquad \text{in } \Omega. \tag{3.7}$$

If we denote $\theta = p - \frac{(p-1)(p-\gamma)}{\alpha}$, we notice that α can be chosen large enough to have $\theta \in (1,p)$, and thus the problem

$$\begin{cases} -\Delta_p \phi = Cd(x)^{-\theta} & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

admits a unique positive solution ϕ thanks to Lemma 3. Moreover, $\phi \leq Cd^{\frac{p-\theta}{p-1}}$ in Ω.

Using (3.7) and the comparison principle, we obtain that $w \leq C\phi \leq Cd^{\frac{p-\gamma}{\alpha}}$, which, after returning to the original function u, provides with $u(x) \geq -(p-\gamma)\log d(x) + C$, the lower inequality in (3.5). This concludes the proof. □

We now prove the behavior (1.3). We perform a local analysis of the solutions near a boundary point, rather similar to the one employed in the proof of Theorem 2. Inequalities (3.5) are important for this aim.

Proof of estimates (1.3). Let $x_0 \in \partial\Omega$. By a translation and a rotation we can always assume that $x_0 = 0$ and $\nu(x_0) = -e_1$, where e_1 is the first vector in the canonical basis of $\mathbb{R}^N$. Let $\{x_n\} \subset \Omega$ be an arbitrary sequence with $x_n \to 0$, and denote by ξ_n the projection of x_n onto $\partial\Omega$. Set $d_n = d(x_n)$, and define the function

$$v_n(y) = u(\xi_n + d_n y) + (p-\gamma)\log d_n$$

for $y \in \Omega_n := \{y \in \mathbb{R}^N : \ \xi_n + d_n y \in \mathcal{U}\}$, where $\mathcal{U}$ is the neighborhood of $x_0 = 0$ given by Lemma 4. Remark that $\Omega_n \to D$ as $n \to \infty$. Now the function v_n verifies

$$\Delta_p v_n = d_n^\gamma a(\xi_n + d_n y) e^{v_n} \qquad \text{in } \Omega_n. \tag{3.8}$$

Notice that $d(\xi_n + d_n y) \sim d_n y_1$, and thus we obtain that

$$d_n^\gamma a(\xi_n + d_n y) \to Q_0 y_1^{-\gamma},$$

where $Q_0 = Q(0)$.

Lemma 4 gives us the necessary estimates to pass to the limit in (3.8). There exist constants C_1, C_2 such that

$$(p-\gamma)\log\left(\frac{d_n}{d(\xi_n + d_n y)}\right) + C_1 \le v_n \le (p-\gamma)\log\left(\frac{d_n}{d(\xi_n + d_n y)}\right) + C_2. \tag{3.9}$$

The terms with the logarithm in (3.9) converge to $-(p-\gamma)\log y_1$ as $n \to \infty$, and thus we obtain that the sequence v_n is uniformly bounded in compacts of D, which in particular entails that the sequence $\{v_n\}$ is precompact in $C^1_{\rm loc}(D)$. Thus we may assume that $v_n \to v$ in $C^1_{\rm loc}(D)$. Passing to the limit in (3.8), we obtain that v verifies

$$\begin{cases} \Delta_p v = Q_0 y_1^{-\gamma_0} e^v & \text{in } D, \\ v = +\infty & \text{on } \partial D \end{cases}$$

and

$$-(p-\gamma)\log y_1 + C_1 \le v \le -(p-\gamma)\log y_1 + C_2.$$

Using Theorem 2, we know that

$$v(y) = -(p-\gamma)\log y_1 + \log\left(\frac{(p-1)(p-\gamma)^{p-1}}{Q_0}\right),$$

and setting $y = e_1$ we arrive at

$$u(x_n) + (p-\gamma)\log d(x_n) \to \log\left(\frac{(p-1)(p-\gamma)^{p-1}}{Q_0}\right),$$

as was to be shown. Since the sequence $\{x_n\}$ is arbitrary, the proof is concluded. □

We finally consider uniqueness.

Proof of uniqueness. Let u, v be weak solutions to (1.1) and choose $\varepsilon > 0$. Consider the set $\Omega_\varepsilon = \{x \in \Omega : \ u(x) > v(x) + \varepsilon\}$ and assume it is nonempty. Since, thanks to estimates (1.3) we have

$$\lim_{x \to x_0} (u(x) - v(x)) = 0 \tag{3.10}$$

for every $x_0 \in \partial\Omega$, it follows that $\Omega_\varepsilon \subset\subset \Omega$ and thus $u = v + \varepsilon$ on $\partial\Omega_\varepsilon$. On the other hand,

$$\Delta_p u = a(x)e^u > a(x)e^{v+\varepsilon} > a(x)e^v = \Delta_p(v+\varepsilon)$$

in Ω_ε, and by the comparison principle $u \leq v + \varepsilon$ in Ω_ε, which is impossible. Thus Ω_ε is empty, and then $u \leq v + \varepsilon$. Letting $\varepsilon \to 0$ we obtain $u \leq v$ and the symmetric argument shows that $u = v$. Uniqueness is proved. □

Remark 2. An alternative proof of uniqueness is possible, which makes use of inequalities (3.5) given by Lemma 4. Indeed, since u, v are bounded from below, we can select $K > 0$ so that $\tilde{u} = u + K, \tilde{v} = v + K$ are greater than $p - 1$ in Ω. Moreover, $\tilde{u}$ and $\tilde{v}$ are solutions to

$$\begin{cases} \Delta_p u = \tilde{a}(x)e^u & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

where $\tilde{a}(x) = a(x)e^{-K}$ verifies the same assumptions as a, with Q replaced by $e^{-K}Q$. On the other hand, notice that (3.5) implies that $\tilde{u}$, $\tilde{v}$ verify

$$\frac{\tilde{u}(x)}{\tilde{v}(x)} \to 1$$

as $d(x) \to 0+$. In particular, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $(1-\varepsilon)\tilde{v} \leq \tilde{u} \leq (1+\varepsilon)\tilde{v}$ in $\Omega_\delta := \{x \in \Omega : \ d(x) < \delta\}$. Consider the problem

$$\begin{cases} \Delta_p w = \tilde{a}(x)e^w & \text{in } \Omega \setminus \Omega_\delta, \\ w = \tilde{u} & \text{on } \partial(\Omega \setminus \Omega_\delta), \end{cases} \tag{3.11}$$

which has $w = \tilde{u}$ as unique solution, thanks to the comparison principle. In addition, since the function e^w/w^{p-1} is increasing when $w > p-1$, it can be easily checked that $(1-\varepsilon)\tilde{v}$ and $(1+\varepsilon)\tilde{v}$ are a sub- and a supersolution to (3.11), and thus uniqueness implies that $(1-\varepsilon)\tilde{v} \leq \tilde{u} \leq (1+\varepsilon)\tilde{v}$ in $\Omega \setminus \Omega_\delta$.

As a conclusion, $(1-\varepsilon)\tilde{v} \leq \tilde{u} \leq (1+\varepsilon)\tilde{v}$ in Ω, and letting $\varepsilon \to 0$ we find $\tilde{u} = \tilde{v}$, that is, $u = v$.

Acknowledgements. The author would like to thank the referee for some valuable suggestions.

References

[1] C. Bandle, M. Marcus, *Sur les solutions maximales de problèmes elliptiques non linéaires: bornes isopérimetriques et comportement asymptotique*, C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), 91–93.

[2] C. Bandle, M. Marcus, *'Large' solutions of semilinear elliptic equations: Existence, uniqueness and asymptotic behaviour*, J. Anal. Math. **58** (1992), 9–24.

[3] L. Bieberbach, *$\Delta u = e^u$ und die automorphen Funktionen*, Math. Ann. **77** (1916), 173–212.

[4] M. Chuaqui, C. Cortázar, M. Elgueta, C. Flores, J. García-Melián, R. Letelier, *On an elliptic problem with boundary blow-up and a singular weight: the radial case*, Proc. Roy. Soc. Edinburgh **133** (2003), 1283–1297

[5] M. Chuaqui, C. Cortázar, M. Elgueta, J. García-Melián, *Uniqueness and boundary behaviour of large solutions to elliptic problems with singular weights*, Comm. Pure Appl. Anal. **3** (2004), 653–662.

[6] F. Cîrstea, Y. Du, *General uniqueness results and variation speed for blow-up solutions of elliptic equations*, Proc. London Math. Soc. **91** (2005), 459–482.

[7] F. Cîrstea, V. Rădulescu, *Uniqueness of the blow-up boundary solution of logistic equations with absorbtion*, C. R. Acad. Sci. Paris Sér. I Math. **335** (5) (2002), 447–452.

[8] F. Cîrstea, V. Rădulescu, *Nonlinear problems with singular boundary conditions arising in population dynamics: a Karamata regular variation theory approach*, Asymptotic Anal. **46** (2006), 275–298.

[9] L. Damascelli, *Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results*, Ann. Ins. H. Poincaré Anal. Non Linéaire **15** (1998), 493–516.

[10] M. Del Pino, R. Letelier, *The influence of domain geometry in boundary blow-up elliptic problems*, Nonlinear Anal. **48** (6) (2002), 897–904.

[11] G. Díaz, R. Letelier, *Explosive solutions of quasilinear elliptic equations: Existence and uniqueness*, Nonlinear Anal. **20** (1993), 97–125.

[12] E. Di Benedetto, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonl. Anal. **7** (1983), 827–850.

[13] Y. Du, *Asymptotic behavior and uniqueness results for boundary blow-up solutions*, Diff. Int. Eqns. **17** (2004), 819–834.

[14] Y. Du, Z. Guo, *Boundary blow-up solutions and their applications in quasilinear elliptic equations*, J. Anal. Math. **89** (2003), 277–302.

[15] Y. Du, Q. Huang, *Blow-up solutions for a class of semilinear elliptic and parabolic equations*, SIAM J. Math. Anal. **31** (1999), 1-18.

[16] J. García-Melián, *Nondegeneracy and uniqueness for boundary blow-up elliptic problems*, J. Diff. Eqns. **223** (2006), 208–227.

[17] J. García-Melián, *Uniqueness for boundary blow-up problems with continuous weights*, Proc. Amer. Math. Soc. **135** (2007), 2785–2793.

[18] J. García-Melián, *Large solutions for equations involving the p-Laplacian and singular weights*, submitted for publication.

[19] J. García-Melián, R. Letelier-Albornoz, J. Sabina de Lis, *Uniqueness and asymptotic behaviour for solutions of semilinear problems with boundary blow-up*, Proc. Amer. Math. Soc. **129** (2001), no. 12, 3593–3602.

[20] J. García-Melián, J. D. Rossi, J. Sabina de Lis, *Large solutions for the Laplacian with a power nonlinearity given by a variable exponent*, to appear in Ann. Inst. H. Poincaré (C), Nonlinear Analysis.

[21] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer–Verlag, 1983.

[22] Z. Guo, J. R. L. Webb, *Structure of boundary blow-up solutions of quasilinear elliptic problems I. Large and small solutions*, Proc. Roy. Soc. Edinburgh **135A** (2005), 615–642.

[23] Z. Guo, J. R. L. Webb, *Structure of boundary blow-up solutions of quasilinear elliptic problems II: small and intermediate solutions*, J. Diff. Eqns. **211** (2005), 187–217

[24] J. B. Keller, *On solutions of $\Delta u = f(u)$*, Comm. Pure Appl. Math. **10** (1957), 503–510.

[25] V. A. Kondrat'ev, V. A. Nikishkin, *Asymptotics, near the boundary, of a solution of a singular boundary value problem for a semilinear elliptic equation*, Differential Equations **26** (1990), 345–348.

[26] G. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonl. Anal. **12** (1988), 1203–1219.

[27] C. Loewner, L. Nirenberg, *Partial differential equations invariant under conformal of projective transformations*, Contributions to Analysis (a collection of papers dedicated to Lipman Bers), Academic Press, New York, 1974, p. 245–272.

[28] J. López-Gómez, *The boundary blow-up rate of large solutions*, J. Diff. Eqns. **195** (2003), 25–45.

[29] J. López-Gómez, *Optimal uniqueness theorems and exact blow-up rates of large solutions*, J. Diff. Eqns. **224** (2006), 385–439.

[30] M. Marcus, L. Véron, *Uniqueness and asymptotic behaviour of solutions with boundary blow-up for a class of nonlinear elliptic equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (2) (1997), 237–274.

[31] J. Matero, *Quasilinear elliptic equations with boundary blow-up*, J. Anal. Math. **69** (1996), 229–247.

[32] P. J. McKenna, W. Reichel, W. Walter, *Symmetry and multiplicity for nonlinear elliptic differential equations with boundary blow-up*, Nonl. Anal. **28** (1997), 1213–1225.

[33] A. Mohammed, *Existence and asymptotic behavior of blow-up solutions to weighted quasilinear equations*, J. Math. Anal. Appl. **298** (2004), 621–637.

[34] A. Mohammed, G. Porcu, G. Porru, *Large solutions to some non-linear O.D.E. with singular coefficients*, Nonlinear Anal. **47** (2001), 513–524.

[35] R. Osserman, *On the inequality $\Delta u \geq f(u)$*, Pacific J. Math. **7** (1957), 1641–1647.

[36] P. Tolksdorf, *The Dirichlet problem in domains with conical boundary points*, Comm. Part. Diff. Eqns. **8** (1983), 773–817.

[37] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Diff. Eqns. **51** (1984), 126–150.

[38] J. L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations.* Appl. Math. Optim. **12** (1984), 191–202.

[39] L. Véron, *Semilinear elliptic equations with uniform blowup on the boundary*, J. Anal. Math. **59** (1992), 231–250.

[40] Z. Zhang, *A remark on the existence of explosive solutions for a class of semilinear elliptic equations*, Nonlinear Anal. **41** (2000), 143–148.

J. García-Melián
Dpto. de Análisis Matemático,
Universidad de La Laguna,
C/ Astrofísico Francisco Sánchez s/n, 38271
La Laguna, SPAIN
E-mail address: jjgarmel@ull.es